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Free vibration analysis of extension-torsion coupled elements using a dynamic finite element (DFE) formulation

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Free Vibration Analysis of Extension-Torsion Coupled Elements Using a Dynamic Finite Element (DFE) Formulation

by

Andrew Roach, B.Eng., Ryerson University, June 2003

A Project
presented to Ryerson University

in partial fulfillment of the
requirements for the degree of
Master of Engineering
in the Program of
Mechanical Engineering

Toronto, Ontario, Canada, 2005

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Andrew Roach

Free Vibration Analysis of Extension-Torsion Coupled Structural Elements Using a Dynamic Finite Element (DFE) Formulation

Andrew Roach, B.Eng., June 2003

Master of Engineering

Ryerson University, Toronto, Ontario, Canada

In this report, the extension-torsion coupled vibration behavior of several structural members is investigated. In order to solve the governing differential equations of motion for the problem, three different approaches, namely the dynamic stiffness matrix (DSM), finite element (FEM), and dynamic finite element (DFE) methods are used. Three different engineering applications of interest are identified, namely, a helical spring, a wire rope and laminated composite beams. For each of these applications, a method for determining the cross-sectional stiffness constants of interest is first introduced. Illustrative examples of each system are then studied where resulting natural frequencies and modes are compared to those available in literature. In order to determine the performance of each solution method in the determination of the dynamic behavior of these systems, all three (DSM, FEM, and DFE) methods are used in the examples, and a comparative study among the results is then carried out to gauge the accuracy of each approach.

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Andrew Roach

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Nomenclature

Symbol	Definition
E_1	Laminate Longitudinal Modulus of Elasticity
E_2	Laminate Transverse Modulus of Elasticity
G_{12}	Laminate Shear Modulus
ν_{12}	Laminate Poisson's ratio
Q_{ij}	Reduced Laminate Stiffness Terms
$\bar{Q}_{ij}$	Laminate Transformed Reduced Stiffness Terms
A_{ij}	Laminate Stiffness Terms
NOL	Number of Layers which make up Laminate
h_k	Thickness of Lamina 'k'
γ_{ij}	In-Plane Strains
Φ_*	Strain Energy Density of Shell
Φ	One-Dimensional Beam Energy Density
$A(s)$	Material Effective Axial Stiffness Term
$B(s)$	Material Effective Extension-Torsion Coupled Stiffness Term
$C(s)$	Material Effective Torsional Stiffness Term
r	Composite Circular Tube Radius
b	Composite Box-Beam Base
h	Composite Box-Beam Height
A_e	Enclosed Area of Cross-Section
P	Perimeter of Section
f_o	Function Value at Beam Root
f_{tip}	Function Value at Beam Tip
f_{avg}	Function Average Value over Interval
c	Taper Ratio
n	Taper Exponent

Symbol	Definition
C_{11}	Axial Stiffness Term
C_{12}	Extension-Torsion Coupled Stiffness Term
C_{22}	Torsional Stiffness Term
C_{33}	Horizontal Bending Stiffness Term
C_{44}	Vertical Bending Stiffness Term
F	Axial Force on Member
T	Torque on Member
u, U	Axial Displacement of Member
θ, Θ	Rotational Displacement of Member
m	Mass per Unit Length
I	Polar Mass Moment of Inertia per Unit Length
ω	Circular Frequency
β_1, β_2	DSM Constant
$\kappa_{\beta_1}, \kappa_{\beta_2}$	DSM Constant
g_{β_1}, g_{β_2}	DSM Constant
h_{β_1}, h_{β_2}	DSM Constant
L	Member Length
ξ	Non-Dimensionalized Length
$[F]$	Element Force Matrix
$[K]$	Element Stiffness Matrix
$[U]$	Element Displacement Vector
$[\bar{K}]$	Global Stiffness Matrix
$[\bar{U}]$	Global Displacement Vector
$\mathbb{R}$	Residual
N	Weighting (Shape) Function
W	Total Virtual Work

Symbol	Definition
W_{int}	Internal Virtual Work
W_{ext}	External Virtual Work
NOE	Number of Elements
e	Element Number
$[M]$	Element Mass Matrix
$[\bar{M}]$	Global Mass Matrix
δ	Virtual Displacement
α	DFE Shape Function Variable (Extension)
τ	DFE Shape Function Variable (Torsion)
J	Total Number of Frequencies Below Trial Value
J_m	Number of Component CL/CL Frequencies Below Trial Value
J_o	Total Number of All Components' CL/CL Frequencies Below Trial Value
$sign[\]$	Number of Negative Elements on Main Diagonal of Matrix; sign count of []
$[\overline{K(w)}^\Delta]$	Upper Triangularized Global Stiffness Matrix
ϕ	Spring Element Helix Angle
r_{spring}	Spring Element Radius
R_{spring}	Spring Element Wire Radius
T_ϕ	Tangent of Spring Helix Angle ‘ ϕ ’
ΔL	Change in Specimen Length
L_o	Undeformed Specimen Length

Chapter 1: Introduction

This report will be primarily concerned with the extension-torsion coupled vibration of various systems (see section 2.1). No matter if the coupling occurs due to geometric or material effects, coupled systems exist everywhere, with extension-torsion coupled systems in particular finding numerous applications in mechanical, structural, and aerospace engineering. This problem has been looked at by various authors [1-7,27,28] who have devised numerous solutions to the problem. Included in this investigation is the derivation of three particular solution methods. First is the Dynamic Stiffness Matrix (DSM) which is an exact formulation based on the algebraic inversion of the compliance matrix (see section 2.2). The second is the Finite Element Method (FEM) which is used extensively throughout the literature and represents an approximate solution to the differential equations of motion for the problem (see section 2.3). The third and final approach used is the Dynamic Finite Element (DFE, see section 2.4) . This method incorporates frequency dependant shape functions in a FEM type formulation. The DFE is also approximate, but is seen to have better convergence rates than the FEM solution, particularly for higher frequencies. These three approaches were all found to deliver valid results as long as an appropriate mesh is used in the solution.

In this research, three categories of structural elements are studied. These “apparently different” systems all exhibit similar coupled vibrational behavior governed by analogous differential equations of motion. The first geometrically coupled system examined is a helical spring (see sections 3.1 and 4.1). For this type of structural member the bending displacements are ignored because springs are typically not loaded/excited in the lateral direction and therefore the behavior in this direction is not of great importance. These types of springs find typical application in various suspension and vibration damping systems, where accurate modeling of the extension-torsion coupled vibrational behavior is paramount, i.e., it determines the performance of the system in application. Determination of the axial, torsional and coupling stiffness constants for a spring is relatively easy with formulae given in references [10,14] in terms of the geometric and material properties of the spring. The end conditions of interest for this application are the CLamped-CLamped (CL/CL) and CLamped-FRee (CL/FR) boundary conditions since both find real world application. The presented uniform helical spring example

demonstrates the applicability of the methods derived to these problems by comparing frequency values to those found in the literature [10]. It also demonstrates the general applicability of the Wittrick-Williams method to solve eigenvalue problems, including the linear ones obtained from the FEM. The effect of boundary conditions on the convergence rates and the superiority of the DFE when compared to the FEM are also demonstrated. The mode shapes are also included in order to give the reader a full appreciation of the coupled nature of the vibration problem that these systems represent.

The second system studied here is a wire rope which also exhibits extension-torsion geometric coupling due to the twisting of the rope strands around each other (sections 3.2 and 4.2). The bending displacements, in this case, are also ignored since these ropes are only used in applications requiring high axial strength/stiffness. The primary application for these types of ropes is in the offshore industry, where they are used to moor ships, attach anchors, and hold in place oil platforms. Due to the diversity of the applications of interest, three particular boundary conditions are reviewed, CL/CL, CLamped-SWivel (CL/SW), where one end is allowed to displace in rotation only, and the SWivel-SWivel (SW/SW), where both ends are allowed to displace in rotation (section 4.2). The inclusion of the CL/SW and SW/SW boundary conditions helps to demonstrate the C^0 continuity of the second order vibration problem through depiction of the modes, and helps to further demonstrate the difference in behavior between a wire rope and a beam in bending governed by a 4th order equation of motion. The effect of wire length on the natural frequency is also reviewed, showing no overall effect on the accuracy of the solution regardless of method. Determining the section properties for this type of wire rope requires experimental testing, with most of the analytical models being based on test data [20]; therefore, an experimental method for determining section stiffness parameters is given for this application [13].

The third category of structural system studied in this research includes thin-walled, closed-section anisotropic composite beams (sections 3.3, 4.3 - 4.7). These types of sections find their primary application in rotor and wind turbine blades, which benefit from the coupling present between the deformation modes [34]. The stiffness model used here is based on a contour analysis method [5] that allows this theory to be used for anisotropic beams with both

rectangular and circular cross-sections, as is done in the present study. The model makes use of numerous assumptions in order to simplify the problem, including that of no internal pressure, as well as negligible shear stress. These assumptions have been verified to be accurate through comparison of theoretical values with the experimental data of references [11,28]. The beam is also assumed to be long and slender, which is applicable to most real world applications for a thin-walled theory. For thick-walled and/or short beams, theories such as those of [27] must be used in order to determine the cross-sections stiffness properties. Classic laminate theory (section 3.3.1) is used in order to determine the mechanical properties for the composite which makes up the beam. In order to fully appreciate the dynamic behavior of these systems, and allow this model to be used for numerous applications, two different geometries of interest are investigated (sections 4.3 to 4.6). The first of these geometries is a rectangular box-beam (sections 4.3 and 4.4). This cross-section is of interest not only because it finds application itself in industry, but also because through the use of equivalent parameters, it can be used in order to approximate the extension, torsion and extension-torsion coupled stiffnesses for a non-circular/rectangular cross-section [18,27]. The second application of interest is a circular tube (sections 4.5 and 4.6), which also finds numerous applications in industry and therefore warrants investigation. These cross-sections were also chosen due to the availability of verified numerical results, allowing models developed to be validated.

In the beam examples of sections 4.3 to 4.6, the effects of the stiffness method used will be demonstrated. This will show how important it is to use consistent methodology for determination of the sections stiffness terms and vibration frequencies (i.e., the inclusion of couplings). The effects that changes in laminate have on the dynamic behavior of these systems is also investigated, and demonstrates the major effect ply-angle changes can have on the response. The use of an average value technique for calculating parameters for non-uniform beams is also introduced (section 3.3.4), with the effects of these non-uniformities on the dynamic behavior investigated (section 4.7). These examples demonstrate that through accurate modeling, the dynamic behavior of these types of beams can be tailored in order to suit the particular application, allowing for more efficient design. The methods developed in this project are shown to be capable of accurately determining the dynamic properties of these composite

beam sections and, due to their relative simplicity and ease of calculation, make them excellent for use in preliminary design and optimization studies.

As will be finally seen in the concluding remarks (see section 5), the models, and numerical methods derived in this work, were used to evaluate the natural frequencies and modes for various extension-torsion coupled systems. The results showed the applicability of the methods developed to any systems which exhibit this type of coupling. Many interesting trends and aspects of coupled vibration were also discovered through the completion of the examples contained in this report and are discussed later. The models developed in this project, even though simple, were able to reproduce validated free vibration frequencies and modes for the applications shown here and can be used in optimization type studies due to their relative simplicity and quickness of calculation.

Chapter 2: Theory

In this chapter, the general theory related to the coupled extension-torsion vibration of various structural elements is presented. First, the differential equations of motion are presented and briefly discussed. Three different solutions to these differential equations are then derived, first the DSM which represents an exact solution to these equations. Next the FEM and DFE methods are introduced, each of these represents an approximate solution to the differential equations of motion and makes use of different types of approximation functions for the independent variables of the problem. Finally, the Wittrick-Williams (WW) root counting algorithm is discussed, allowing for a solution to the non-linear eigenvalue problems that the DSM and DFE methods yield.

2.1 Extension-Torsion Coupled Vibration

Due to the coupled nature of the extension and torsion deformations of the types of systems studied here, the differential equations, and therefore their solutions are more difficult to obtain than those of uncoupled vibration. First, the sign convention which is used throughout the project is shown, then the differential equations of motion are written with regard to this convention. Next the time dependence of the system is removed using a standard sine wave and circular frequency assumption (i.e., harmonic motion assumptions). Three methods, the DSM, FEM, and DFE are then used in order to find either exact or approximate solutions to these frequency dependant differential equations of motion. These three methods are used to analyze the examples of the project, with the results being compared to those available in the literature.

The sign convention used in this derivation for displacement, rotation, force and moment is shown in Figure 1 [1-5,10,14].

Figure 1: Sign Convention Used in Extension-Torsion Coupled Model

The governing differential equations of motion for free vibration of an extension-torsion coupled member are written as [1-5,10,14,15,26-28]:

$$C_{11}u_{,xx} + C_{12}\theta_{,xx} = mu_{,tt} \quad (1)$$

$$C_{12}u_{,xx} + C_{22}\theta_{,xx} = I\theta_{,tt} \quad (2)$$

with ' m ' being the mass per unit length and ' I ' being the polar mass moment of inertia per unit length of the element. ' C_{11} ', ' C_{12} ', and ' C_{22} ' are the extensional, extension-torsion coupling and torsional stiffness terms, respectively.

Assuming small deformations and a linear relationship between force and strains, one can write [1,10,14],

$$F = C_{11}\varepsilon + C_{12}\gamma \quad (3)$$

$$T = C_{12}\varepsilon + C_{22}\gamma \quad (4)$$

where ' F ' and ' T ', respectively, are the axial force and torque present in the member and,

$$\varepsilon = u_{,x} \quad (5)$$

$$\gamma = \theta_{,x} \quad (6)$$

where ' ε ' and ' γ ' are the axial and torsional strains, respectively.

It is also assumed that the motion takes the form of a sine wave with amplitudes ' $U(x)$ ' and ' $\Theta(x)$ ', and circular frequency ' ω ':

$$u(x,t) = U(x) \sin \omega t \quad (7)$$

$$\theta(x,t) = \Theta(x) \sin \omega t \quad (8)$$

This allows the removal of the time dependence within the system. Substituting the assumed frequency equations (7) and (8) into the original governing differential equations (1) and (2) one then obtains the following set of differential equations governing the extension-torsion coupled vibration of structural elements:

$$C_{11} U_{,xx} + C_{12} \Theta_{,xx} = -m\omega^2 U \quad (9)$$

$$C_{12} U_{,xx} + C_{22} \Theta_{,xx} = -I\omega^2 \Theta \quad (10)$$

2.2 Derivation of the Dynamic Stiffness Matrix (DSM) Method

In this section, the development of the DSM stiffness matrix method is presented [1,10]. This method represents an exact solution to the differential equations of motion, allowing one to make use of only one element in order to determine any number of the natural frequencies of the system. The DSM method results in a frequency dependant stiffness matrix, requiring the use of a non-linear eigenvalue solution method, like the WW [32], in order to determine a result. The DSM is used as a benchmark in order to validate results and determine the convergence rates of the FEM and DFE solutions. First, the DSM results were compared to those available in literature in order to ensure the correctness of the DSM-based software developed in this project.

Solving the frequency dependant equations of motion (9) and (10) by eliminating either ‘ U ’ or ‘ Θ ’ leads to the following fourth order equation,

$$\{(C_{11}C_{22} - C_{12}^2)D^4 + (C_{11}I + C_{22}m)\omega^2 L^2 D^2 + mI\omega^4 L^4\}W = 0 \quad (11)$$

where: $W = U \text{ or } \Theta$; $D = \frac{\partial}{\partial \xi}$; $\xi = \frac{x}{L}$

This can then be expressed as,

$$(D^2 + \beta_1^2)(D^2 + \beta_2^2)W = 0 \quad (12)$$

where ' β_1^2 ' and ' β_2^2 ' are given by applying the quadratic formula to equation (11) giving:

$$\left. \begin{matrix} \beta_1^2 \\ \beta_2^2 \end{matrix} \right\} = \frac{\omega^2 L^2 [(C_{11}I + C_{22}m) \pm \{(C_{11}I - C_{22}m)^2 + 4mIC_{12}^2\}^{1/2}]}{2(C_{11}C_{22} - C_{12}^2)} \quad (13)$$

Since both ' β_1 ' and ' β_2 ' are real numbers, their squares must be positive values. This means that equation (13) must always give a positive result. Since the numerator of (13) can not be negative [1,10,13,14], one should have ' $C_{11}C_{22} > C_{12}^2$ ' in order for the denominator of this equation to remain positive.

The general solution to the differential equation (11) can then be written in terms of both ' U ' and ' Θ ' as [1,10]:

$$U(\xi) = A_1 \sin \beta_1 \xi + A_2 \cos \beta_1 \xi + A_3 \sin \beta_2 \xi + A_4 \cos \beta_2 \xi \quad (14)$$

$$\Theta(\xi) = B_1 \sin \beta_1 \xi + B_2 \cos \beta_1 \xi + B_3 \sin \beta_2 \xi + B_4 \cos \beta_2 \xi \quad (15)$$

If equations (14) and (15) are substituted into either equation (9) or (10) one can then find that the constants ' $A_1 - A_4$ ' and ' $B_1 - B_4$ ' can be related by,

$$B_1 = \kappa_{\beta_1} A_1 \quad B_2 = \kappa_{\beta_1} A_2 \quad B_3 = \kappa_{\beta_2} A_3 \quad B_4 = \kappa_{\beta_2} A_4 \quad (16)$$

with:

$$\kappa_{\beta_1} = \frac{C_{12}\beta_1^2}{(I\omega^2 L^2 - C_{22}\beta_1^2)} = \frac{(m\omega^2 L^2 - C_{11}\beta_1^2)}{C_{12}\beta_1^2} \quad (17)$$

$$\kappa_{\beta_2} = \frac{C_{12}\beta_2^2}{(I\omega^2 L^2 - C_{22}\beta_2^2)} = \frac{(m\omega^2 L^2 - C_{11}\beta_2^2)}{C_{12}\beta_2^2} \quad (18)$$

The relationship between these two sets of constants allows one to define the force and torque on the system in terms of either the displacement or rotation only. If the force and torque equations (3) and (4) are expressed in terms of the displacement constants only they become,

$$F(\xi) = A_1 g_{\beta_1} \cos \beta_1 \xi - A_2 g_{\beta_1} \sin \beta_1 \xi + A_3 g_{\beta_2} \cos \beta_2 \xi - A_4 g_{\beta_2} \sin \beta_2 \xi \quad (19)$$

$$T(\xi) = A_1 h_{\beta_1} \cos \beta_1 \xi - A_2 h_{\beta_1} \sin \beta_1 \xi + A_3 h_{\beta_2} \cos \beta_2 \xi - A_4 h_{\beta_2} \sin \beta_2 \xi \quad (20)$$

where:

$$g_{\beta_1} = \frac{\beta_1 (C_{11} + \kappa_{\beta_1} C_{12})}{L} \quad (21)$$

$$g_{\beta_2} = \frac{\beta_2 (C_{11} + \kappa_{\beta_2} C_{12})}{L} \quad (22)$$

$$h_{\beta_1} = \frac{\beta_1 (C_{12} + \kappa_{\beta_1} C_{22})}{L} \quad (23)$$

$$h_{\beta_2} = \frac{\beta_2 (C_{12} + \kappa_{\beta_2} C_{22})}{L} \quad (24)$$

Equations (21) through (24) are constants that relate the force in the member to the torsion through the value of the stiffness constants and ‘ κ_{β_1} ’ and ‘ κ_{β_2} ’.

The end conditions for the displacement and rotation using Figure 2 are:

$$\text{at end 1 (i.e. } \xi = 0): U = U_1 \text{ and } \Theta = \Theta_1 \quad (25)$$

$$\text{at end 2 (i.e. } \xi = 1): U = U_2 \text{ and } \Theta = \Theta_2 \quad (26)$$

Figure 2: Nodal Displacements, Rotations, Forces and Torques

Making use of the sign convention shown in Figure 1, the forces and moments on the end of the element are given as:

$$\text{at end 1 (i.e. } \xi = 0 \text{): } F = -F_1 \text{ and } T = -T_1 \quad (27)$$

$$\text{at end 2 (i.e. } \xi = 1 \text{): } F = F_2 \text{ and } T = T_2 \quad (28)$$

Combing the end conditions for displacement and rotation (25) and (26) with the displacement and rotation equations (14) and (15) and applying the relationship between the displacement and rotation constants (17) and (18) leads to the following matrix equation,

$$\begin{bmatrix} U_1 \\ \Theta_1 \\ U_2 \\ \Theta_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & \kappa_{\beta_1} & 0 & \kappa_{\beta_2} \\ \sin \beta_1 & \cos \beta_1 & \sin \beta_2 & \cos \beta_2 \\ \kappa_{\beta_1} \sin \beta_1 & \kappa_{\beta_1} \cos \beta_1 & \kappa_{\beta_2} \sin \beta_2 & \kappa_{\beta_2} \cos \beta_2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} \quad (29)$$

which can be written as:

$$[U] = [B][A] \quad (30)$$

Performing a similar substitution using the end conditions for force and torque (27) and (28), the force and torque equations (19) and (20), and the relationship between the force and torque constants (21)-(24) one then obtains,

$$\begin{bmatrix} F_1 \\ T_1 \\ F_2 \\ T_2 \end{bmatrix} = \begin{bmatrix} -g_{\beta_1} & 0 & -g_{\beta_2} & 0 \\ -h_{\beta_1} & 0 & -h_{\beta_2} & 0 \\ g_{\beta_1} \cos \beta_1 & -g_{\beta_1} \sin \beta_1 & g_{\beta_2} \cos \beta_2 & -g_{\beta_2} \sin \beta_2 \\ h_{\beta_1} \cos \beta_1 & -h_{\beta_1} \sin \beta_1 & h_{\beta_2} \cos \beta_2 & -h_{\beta_2} \sin \beta_2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} \quad (31)$$

which can be expressed as:

$$[F] = [D][A] \quad (32)$$

Equations (30) and (32) can be solved in order to give the frequency dependant stiffness matrix for a single DSM element as,

$$[F] = [K][U] \quad (33)$$

$$\begin{bmatrix} F_1 \\ T_1 \\ F_2 \\ T_2 \end{bmatrix} = \begin{bmatrix} K_{1,1} & K_{1,2} & K_{1,3} & K_{1,4} \\ & K_{2,2} & K_{2,3} & K_{2,4} \\ & & K_{3,3} & K_{3,4} \\ \text{symmetry} & & & K_{4,4} \end{bmatrix} \begin{bmatrix} U_1 \\ \Theta_1 \\ U_2 \\ \Theta_2 \end{bmatrix} \quad (34)$$

where:

$$[K] = [D][B]^{-1} \quad (35)$$

This matrix inversion and premultiplication was carried out algebraically in references [1,10] and the results for each entry in matrix can be written as,

$$K_{1,1} = K_{3,3} = \frac{(k_{\beta_1} g_{\beta_2} \sin \beta_1 \cos \beta_2 - k_{\beta_2} g_{\beta_1} \cos \beta_1 \sin \beta_2)}{\Delta} \quad (36)$$

$$K_{1,2} = K_{3,4} = \frac{(g_{\beta_1} \cos \beta_1 \sin \beta_2 - g_{\beta_2} \sin \beta_1 \cos \beta_2)}{\Delta} \quad (37)$$

$$K_{1,3} = -\frac{(k_{\beta_1} g_{\beta_2} \sin \beta_1 - k_{\beta_2} g_{\beta_1} \sin \beta_2)}{\Delta} \quad (38)$$

$$K_{1,4} = K_{2,3} = -\frac{(g_{\beta_1} \sin \beta_2 - g_{\beta_2} \sin \beta_1)}{\Delta} \quad (39)$$

$$K_{2,2} = K_{4,4} = \frac{(h_{\beta_1} \cos \beta_1 \sin \beta_2 - h_{\beta_2} \sin \beta_1 \cos \beta_2)}{\Delta} \quad (40)$$

$$K_{2,4} = -\frac{(h_{\beta_1} \sin \beta_2 - h_{\beta_2} \sin \beta_1)}{\Delta} \quad (41)$$

with:
$$\Delta = (k_{\beta_1} - k_{\beta_2}) \sin \beta_1 \sin \beta_2 \quad (42)$$

This single element equation can then be used in order to assemble a global stiffness matrix for a multi-element solution using the standard FEM assembly method. This is required if there are changes in geometry or material along the length of the member or if one desires to use the DSM method in order to determine the natural modes of structures assembled from extension-torsion coupled elements. Once assembled, this method leads to the non-linear eigenvalue problem,

$$[\bar{K}(\omega)][\bar{U}] = [0] \quad (43)$$

which, after the application of appropriate boundary conditions, must be solved using an iterative method such as the one develop by Wittrick and Williams [32] introduced later in this report.

2.3 Derivation of the Finite Element Model (FEM)

The finite element method is now used in order to determine an approximate solution to the equations of motion governing the coupled extension-torsion vibrations of structural members. This general approach is used widely throughout industry and is based on established algorithms, allowing it to be applied to numerous types of systems. Making use of a weighted residual technique and linear shape functions [17] for the independent variables, this method yields constant value matrices for both the stiffness and inertial terms. These matrices can then be assembled using the standard technique, in order to obtain a global linear eigen-value problem which can be solved using many numerical methods [33].

The frequency dependant equations of motion governing the coupled extension-torsion free vibration of structural members are:

$$C_{11} U_{,xx} + C_{12} \Theta_{,xx} = -m\omega^2 U \quad (44)$$

$$C_{12} U_{,xx} + C_{22} \Theta_{,xx} = -I\omega^2 \Theta \quad (45)$$

If a function is chosen to approximate the independent variables ' U ' and ' Θ ' of (44) and (45), it will not in general satisfy the equations exactly, but instead will leave some residual value, $\mathbb{R}_1$ and $\mathbb{R}_2$, [15] as:

$$m\omega^2 U + C_{11} U_{,xx} + C_{12} \Theta_{,xx} = \mathbb{R}_1 \quad (46)$$

$$I\omega^2 \Theta + C_{12} U_{,xx} + C_{22} \Theta_{,xx} = \mathbb{R}_2 \quad (47)$$

The Galerkin type weighted residual formulation, using appropriate weighting functions, allows one to define the integral equations associated with (46) and (47). These are set to zero when evaluated over the entire system:

$$\int_L \delta U \mathbb{R}_1 dx = 0 \quad \text{and} \quad \int_L \delta \Theta \mathbb{R}_2 dx = 0 \quad (48)$$

Applying Galerkin's method and allowing the weighting functions ' δU ' and ' $\delta \Theta$ ' to be the same as the shape functions, ' N ' (see reference [15]), leads to:

$$\int_0^L m\omega^2 U N dx + \int_0^L C_{11} U_{,xx} N dx + \int_0^L C_{12} \Theta_{,xx} N dx = 0 \quad (49)$$

$$\int_0^L I\omega^2 \Theta N dx + \int_0^L C_{12} U_{,xx} N dx + \int_0^L C_{22} \Theta_{,xx} N dx = 0 \quad (50)$$

By applying an appropriate number of integration by parts to both (49) and (50) one is able to reduce the order of the required inter-element continuity from C^1 to C^0 , while also making ‘ N_i ’, ‘ Θ ’ and ‘ U ’ of the same order. Performing one integration by parts, in this case leads to:

$$W_a = \int_0^L m\omega^2 U N dx + [C_{11} U_{,x} N]_0^L - \int_0^L C_{11} U_{,x} N_{,x} dx + [C_{12} \Theta_{,x} N]_0^L - \int_0^L C_{12} \Theta_{,x} N_{,x} dx = 0 \quad (51)$$

$$W_t = \int_0^L I\omega^2 \Theta N dx + [C_{12} U_{,x} N]_0^L - \int_0^L C_{12} U_{,x} N_{,x} dx + [C_{22} \Theta_{,x} N]_0^L - \int_0^L C_{22} \Theta_{,x} N_{,x} dx = 0 \quad (52)$$

The boundary terms of (51) and (52) can be found equal to zero [33] giving:

$$[C_{11} U_{,x} N]_0^L = [C_{12} \Theta_{,x} N]_0^L = [C_{12} U_{,x} N]_0^L = [C_{22} \Theta_{,x} N]_0^L = 0$$

Expressions (51) and (52) also satisfy the Principle of Virtual Work (PVW) for the entire system written as:

$$W = W_{Ext} + W_{Int} \quad (53)$$

Since there is no applied force in the system (i.e. free vibration):

$$W_{Ext} = 0 \Rightarrow W = W_{Int} \quad (54)$$

Discretizing the problem into a certain number of elements using $\xi = \frac{x}{L}$ (and therefore $dx = L d\xi$) and substituting into (51) and (52) one obtains (see Figure 3),

Figure 3 : Discretization Scheme

$$W_a^e = \int_0^1 m\omega^2 U N d\xi - \int_0^1 \frac{C_{11} U_{,x} N_{,x}}{L} d\xi - \int_0^1 \frac{C_{12} \Theta_{,x} N_{,x}}{L} d\xi \quad (55)$$

$$W_t^e = \int_0^1 I\omega^2 \Theta N d\xi - \int_0^1 \frac{C_{12} U_{,x} N_{,x}}{L} d\xi - \int_0^1 \frac{C_{22} \Theta_{,x} N_{,x}}{L} d\xi \quad (56)$$

where the PVW for the element is then written as:

$$W = W_{int} = \sum_0^{NOE} W_{int}^e \quad (57)$$

$$W_{int}^e = W_a^e + W_t^e \quad (58)$$

Using a linear approximation for both variables and the shape (weighting) functions,

$$N = \begin{bmatrix} 1-\xi & \xi \end{bmatrix} \quad U = N \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \quad \Theta = N \begin{bmatrix} \Theta_1 \\ \Theta_2 \end{bmatrix}$$

and substituting back into (55) and (56) one obtains:

$$W_a^e = m\omega^2 L \int_0^1 \begin{bmatrix} 1-\xi \\ \xi \end{bmatrix} \begin{bmatrix} 1-\xi & \xi \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} d\xi - \int_0^1 \frac{C_{11}}{L} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} d\xi - \int_0^1 \frac{C_{12}}{L} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} \Theta_1 \\ \Theta_2 \end{bmatrix} d\xi \quad (59)$$

$$W_t^e = I\omega^2 L \int_0^1 \begin{bmatrix} 1-\xi \\ \xi \end{bmatrix} \begin{bmatrix} 1-\xi & \xi \end{bmatrix} \begin{bmatrix} \Theta_1 \\ \Theta_2 \end{bmatrix} d\xi - \int_0^1 \frac{C_{12}}{L} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} d\xi - \int_0^1 \frac{C_{22}}{L} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} \Theta_1 \\ \Theta_2 \end{bmatrix} d\xi \quad (60)$$

Integrating and combining the extension and torsion matrices into a single matrix one then obtains the final expression for the FEM as,

$$W_{int}^e = \left(\frac{1}{L} \begin{bmatrix} C_{11} & C_{12} & -C_{11} & -C_{12} \\ C_{12} & C_{22} & -C_{12} & -C_{22} \\ -C_{11} & -C_{12} & C_{11} & C_{12} \\ -C_{12} & -C_{22} & C_{12} & C_{22} \end{bmatrix} - \omega^2 \begin{bmatrix} \frac{mL}{3} & 0 & \frac{mL}{6} & 0 \\ 0 & \frac{IL}{3} & 0 & \frac{IL}{6} \\ \frac{mL}{6} & 0 & \frac{mL}{3} & 0 \\ 0 & \frac{IL}{6} & 0 & \frac{IL}{3} \end{bmatrix} \right) \begin{bmatrix} U_1 \\ \Theta_1 \\ U_2 \\ \Theta_2 \end{bmatrix}$$

or:

$$W_{int}^e = ([K] - \omega^2 [M])[U] \quad (61)$$

Element equations (61) are then assembled and the boundary conditions applied, to find the overall (global) system equation:

$$W_{Int} = \sum^{NOE} W_{Int}^e = ([\bar{K}] - \omega^2 [\bar{M}])[\bar{U}] = 0 \quad (62)$$

This represents a linear eigenproblem in terms of ω^2 which can easily be solved using many well known techniques [33].

2.4 Derivation of the Dynamic Finite Element (DFE) Model

The DFE is the last solution method developed here for the analysis of coupled extension-torsion free vibration of structural members. This approach is a mixture of the FEM and DSM methods, with the derivation starting with the use of a weighted residual method and following a similar procedure to that of the FEM formulation. The DFE, unlike the FEM, makes use of trigonometric interpolation (shape) functions, inspired from the DSM [1,10], in order to approximate the independent variables in the problem. The DFE solution yields a frequency dependant stiffness matrix, and consequently a non-linear eigenvalue problem, which requires the use of the Wittrick-Williams method or another root counting technique in order to calculate the systems frequency values. The DFE method has been developed and used to analyze various types of differential equations; in particular, other coupled free vibration applications can be seen in references [12,22,29,31] and demonstrate the general applicability of this method to these types of problems.

The equations of motion for the free vibration of extension-torsion coupled systems are:

$$C_{11} U_{,xx} + C_{12} \Theta_{,xx} = -m\omega^2 U \quad (63)$$

$$C_{12} U_{,xx} + C_{22} \Theta_{,xx} = -I\omega^2 \Theta \quad (64)$$

A Galerkin type weighted residual formulation, using the appropriate weighting functions allows one to define the integral equations associated with (63) and (64). These integral equations can now be set to zero, if evaluated over the entire system, yielding:

$$\int_0^L \delta U (m\omega^2 U) dx + \int_0^L \delta U (C_{11} U_{,xx}) dx + \int_0^L \delta U (C_{12} \Theta_{,xx}) dx = 0 \quad (65)$$

$$\int_0^L \delta \Theta (I\omega^2 \Theta) dx + \int_0^L \delta \Theta (C_{12} U_{,xx}) dx + \int_0^L \delta \Theta (C_{22} \Theta_{,xx}) dx = 0 \quad (66)$$

In order to fully define the problem, the boundary conditions must be applied. The application of integration by parts to (65) and (66) causes the boundary terms to appear in the equations. This is also in accordance with the Galerkin FEM formulation and reduces the inter-element continuity requirement from C^1 to C^0 (see also eqs (51) and (52)):

$$\int_0^L \delta U (m\omega^2 U) dx + [C_{11} \delta U U_{,x}]_0^L - \int_0^L \delta U_{,x} (C_{11} U_{,x}) dx + [C_{12} \delta U \Theta_{,x}]_0^L - \int_0^L \delta U_{,x} (C_{12} \Theta_{,x}) dx = 0 \quad (67)$$

$$\int_0^L \delta \Theta (I\omega^2 \Theta) dx + [C_{12} \delta \Theta U_{,x}]_0^L - \int_0^L \delta \Theta_{,x} (C_{12} U_{,x}) dx + [C_{22} \delta \Theta \Theta_{,x}]_0^L - \int_0^L \delta \Theta_{,x} (C_{22} \Theta_{,x}) dx = 0 \quad (68)$$

The boundary terms of (67) and (68) can be found equal to zero (see reference [33]).

Using the PVW seen in (53) and (54) one can write:

$$W_a = \int_0^L \delta U (m\omega^2 U) dx - \int_0^L \delta U_{,x} (C_{11} U_{,x}) dx - \int_0^L \delta U_{,x} (C_{12} \Theta_{,x}) dx = 0 \quad (69)$$

$$W_t = \int_0^L \delta \Theta (I\omega^2 \Theta) dx - \int_0^L \delta \Theta_{,x} (C_{12} U_{,x}) dx - \int_0^L \delta \Theta_{,x} (C_{22} \Theta_{,x}) dx = 0 \quad (70)$$

Discretizing the problem into elements (see Figure 3) using $\xi = \frac{x}{L}$, $dx = Ld\xi$ and $\frac{d()}{dx} = \frac{1}{L} \frac{d()}{d\xi}$ results in:

$$W_a^e = \int_0^1 \delta U (m\omega^2 U) L d\xi - \int_0^1 \frac{\delta U_{,\xi}}{L} \frac{(C_{11} U_{,\xi})}{L} L d\xi - \int_0^1 \frac{\delta U_{,\xi}}{L} \frac{(C_{11} \Theta_{,\xi})}{L} L d\xi \quad (71)$$

$$W_t^e = \int_0^1 \delta \Theta (I\omega^2 \Theta) L d\xi - \int_0^1 \frac{\delta \Theta_{,\xi}}{L} \frac{(C_{12} U_{,\xi})}{L} L d\xi - \int_0^1 \frac{\delta \Theta_{,\xi}}{L} \frac{(C_{22} \Theta_{,\xi})}{L} L d\xi \quad (72)$$

A second integration by parts is now applied to the element equations (71) and (72) leading to,

$$W_a^e = \int_0^1 \delta U (m\omega^2 U) L d\xi - \left[\frac{C_{11}}{L} \delta U_{,\xi} U \right]_0^1 + \int_0^1 \frac{C_{11}}{L} U \delta U_{,\xi\xi} d\xi - \int_0^1 \frac{C_{12}}{L} \delta U_{,\xi} \Theta_{,\xi} d\xi \quad (73)$$

$$W_t^e = \int_0^1 \delta \Theta (I\omega^2 \Theta) L d\xi - \left[\frac{C_{22}}{L} \delta \Theta_{,\xi} \Theta \right]_0^1 + \int_0^1 \frac{C_{22}}{L} \Theta \delta \Theta_{,\xi\xi} d\xi - \int_0^1 \frac{C_{12}}{L} \delta \Theta_{,\xi} U_{,\xi} d\xi \quad (74)$$

These expressions can now be rearranged by grouping like terms to yield:

$$W_a^e = \underbrace{\int_0^1 \left[m\omega^2 \delta U + \frac{C_{11} \delta U_{,\xi\xi}}{L^2} \right] U L d\xi}_{(*)} - \left[\frac{C_{11}}{L} \delta U_{,\xi} U \right]_0^1 - \int_0^1 \frac{C_{12}}{L} \delta U_{,\xi} \Theta_{,\xi} d\xi \quad (75)$$

$$W_t^e = \underbrace{\int_0^1 \left[I\omega^2 \delta \Theta + \frac{C_{22} \delta \Theta_{,\xi\xi}}{L^2} \right] \Theta L d\xi}_{(**)} - \left[\frac{C_{22}}{L} \delta \Theta_{,\xi} \Theta \right]_0^1 - \int_0^1 \frac{C_{12}}{L} \delta \Theta_{,\xi} U_{,\xi} d\xi \quad (76)$$

The expression within the first integral (*) of (75) and similarly (**) of (76) can be evaluated to give zero as seen in [16]. In what follows, this process is briefly explained.

The interpolation (shape) functions can be found for each independent variable through the appropriate choice of basis functions that make, in this case, the integral terms (*) and (**) zero. This can be done using the solutions to (*) and (**) as the basis functions of

approximation. This allows for the derivation of trigonometric shape functions that are continuous not only over the element but also over the structure [12]. One can derive the shape functions using the following matrix technique.

Assuming the following basis functions of approximation space for axial displacements (extension),

$$\langle P_n(x) \rangle = \left\langle \cos \alpha \frac{x}{L} \quad \frac{\sin \alpha \frac{x}{L}}{\alpha} \right\rangle \quad (77)$$

and substituting $x = L\xi$ for the reference element, one obtains:

$$\langle P_n(\xi) \rangle = \left\langle \cos \alpha \xi \quad \frac{\sin \alpha \xi}{\alpha} \right\rangle \quad (78)$$

Then using the following approach one is able to derive the shape functions for use in the DFE formulation. The nodal approximation for ‘ $\delta U(\xi)$ ’ can be written as:

$$\delta U(\xi) = \langle P_n(\xi) \rangle \{ \delta a \} \quad (79)$$

Introducing the boundary values then leads to,

$$\{ \delta U \} = [P_n] \{ \delta a \} \quad (80)$$

where the matrix $[P_n]$ is defined such that:

$$[P_n] = \begin{bmatrix} \langle P_n(\xi = 0) \rangle \\ \langle P_n(\xi = 1) \rangle \end{bmatrix} \quad (81)$$

The nodal approximation for ' $\delta U(\xi)$ ' is then obtained as,

$$\delta U(\xi) = \langle Pn(\xi) \rangle [Pn]^{-1} \{ \delta U \} = \langle N(\xi, \omega) \rangle \{ \delta U \} \quad (82)$$

where the dynamic (frequency dependant) shape functions for the axial displacement approximation are:

$$\langle N(\xi, \omega) \rangle = \langle Pn(\xi) \rangle [Pn]^{-1} = \left\langle \frac{\sin \alpha(1-\xi)}{\sin \alpha} \quad \frac{\sin \alpha \xi}{\sin \alpha} \right\rangle = \langle N_1 \quad N_2 \rangle \quad (83)$$

The following procedure must also be performed in terms of ' τ ' in order to develop the torsional dynamic shape functions for the second differential equation (eq (64)). This is not included since the answers are analogous to those found using ' α '. The term ' α ' is solved for by making the term (*) in (75) zero and ' τ ' is given in a similar way using term (**) in (76). The derivation of these terms is performed in [16] with the results given here as:

$$\alpha = \omega L \sqrt{\frac{m}{C_{11}}} \qquad \tau = \omega L \sqrt{\frac{I}{C_{22}}}$$

The definition of shape functions allows one to write the nodal approximations of the variables as:

$$\begin{aligned} U &= [N_1 \quad N_2]_{\alpha} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} & U_{,\xi} &= [N_{1,\xi} \quad N_{2,\xi}]_{\alpha} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \\ \Theta &= [N_1 \quad N_2]_{\tau} \begin{bmatrix} \Theta_1 \\ \Theta_2 \end{bmatrix} & \Theta_{,\xi} &= [N_{1,\xi} \quad N_{2,\xi}]_{\tau} \begin{bmatrix} \Theta_1 \\ \Theta_2 \end{bmatrix} \end{aligned}$$

If the variables and weighting functions are defined within the same approximation space (i.e. Galerkin method), then using $(NU)^T = U^T N^T$ leads to:

$$\begin{aligned}\delta U_{,\xi} &= \begin{bmatrix} N_{1,\xi} & N_{2,\xi} \end{bmatrix}_{\alpha} \begin{bmatrix} \delta U_1 \\ \delta U_2 \end{bmatrix} = \begin{bmatrix} N_{1,\xi} \\ N_{2,\xi} \end{bmatrix}_{\alpha} [\delta U_1 \quad \delta U_2] \\ \delta \Theta_{,\xi} &= \begin{bmatrix} N_{1,\xi} & N_{2,\xi} \end{bmatrix}_{\tau} \begin{bmatrix} \delta \Theta_1 \\ \delta \Theta_2 \end{bmatrix} = \begin{bmatrix} N_{1,\xi} \\ N_{2,\xi} \end{bmatrix}_{\tau} [\delta \Theta_1 \quad \delta \Theta_2]\end{aligned}$$

Then, substituting the expressions for the interpolation functions and evaluating the boundary terms of both (75) and (76) gives:

$$W_a^e = \frac{-C_{11}}{L} [\delta U_{,\xi=1} U_{\xi=1} - \delta U_{,\xi=0} U_{\xi=0}] - \frac{C_{12}}{L} [\delta U_1 \quad \delta U_2] \int_0^1 \begin{bmatrix} N_{1,\xi} \\ N_{2,\xi} \end{bmatrix}_{\alpha} \begin{bmatrix} N_{1,\xi} & N_{2,\xi} \end{bmatrix}_{\tau} d\xi \begin{bmatrix} \Theta_1 \\ \Theta_2 \end{bmatrix} \quad (84)$$

and,

$$W_t^e = \frac{-C_{22}}{L} [\delta \Theta_{,\xi=1} \Theta_{\xi=1} - \delta \Theta_{,\xi=0} \Theta_{\xi=0}] - \frac{C_{12}}{L} [\delta \Theta_1 \quad \delta \Theta_2] \int_0^1 \begin{bmatrix} N_{1,\xi} \\ N_{2,\xi} \end{bmatrix}_{\tau} \begin{bmatrix} N_{1,\xi} & N_{2,\xi} \end{bmatrix}_{\alpha} d\xi \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \quad (85)$$

Expanding and simplifying (84) and (85) then leads to:

$$\begin{aligned}W_a^e &= \frac{-C_{11}}{L} [\delta U_1 \quad \delta U_2] \left[\begin{bmatrix} N_{1,\xi} N_1 & N_{1,\xi} N_2 \\ N_{2,\xi} N_1 & N_{2,\xi} N_2 \end{bmatrix}_{\alpha, \xi=1} - \begin{bmatrix} N_{1,\xi} N_1 & N_{1,\xi} N_2 \\ N_{2,\xi} N_1 & N_{2,\xi} N_2 \end{bmatrix}_{\alpha, \xi=0} \right] \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \\ &\quad - \frac{C_{12}}{L} [\delta U_1 \quad \delta U_2] \int_0^1 \begin{bmatrix} N_{1,\xi} \\ N_{2,\xi} \end{bmatrix}_{\alpha} \begin{bmatrix} N_{1,\xi} & N_{2,\xi} \end{bmatrix}_{\tau} d\xi \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}\end{aligned} \quad (86)$$

$$\begin{aligned}W_t^e &= \frac{-C_{22}}{L} [\delta \Theta_1 \quad \delta \Theta_2] \left[\begin{bmatrix} N_{1,\xi} N_1 & N_{1,\xi} N_2 \\ N_{2,\xi} N_1 & N_{2,\xi} N_2 \end{bmatrix}_{\tau, \xi=1} - \begin{bmatrix} N_{1,\xi} N_1 & N_{1,\xi} N_2 \\ N_{2,\xi} N_1 & N_{2,\xi} N_2 \end{bmatrix}_{\tau, \xi=0} \right] \begin{bmatrix} \Theta_1 \\ \Theta_2 \end{bmatrix} \\ &\quad - \frac{C_{12}}{L} [\delta \Theta_1 \quad \delta \Theta_2] \int_0^1 \begin{bmatrix} N_{1,\xi} \\ N_{2,\xi} \end{bmatrix}_{\tau} \begin{bmatrix} N_{1,\xi} & N_{2,\xi} \end{bmatrix}_{\alpha} d\xi \begin{bmatrix} \Theta_1 \\ \Theta_2 \end{bmatrix}\end{aligned} \quad (87)$$

where the boundary values of the shape functions are known allowing for further simplification of (86) and (87) using,

$$@ \xi = 0 : \quad N_{1\alpha} = N_{1r} = 1, \quad \text{and} \quad N_{2\alpha} = N_{2r} = 0,$$

$$@ \xi = 1 : \quad N_{1\alpha} = N_{1r} = 0, \quad \text{and} \quad N_{2\alpha} = N_{2r} = 1,$$

which gives:

$$W_a^e = \frac{-C_{11}}{L} [\delta U_1 \quad \delta U_2] \begin{bmatrix} -N_{1,\alpha\xi=0} & 0 & N_{1,\alpha\xi=1} & 0 \\ 0 & 0 & 0 & 0 \\ -N_{2,\alpha\xi=0} & 0 & N_{2,\alpha\xi=1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} U_1 \\ \Theta_1 \\ U_2 \\ \Theta_2 \end{bmatrix} \\ - \frac{C_{12}}{L} [\delta U_1 \quad \delta U_2] \int_0^1 \begin{bmatrix} 0 & N_{1,\alpha} N_{1,\tau} & 0 & N_{1,\alpha} N_{2,\tau} \\ 0 & 0 & 0 & 0 \\ 0 & N_{2,\alpha} N_{1,\tau} & 0 & N_{2,\alpha} N_{2,\tau} \\ 0 & 0 & 0 & 0 \end{bmatrix} d\xi \begin{bmatrix} U_1 \\ \Theta_1 \\ U_2 \\ \Theta_2 \end{bmatrix} \quad (88)$$

and,

$$W_t^e = \frac{-C_{22}}{L} [\delta \Theta_1 \quad \delta \Theta_2] \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -N_{1,\tau\xi=0} & 0 & N_{1,\tau\xi=1} \\ 0 & 0 & 0 & 0 \\ 0 & -N_{2,\tau\xi=0} & 0 & N_{2,\tau\xi=1} \end{bmatrix} \begin{bmatrix} U_1 \\ \Theta_1 \\ U_2 \\ \Theta_2 \end{bmatrix} \\ - \frac{C_{12}}{L} [\delta \Theta_1 \quad \delta \Theta_2] \int_0^1 \begin{bmatrix} 0 & 0 & 0 & 0 \\ N_{1,\tau} N_{1,\alpha} & 0 & N_{1,\tau} N_{2,\alpha} & 0 \\ 0 & 0 & 0 & 0 \\ N_{2,\tau} N_{1,\alpha} & 0 & N_{2,\tau} N_{2,\alpha} & 0 \end{bmatrix} d\xi \begin{bmatrix} U_1 \\ \Theta_1 \\ U_2 \\ \Theta_2 \end{bmatrix} \quad (89)$$

Finally, expressions (88) and (89) can be combined together to give the final element dynamic stiffness matrix to be used in the DFE model as:

$$\begin{aligned}
 W_{Int}^e = & \frac{1}{L} \begin{bmatrix} \delta U_1 & 0 & \delta U_2 & 0 \\ 0 & \delta \Theta_1 & 0 & \delta \Theta_2 \end{bmatrix} \begin{bmatrix} -C_{11}N_{1,\alpha} \xi=0 & 0 & C_{11}N_{1,\alpha} \xi=1 & 0 \\ 0 & -C_{22}N_{1,\tau} \xi=0 & 0 & C_{22}N_{1,\tau} \xi=1 \\ -C_{11}N_{2,\alpha} \xi=0 & 0 & C_{11}N_{2,\alpha} \xi=1 & 0 \\ 0 & -C_{22}N_{2,\tau} \xi=0 & 0 & C_{22}N_{2,\tau} \xi=1 \end{bmatrix} \begin{bmatrix} U_1 \\ \Theta_1 \\ U_2 \\ \Theta_2 \end{bmatrix} \\
 & + \frac{C_{12}}{L} \begin{bmatrix} \delta U_1 & 0 & \delta U_2 & 0 \\ 0 & \delta \Theta_1 & 0 & \delta \Theta_2 \end{bmatrix} \int_0^1 \begin{bmatrix} 0 & N_{1,\alpha}N_{1,\tau} & 0 & N_{1,\alpha}N_{2,\tau} \\ N_{1,\tau}N_{1,\alpha} & 0 & N_{1,\tau}N_{2,\alpha} & 0 \\ 0 & N_{2,\alpha}N_{1,\tau} & 0 & N_{2,\alpha}N_{2,\tau} \\ N_{2,\tau}N_{1,\alpha} & 0 & N_{2,\tau}N_{2,\alpha} & 0 \end{bmatrix} d\xi \begin{bmatrix} U_1 \\ \Theta_1 \\ U_2 \\ \Theta_2 \end{bmatrix}
 \end{aligned} \tag{90}$$

The element virtual works (90) are then assembled in the standard way to find the total virtual work for the system (see eqs (57) and (58)). Since the weighting functions ‘ δU ’ and ‘ $\delta \Theta$ ’ are arbitrary, the process of assembling the global stiffness matrix for the system leads to the eigenvalue problem,

$$[\bar{K}(\omega)][\bar{U}] = [0] \tag{91}$$

where again the boundary conditions on the problem need to be taken into consideration.

2.5 Wittrick-Williams Method

Once the global eigenvalue problem is defined, it must be solved in order to determine the natural frequencies and modes of vibration of the system. To do this, one can make use of the Wittrick-Williams (WW) algorithm [32], which is a simple method of calculating the number of natural frequencies of a system that are below a given trial frequency value. The basic equation of the WW method is,

$$J = J_o + \text{sign}[\bar{K}(\omega)^\Delta] \quad (92)$$

where 'J' is the number of natural frequencies below the trial value. The term ' J_o ' is defined by:

$$J_o = \sum^{NOE} J_m \quad (93)$$

where ' J_m ' is the number of clamped-clamped natural frequencies of any component member of the structure that lies below the trial frequency. $\text{sign}[\bar{K}(\omega)^\Delta]$ in (92) is the number of negative elements on the leading diagonal of $[\bar{K}(\omega)^\Delta]$, where $[\bar{K}(\omega)^\Delta]$ is the upper triangulated matrix found by using Gaussian elimination (with no column pivoting [32]) on $[\bar{K}(\omega)]$ evaluated at the trial value.

Formula (92) can then be used in a simple bisection solver in order to converge on any natural frequency, to any desired accuracy. This allows one to solve for any specific frequency number without having to solve for all previous frequencies, which is the requirement of some linear eigenvalue solvers. One should note that the iterative nature of the bisection solver used in conjunction with this method and the requirement of the evaluation of the stiffness matrix at each trial frequency makes it require increased computational time when compared to most linear eigenvalue methods.

2.5.1 Solution for Clamped-Clamped Natural Frequency

In order to make use of the WW method at each trial frequency, one must determine the CL/CL natural frequencies of vibration for all the component members of the discretized system. In order to determine these frequencies one can make use of the exact solution for the CL/CL extension-torsion coupled natural frequencies given in references [1,10].

The CL/CL natural frequencies for an extension-torsion coupled element correspond to the value of ' ω ' that yield infinities in the DSM frequency dependant stiffness matrix. Upon review of equations (36)-(41) one should note that the term ' Δ ' occurs in the denominator of all of them, therefore the elements of the DSM stiffness matrix go to infinity when $\Delta = 0$, where:

$$\Delta = (k_{\beta_1} - k_{\beta_2}) \sin \beta_1 \sin \beta_2 = 0 \quad (94)$$

In equation (94) the term $(k_{\beta_1} - k_{\beta_2})$ can not be equal to zero (see eqs (17) and (18)), therefore $\sin \beta_1 \sin \beta_2$ must be equal to zero. The solution to this equation is easily found from inspection and is given by:

$$\beta_1 = n\pi \text{ or } \beta_2 = n\pi \quad (95)$$

The solution for the CL/CL frequencies can now be obtained by substituting (95) into equation (13) and are given by [1,10]:

$$\omega_{\beta_1}^2 = \frac{2(\kappa_1 \kappa_4 - \kappa_2^2) n\pi}{L^2 [(\kappa_1 I + \kappa_4 m) + \{(\kappa_1 I - \kappa_4 m)^2 + 4mI\kappa_2^2\}^{1/2}]} \quad (96)$$

$$\omega_{\beta_2}^2 = \frac{2(\kappa_1 \kappa_4 - \kappa_2^2) n\pi}{L^2 [(\kappa_1 I + \kappa_4 m) - \{(\kappa_1 I - \kappa_4 m)^2 + 4mI\kappa_2^2\}^{1/2}]} \quad (97)$$

Chapter 3: Applications

In order to make use of the extension-torsion coupled vibration theory developed here, one must find applications which exhibit this type of coupled behavior. The first engineering application is the vibration analysis of a helical spring, which exhibits geometric extension-torsion coupling due to the shape of the spring. In this section, simple, explicit equations in terms of the geometric and material properties of the spring are given for the stiffness terms of interest. The next application is the extension-torsion coupled vibration of wire ropes resulting from the braiding/twisting of the strands making up the rope cross-section. The complications associated with developing a theoretical model for these types of systems is briefly discussed, and an experimental method for determining the appropriate stiffness coefficients is also reviewed. The final application of interest is the vibration analysis of materially coupled composite beams. First classic laminate theory is introduced to allow for the determination of the mechanical behavior of the laminate which make up these beams. Then a method for determining the cross-sectional stiffness parameters for these members is developed, with particular interest paid to extension-torsion coupled box-beams and circular tubes. Examples demonstrating these applications are then shown in the next chapter to verify the applicability of the proposed methods and the precision of results.

3.1 *Helical Spring*

The first type of extension-torsion coupled system which is investigated is that of a helical spring. This type of system exhibits geometrically coupled extension-torsion vibrations due to the way the cross-section deforms under load. First, the equations of motion for this type of system are shown in order to verify the extension-torsion coupled nature of the problem, then the appropriate cross-sectional stiffnesses are given. Due to the numerous applications for these springs both the CL/CL and the CL/FR boundary conditions are investigated in the example problem presented in the next chapter.

According to references [10,14] the differential equations of motion for the deformation of a helical spring as shown in Figure 4 can be written as:

$$C_{11}u_{,xx} + C_{12}\theta_{,xx} = mu_{,tt} \quad (98)$$

$$C_{12}u_{,xx} + C_{22}\theta_{,xx} = I\theta_{,tt} \quad (99)$$

This means that the spring exhibits extension-torsion coupling, while not being allowed to deform (bend) in the lateral directions. The lack of accounting for deformation in the lateral direction makes sense from a physical standpoint in that springs are not normally loaded in these directions.

Figure 4: Model of Helix Spring Element

θ = spring rotation angle

ϕ = helix angle

x = length along spring

u = displacement of spring

r_{spring} = helix radius

R_{spring} = wire diameter

L = spring length

In order to determine the relevant stiffness terms for a helical spring cross-section, the following method is used (see references [10,14]). To make use of this method the spring element must be of constant cross-section (i.e. helix radius $r_{spring} = \text{constant}$), constant helix angle (i.e. $\phi = \text{constant}$) and must have constant material properties. The following formulation

also makes the assumptions that $r_{spring} \gg R_{spring}$ which makes $\frac{R_{spring}^2}{r_{spring}^2} \approx 0$ [10,14]. This guarantees the symmetry of the stiffness matrix. In principle this is a valid assumption since for most configurations $r_{spring} \gg R_{spring}$; however, if ' R_{spring} ' is a large value then $\frac{R_{spring}^2}{r_{spring}^2} \neq 0$ and this method will only give approximate results. The assumptions of small deformation and linearly elastic material allow the force-strain relationship to be approximated with a linear function. However, if large deformations are present then a different non-linear model is required. The stiffness constants for a helical spring are given by:

$$C_{11} = \frac{\pi E R_{spring}^4 \{1 + T_\phi^2 (1 + \nu)\}}{r_{spring}^2 \Gamma} \quad (100)$$

$$C_{12} = \frac{-\pi \nu E R_{spring}^4 T_\phi}{r_{spring} \Gamma} \quad (101)$$

$$C_{22} = \frac{\pi E R_{spring}^4 \{4(1 + \nu + T_\phi^2) + T_\phi^2 R_{spring}^2 / r_{spring}^2\}}{4\Gamma} \quad (102)$$

where,

$$\Gamma = \frac{[4(1 + \nu)(1 + T_\phi^2)^2 + (T_\phi^2 R_{spring}^2 / r_{spring}^2) \times \{(1 - \nu) + T_\phi^2 (1 + \nu)\}]}{T_\phi (1 + T_\phi^2)^{1/2}} \quad (103)$$

and, T_ϕ = tangent of helix angle ϕ ,

E = Young's Modulus of elasticity,

ν = Poisson's ratio.

3.2 Wire Rope

The second application of interest is a braided/twisted wire rope as shown in Figure 5 , which exhibits geometric coupling due to the twisting of the strands making up the rope cross-section. The equations of motion for this type of system are first reviewed, and a method for the experimental determination of the cross-sectional stiffness constants is then briefly discussed. Due to the fact that these types of systems find their primary application in the off shore industry, the boundary conditions of interest for this example are particularly interesting. The CL/SW and SW/SW boundary conditions of this problem are different than those found in the other examples and tend to occur only in these applications, along with the CL/CL conditions.

Figure 5 : Wire Rope Section [13]

The differential equations of motion for a wire rope are given in reference [13] as:

$$C_{11}u_{,xx} + C_{12}\theta_{,xx} = mu_{,tt} \quad (104)$$

$$C_{12}u_{,xx} + C_{22}\theta_{,xx} = I\theta_{,tt} \quad (105)$$

As can be seen, these equations are similar to those of the helical spring, exhibiting extension-torsion coupling and no displacements in the lateral directions. The reason and justification for the removal of these degrees of freedom is due to the nature of the applications for these ropes.

There are many different theoretical methods available for the determination of the stiffness parameters for a wire rope cross-section. Most of these are based in some way on experimental results, either of the entire rope or the properties of the individual strands. These models vary depending on the size of the strands as well as the number that make up the cross-section [20]. This makes the correct determination of the stiffness terms difficult, as the selection of the correct model is paramount. For this reason, most researchers [13] use the stiffness parameters determined experimentally from the force displacement behavior of the entire rope cross-section.

In order to determine the stiffness parameters of the cross-section first one must write the constitutive relations of interest for the wire rope, namely:

$$F = C_{11} \frac{\Delta L}{L_o} + C_{12} \frac{\Delta \theta}{L_o} \quad (106)$$

$$T = C_{12} \frac{\Delta L}{L_o} + C_{22} \frac{\Delta \theta}{L_o} \quad (107)$$

It can be seen from these equations that if a tension-elongation and a torque-elongation test is carried out with the one end prevented from rotating then the axial stiffness (C_{11}) and extension-torsion coupling stiffness (C_{12}) terms can easily be determined knowing the elongation of the test specimen ' ΔL ' and, the gauge length of the rope ' L_o ' used in the tests. Typical results for these tests are taken from reference [13] and shown in Figures 6 and demonstrate the simplicity associated with determining these stiffness terms. It should also be noted that experiments of this type could be used to determine the stiffness constants for any extension-torsion coupled member and are not limited to wire rope cross-sections.

Fig. A2 Tension versus elongation test data showing loading and unloading curves as broken lines. The solid line represents average slope.

Fig. A3 Torque versus elongation test data showing loading and unloading curves as broken lines. The solid line represents average slope.

Figures 6 : CL/CL Tension and Torque Tests; Wire Rope [13]

The torsional stiffness constant ' C_{22} ' can now be determined using the slope of the elongation-rotation curves as depicted in Figure 7.

Fig. A5 Elongation versus end rotation data collected at discrete points. The solid lines represent slopes for data collected with zero and 300 in. lb torque applied to the swivel.

Figure 7 : CL/SW Elongation versus Rotation Test; Wire Rope [13]

This demonstrates that it is relatively easy to experimentally determine the stiffness constants of interest for extension-torsion coupled members, requiring only simple tests and equipment.

3.3 Composite Box Beam and Circular Tube

Next, the dynamic behavior of flexible composite rectangular box-beam and tubular structural elements is examined. These cross-sections find numerous practical engineering applications varying from helicopter rotor blades to shafts for golf clubs, and many researchers [1-5,23-28] have developed techniques for calculating the appropriate stiffness constants of these elements. These methods all make use of the stiffness parameters given by classical laminate theory, in order to develop effective material stiffness terms for the cross-section. This allows for a reduction of the dimensionality of the problem, making a traditional one dimensional beam energy approach appropriate. In the following sections, the method presented in references [4,5] is used in order to develop closed form solutions for the stiffness constants of both rectangular and circular Circumferentially Uniform Stiffness (CUS) composite cross-sections.

3.3.1 Classic Laminate Theory

The use of classic laminate theory allows one to determine the stiffness parameters of a laminate using the plane stress assumption, which is acceptable for most laminates used in the composite members of interest. The in-plane lamina reduced stiffness terms (Q_{ij}) are defined using the plane stress assumption and the anisotropic nature of the material as [19] (see Figure 8):

$$Q_{11} = \frac{E_1}{1 - \left(\frac{E_2}{E_1}\right)\nu_{12}^2}; \quad Q_{12} = \nu_{12}Q_{22}; \quad Q_{22} = \frac{E_2}{E_1}Q_{11}; \quad Q_{66} = G_{12} \quad (108)$$

where E_1 , E_2 , ν_{12} , and G_{12} are the longitudinal modulus of elasticity, transverse modulus of elasticity, longitudinal-transverse poison ratio, and the longitudinal-transverse shear modulus of the lamina, respectively. These lamina reduced stiffnesses are expressed in the material directions (1 and 2).

Figure 8: Stresses Acting on Fiber Reinforced Element

Since the lamina that make up a laminate do not necessarily share the same material axis, it is convenient to transfer their mechanical properties into a common axis. For convenience the principal load direction is frequently used as the common axis (refer to Figure 9).

Figure 9 : Composite Material Axis Transform

The lamina transformed reduced stiffnesses ($\bar{Q}_{ij}$) are then given by [19]:

$$\bar{Q}_{11} = Q_{11} \cos(\theta)^4 + 2(Q_{12} + 2Q_{66}) \sin(\theta)^2 \cos(\theta)^2 + Q_{22} \sin(\theta)^4 \quad (109)$$

$$\bar{Q}_{12} = (Q_{11} + Q_{22} - 4Q_{66}) \sin(\theta)^2 \cos(\theta)^2 + Q_{12} (\sin(\theta)^4 + \cos(\theta)^4) \quad (110)$$

$$\bar{Q}_{16} = (Q_{11} - Q_{12} - 2Q_{66}) \sin(\theta) \cos(\theta)^3 + (Q_{12} - Q_{22} + 2Q_{66}) \sin(\theta)^3 \cos(\theta) \quad (111)$$

$$\bar{Q}_{22} = Q_{11} \sin(\theta)^4 + 2(Q_{12} + 2Q_{66}) \sin(\theta)^2 \cos(\theta)^2 + Q_{22} \cos(\theta)^4 \quad (112)$$

$$\bar{Q}_{26} = (Q_{11} - Q_{12} - 2Q_{66}) \sin(\theta)^3 \cos(\theta) + (Q_{12} - Q_{22} + 2Q_{66}) \sin(\theta) \cos(\theta)^3 \quad (113)$$

$$\bar{Q}_{66} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66}) \sin(\theta)^2 \cos(\theta)^2 + Q_{66} (\sin(\theta)^4 + \cos(\theta)^4) \quad (114)$$

The in-plane laminate stiffness coefficients (A_{ij}) are then defined as the sum of the transformed reduced stiffness terms of each of the lamina multiplied by the thickness of the lamina. Mathematically this is expressed as [19]:

$$A_{ij} = \sum_{k=1}^{NOL} \bar{Q}_{ij} h_k \quad (115)$$

where ' NOL ' is the total number of lamina which make up the laminate and ' h_k ' is the thickness of each individual lamina ' k '. The ' A_{ij} ' stiffness terms can then be used to determine the effective material stiffness parameters of the laminate.

3.3.2 Anisotropic, Slender, Thin-Walled, Closed-Section Beam Theory

In order to determine the cross-sectional stiffness constants, a general anisotropic beam theory has been developed [5]. First, the dimensionality of the problem is reduced by making the assumption of no internal pressure and no transverse shear effects at the laminate level. These assumptions are similar to those made by other authors [1-5, 23-29] and have been confirmed by experimental data [5,11,28] to be valid for most cross-sections and laminates. A displacement field is then assumed, taking into account relevant warping terms, in order to allow the strain on the cross-section to be calculated. These strains can then be used in a one-dimensional beam energy equation, allowing for the constitutive relations to be determined. These equations yield integral formulae for the stiffness parameters of the cross-section.

Figure 10: Anisotropic Shell

The classical solution for the strain energy density of a two-dimensional anisotropic shell undergoing membrane behavior (Figure 10) can be used in order to determine the materials effective extension, torsion and coupling stiffness coefficients for the laminate being used. The strain energy density for the shell is written as [5]:

$$2\Phi_* = A_{11}\gamma_{11}^2 + A_{22}\gamma_{22}^2 + 4A_{66}\gamma_{12}^2 + 2A_{12}\gamma_{11}\gamma_{22} + 4A_{16}\gamma_{11}\gamma_{12} + 4A_{26}\gamma_{22}\gamma_{12} \quad (116)$$

where ' A_{ij} ' are the in-plane laminate stiffness constants obtained from classical laminate theory and ' γ_{ij} ' are the in-plane strains. Then using the additional assumption of no internal pressure and therefore no resultant hoop stress one can write,

$$N_{22} = \frac{\partial \Phi_*}{\partial \gamma_{22}} = \frac{1}{2}(2A_{22}\gamma_{22}) + \frac{1}{2}(2A_{12}\gamma_{11}) + \frac{1}{2}(4A_{26}\gamma_{12}) = 0 \quad (117)$$

where N_{22} is the hoop stress in the shell.

This allows one to solve for ' γ_{22} ' in terms of the extensional and torsional strains as:

$$\gamma_{22} = -\left(\frac{1}{A_{22}}\right)[A_{12}\gamma_{11} + 2A_{26}\gamma_{12}] \quad (118)$$

The strain energy density of the shell can then be expressed as:

$$2\Phi_* = \left(A_{11} - \frac{A_{12}^2}{A_{22}}\right)\gamma_{11}^2 + 2\left(A_{16} - \frac{A_{12}A_{26}}{A_{22}}\right)\gamma_{11}\gamma_{12} + 4\left(A_{66} - \frac{A_{26}^2}{A_{22}}\right)\gamma_{12}^2 = A(s)\gamma_{11}^2 - B(s)\gamma_{11}\gamma_{12} + C(s)\gamma_{12}^2 \quad (119)$$

This allows for the determination of the materials effective stiffness parameters as the extensional stiffness ' $A(s)$ ', the extension-torsion coupling stiffness ' $B(s)$ ' and the torsional stiffness ' $C(s)$ '. These effective material stiffness terms are expressed in terms of the laminate in-plane composite material stiffness constants given by classic laminate theory. The values of ' A ', ' B ' and ' C ' at each point along the cross-section are therefore:

$$A(s) = A_{11} - \frac{A_{12}^2}{A_{22}} \quad (120)$$

$$B(s) = 2\left(\frac{A_{12}A_{26}}{A_{22}} - A_{16}\right) \quad (121)$$

$$C(s) = 4\left(A_{66} - \frac{A_{26}^2}{A_{22}}\right) \quad (122)$$

Another method for developing the materials effective stiffness parameters is demonstrated in references [2,26-28] and yields the same results. This method is based on the same assumptions except that it makes use of the constitutive relations for the laminate instead of the strain energy functional to allow for the elimination of ' γ_{22} ' from the problem.

The displacement field chosen here was taken from references [4,5] and accounts for various types of warping. This field is similar to those used by other authors [2,3,7,18,21,26-28] and is based on the following contour form,

$$\begin{aligned} v_1 = & U_1(x) - y(s)U_{2,x}(x) - z(s)U_{3,x}(x) + G(s)\theta(x) \\ & + g_1(s)U_{1,x}(x) + g_2(s)U_{2,xx}(x) + g_3(s)U_{3,xx}(x) \end{aligned} \quad (123)$$

$$v_2 = U_2(x)\frac{dy}{ds} + U_3(x)\frac{dz}{ds} + \theta(x)r_n \quad (124)$$

$$v = U_2(x)\frac{dz}{ds} - U_3(x)\frac{dy}{ds} - \theta(x)r_t \quad (125)$$

where the coordinates of the cross-section are $y(s)$ and $z(s)$, r_n and r_t are projections of the distance to the point on the contour 'R' and the other terms of (123)-(125) are defined with regards to Figure 10.

The terms $g_1(s)$, $g_2(s)$ and $g_3(s)$ in this field represent the out-of-plane warping of the cross-section due to axial strain, and horizontal and vertical bending curvatures, respectively. The product term $G(s)\phi(x)$ represent out-of-plane warping due to torsional twist rate. The present analysis does not however include the restrained warping effects due to the fixed end. As reported in reference [18], inclusion of these end restraints causes an increase in the coupling and torsional stiffness of the cross-section.

Various forms for these warping functions are given by numerous authors (see [2,3,18,21,26-28]). The particular forms used in the present study are those presented by Berdichevsky *et al* [4,5] as,

$$\begin{aligned}
 G(s) &= \int_0^s \left[\frac{2A_e}{I\bar{c}} c(\tau) - r_n(\tau) \right] d\tau & g_1(s) &= \int_0^s \left[b(\tau) - \frac{\bar{b}}{\bar{c}} c(\tau) \right] d\tau \\
 g_2(s) &= - \int_0^s \left[b(\tau) y(\tau) - \frac{\bar{b}\bar{y}}{\bar{c}} c(\tau) \right] d\tau & g_3(s) &= - \int_0^s \left[b(\tau) z(\tau) - \frac{\bar{b}\bar{z}}{\bar{c}} c(\tau) \right] d\tau \quad (126) \\
 b(s) &= - \frac{2B(s)}{C(s)} & c(s) &= \frac{1}{C(s)}
 \end{aligned}$$

where 's' is the length along the contour, and 'τ' represents the current position on the contour. The bar over certain quantities represents the average value of that quantity over the entire perimeter.

The assumed displacement field can now be used in order to calculate the strain on the thin-walled beam cross-section leading to:

$$\gamma_{11} = \frac{\partial v_1}{\partial x} = U_{1,x}(x) - y(s)U_{2,x}(x) - z(s)y(s)U_{3,x}(x) \quad (127)$$

$$\begin{aligned}
 2\gamma_{12} &= \frac{\partial v_1}{\partial s} + \frac{\partial v_2}{\partial x} = \frac{2A_e}{P\bar{c}} c(s)\phi_{,x}(x) + \left[b(s) - \frac{\bar{b}}{\bar{c}} c(s) \right] U_{1,x}(x) - \\
 &\quad \left[b(s)y(s) - \frac{\bar{b}\bar{y}}{\bar{c}} c(s) \right] U_{2,x}(x) - \left[b(s)z(s) - \frac{\bar{b}\bar{z}}{\bar{c}} c(s) \right] U_{3,x}(x) \quad (128)
 \end{aligned}$$

$$\gamma_{22} = \frac{\partial v_2}{\partial s} + \frac{v}{R} = 0 \quad (129)$$

It should be noted that in the derivation of this strain field, higher order terms associated with $G\phi''(x)$ were neglected. As reported in reference [5], this assumption is valid as long as:

$$\frac{A}{C} \left(\frac{b}{L} \right) \ll 1 \quad \text{and} \quad \frac{B}{C} \left(\frac{b}{L} \right) \ll 1$$

This is similar to the long slender beam assumption since these inequalities are satisfied regardless of material used if:

$$\left(\frac{b}{L}\right) \ll 1 \quad .$$

Substituting the strain field into (119) and integrating over the contour 's' allows one to write the traditional one-dimensional beam energy equation as,

$$I = \int_0^L \Phi dx - \int P_i u_i dx ds \quad (130)$$

with,

$$\begin{aligned} \Phi = & \frac{1}{2} [C_{11}(U_{1,x})^2 + C_{22}(\phi_{,x})^2 + C_{33}(U_{3,x})^2 + C_{44}(U_{4,x})^2] + \\ & C_{12}U_{1,x}\phi_{,x} + C_{13}U_{1,x}U_{3,x} + C_{14}U_{1,x}U_{2,x} + C_{23}\phi_{,x}U_{3,x} + C_{24}\phi_{,x}U_{2,x} + C_{34}U_{2,x}U_{3,x} \end{aligned}$$

and P_i being the surface tractions on the faces of the shell.

The constitutive relations can now be written in terms of the resultant stresses by differentiating the beam energy with respect to the different kinematic variables in the beam energy equation (130). Similarly, it can be derived by integrating the axial stress and shear flow of the shell as:

$$T = \frac{\partial \Phi}{\partial U_{1,x}} = \oint N_{11} ds \quad (131)$$

$$M_x = \frac{\partial \Phi}{\partial \phi_{,x}} = \oint N_{12} r_n(s) ds \quad (132)$$

$$M_y = \frac{\partial \Phi}{\partial U_{3,x}} = -\oint N_{11} z(s) ds \quad (133)$$

$$M_z = \frac{\partial \Phi}{\partial U_{2,x}} = -\oint N_{11} y(s) ds \quad (134)$$

where ' N_{11} ' and ' N_{12} ' are the axial and shear stresses calculated through the differentiation of the strain energy density of the membrane (119) using:

$$N_{11} = \frac{\partial \Phi_*}{\partial \gamma_{11}} = A(s)\gamma_{11} + B(s)\gamma_{12} \quad (135)$$

$$N_{12} = \frac{\partial \Phi_*}{\partial \gamma_{12}} = \frac{1}{2}(B(s)\gamma_{11} + C(s)\gamma_{12}) = \text{const} \quad (136)$$

The constitutive relation for the beam cross-section can now be written as:

$$\begin{pmatrix} T \\ M_x \\ M_y \\ M_z \end{pmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{21} & C_{22} & C_{23} & C_{24} \\ C_{31} & C_{32} & C_{33} & C_{34} \\ C_{41} & C_{42} & C_{43} & C_{44} \end{bmatrix} \begin{bmatrix} U_{1,x} \\ \phi_{,x} \\ U_{3,x} \\ U_{2,x} \end{bmatrix} \quad (137)$$

Integral forms of the terms C_{ij} are given in Appendix A for reference.

3.3.3 CUS configuration Composite Sections

Of particular interest in this project are composite beams which exhibit extension-torsion material coupling. These are commonly referred to as Circumferentially Uniform Stiffness (CUS) since the effective material stiffness terms ' $A(s)$ ', ' $B(s)$ ' and ' $C(s)$ ' are constant around the cross-section. For these particular applications the constitutive relationship (137) reduces to [4,5],

$$\begin{pmatrix} T \\ M_x \\ M_y \\ M_z \end{pmatrix} = \begin{bmatrix} C_{11} & C_{12} & 0 & 0 \\ C_{21} & C_{22} & 0 & 0 \\ 0 & 0 & C_{33} & 0 \\ 0 & 0 & 0 & C_{44} \end{bmatrix} \begin{bmatrix} U_{1,x} \\ \phi_{,x} \\ U_{3,x} \\ U_{2,x} \end{bmatrix} \quad (138)$$

which demonstrates that the CUS configuration exhibits extension-torsion displacement coupling and two independent, uncoupled bending displacements. The expressions for the five stiffness constants of interest (C_{11} , C_{12} , C_{22} , C_{33} , and C_{44}) are given in references [3-5] and can be expressed in closed form as,

$$C_{11} = \oint \left(A - \frac{B^2}{C} \right) ds + \frac{\left[\oint \left(\frac{B}{C} \right) ds \right]^2}{\oint \left(\frac{1}{C} \right) ds} = AP \quad (139)$$

$$C_{12} = \frac{\oint \left(\frac{B}{C} \right) ds}{\oint \left(\frac{1}{C} \right) ds} A_e = BA_e \quad (140)$$

$$C_{22} = \frac{1}{\oint \left(\frac{1}{C} \right) ds} A_e^2 = \frac{C}{P} A_e^2 \quad (141)$$

$$C_{33} = \oint \left(A - \frac{B^2}{C} \right) z^2 ds + \frac{\left[\oint \left(\frac{B}{C} \right) z ds \right]^2}{\oint \left(\frac{1}{C} \right) ds} = \begin{cases} C_{33_{Box-Beam}} = \left(A - \frac{B^2}{C} \right) \left[\frac{h^3}{6} + 2b \left(\frac{h}{2} \right)^2 \right] \\ C_{33_{tube}} = \left(A - \frac{B^2}{C} \right) \frac{2\pi r^3}{3} \end{cases} \quad (142)$$

$$C_{44} = \oint \left(A - \frac{B^2}{C} \right) y^2 ds + \frac{\left[\oint \left(\frac{B}{C} \right) y ds \right]^2}{\oint \left(\frac{1}{C} \right) ds} = \begin{cases} C_{44_{Box-Beam}} = \left(A - \frac{B^2}{C} \right) \left[\frac{b^3}{6} + 2h \left(\frac{b}{2} \right)^2 \right] \\ C_{44_{tube}} = \left(A - \frac{B^2}{C} \right) \frac{2\pi r^3}{3} \end{cases} \quad (143)$$

where ' A_e ' and ' P ' are the area enclosed by the centre-line of the cross-section and its perimeter, respectively.

The primary application of CUS composite beams is in helicopter rotor and wind mill blades, where the material coupling can be used in a beneficial way to improve the aeroelastic performance of the blade [34]. In what follows two different cross-sections will be examined. First a composite box-beam and then a circular tube, where the geometries are defined with reference to Figure 11. The reason why these two geometries were chosen is due to the fact that they can be extended in order to evaluate the stiffness terms for a true airfoil section. It has been shown in references [18,27] that equivalent area and perimeter may be used in order to evaluate the extension, torsion and coupling stiffness terms for non-rectangular/circular cross sections with reasonable accuracy. The determination of the correct bending stiffness depends highly on the actual shape of the section and therefore an equivalence method can not be used.

Figure 11: Box Beam and Circular Tube Cross Sections

The pertinent geometric properties for both the box-beam and circular tube are,

$$A_{e_{BoxBeam}} = (b-t)(h-t) \quad (144)$$

$$P_{BoxBeam} = 2(b-t) + 2(h-t) \quad (145)$$

$$m_{BoxBeam} = 2\rho t(b-t) + 2\rho t(h-t) \quad (146)$$

$$I_{BoxBeam} = \frac{1}{6} \rho t (b-t) \left((b-t)^2 + t^2 \right) + 2 \rho t (b-t) \left[\frac{(h-t)}{2} \right]^2 + \frac{1}{6} \rho t (h-t) \left((h-t)^2 + t^2 \right) + 2 \rho t (h-t) \left[\frac{(b-t)}{2} \right]^2 \quad (147)$$

and,

$$P_{CircularTube} = 2\pi \left(r - \frac{t}{2} \right) \quad (148)$$

$$A_{eCircularTube} = \pi \left(r - \frac{t}{2} \right)^2 \quad (149)$$

$$m_{CircularTube} = 2\pi \rho r t \quad (150)$$

$$I_{CircularTube} = 2\pi \rho t r^3 \quad (151)$$

where ‘ m ’ and ‘ I ’ are the mass per unit length and the mass moment of inertia per unit length, respectively.

3.3.4 Geometric Property Variation

In order to expand the current theory to non-uniform tapered composite beams, an average value method is used to determine the geometric properties and therefore the stiffness coefficients for each element. For many non-uniform geometries, a function of the following form can be used to represent the variation of the dimensional variables along the length of the beam [9],

$$f(x) = f_o \left[1 + c \left(\frac{x}{L} \right) \right]^n \quad (152)$$

where 'c' is the taper coefficient and is defined by:

$$c = \frac{f_{tip}}{f_o} \quad c = \begin{cases} -c & \text{if } f_{tip} < f_o \\ c & \text{if } f_{tip} > f_o \end{cases} \quad (153)$$

' f_{tip} ' represents the value of the function at the tip of the beam and ' f_o ' is the value of the function at the root of the beam. 'n' is the taper exponent and defines the way in which the beam geometry changes along its length, i.e., as a linear, quadratic or cubic function. Then the well-known averaging technique can be used, with variables being defined in Figure 12, in order to find the values for the geometric properties for each element along the beam.

$$f_{avg} = \frac{1}{j-i} \int_i^j f(x) dx \quad (154)$$

Figure 12: Averaging Technique

3.3.5 Lay-up and Ply-angle effects

In order to determine the material properties and section stiffness constants for the composite beams studied in this report, a Matlab pre-processor was first created. This pre-processor acts separately from and prior to the vibration analysis of the problem. The pre-processor and the effects of changes in parameters on the values it calculates are examined in this section. First, the effect of ply-angle changes on the materials reduced stiffness constants, obtained from classical laminate theory, is examined and values from different lay-ups are shown. Next the effect of these changes on the materials effective stiffness parameters (A , B , and C) is examined. Overall cross-sectional stiffness constants which come from this pre-processor are compared in the box beam examples to those published in literature in order to fully verify the program created.

When designing composite beams, the fiber and epoxy chosen for a particular application are not the only important material concerns. Also of key importance is the lamina ply-angles and the stacking sequence. There are many different classes of composite laminates, of primary concern here are anti-symmetric lay-ups since they exhibit extension-torsion coupling. An anti-symmetric lay-up is one made up of plies which are not symmetric about the laminates mid-plane; these types of laminates are used in the examples of this paper. One could also design using a symmetric or a cross-ply lay-up. A symmetric lay-up is made up of plies which are symmetric about the laminates mid-plane; a cross-ply lay-up is made of plies which are oriented along or at a right angle to the principle load direction. Symmetric and cross-ply circular tube examples are studied in Appendix H and Appendix I with final frequency values are compared to published results. The effect of difference in lay-up on the reduced stiffness coefficients of the laminate can be seen in Table 1. The first and third columns contain results for typical anti-symmetric lay-ups and demonstrate coupling between extension and torsion through the non-zero ' A_{16} ' and ' A_{26} ', which in turn makes the effective coupling stiffness ' $B(s)$ ' non-zero. These lay-ups are the ones used in the examples in the main section of this report.

Lay-up	[20/-70/20/-70 ₂ /20] _T	[20] ₆	[60] ₆	[45/-45/-45/45] _T	[0/0/0/90] _T
A ₁₁ (lb/in)	320,000	502,690	89,706	155,000	364,000
A ₁₂ (lb/in)	82,700	72,632	117,080	114,000	13,500
A ₁₆ (lb/in)	73,900	158,530	68,909	0	0
A ₂₂ (lb/in)	313,000	51,316	380,800	144,000	145,000
A ₂₆ (lb/in)	-74,100	28,581	183,180	0	0
A ₆₆ (lb/in)	93,100	80,619	125,060	121,000	20,500
A(s) (Lb)	298,000	399,880	53,711	65,100	363,000
B(s) (Lbin)	-187,000	-236,150	-25,180	0	0
C(s) (Lbin ²)	302,000	258,800	147,770	482,000	81,800

Table 1: Material Reduced Stiffness Terms for Different Lay-ups

The lay-ups in the fourth and fifth column represent a symmetric and a cross ply lay-ups respectively. As can be seen, these lay-ups do not exhibit the desired extension-torsion material couplings due to the fact that the plies cancel out each others coupling effects.

The effect of ply-angle on the overall laminate properties can be investigated by plotting the reduced stiffness terms versus ply-angle. For the $[\eta / \eta -90 / \eta / \eta -90 / \eta -90 / \eta]$ configuration (Figure 13) the reduced stiffness terms vary significantly with changes in ply-angle, which in turn affect the magnitude as well as the direction of the material couplings present.

Figure 13: Reduced Stiffness Variation with Ply-Angle $[\eta / \eta -90 / \eta / \eta -90 / \eta -90 / \eta]$ Lay-up

The resulting changes in laminate stiffness terms directly affect the material effective stiffness parameters 'A', 'B', and 'C' used in the analysis, and the effect is quite substantial (see Figure 14).

Figure 14: Effective Material Stiffness Terms versus Ply-Angle [$\eta / \eta -90 / \eta / \eta -90 / \eta -90 / \eta$] Lay-up

As it can be seen from Figure 14, the coupling stiffness 'B' can be made to be zero, which would make even an un-symmetric laminate exhibit uncoupled extension and torsion behavior. It should also be noted that 'B' changes sign with ply-angle (η), and therefore the coupling will change direction. In other words, a positive extension causes a negative twist for $0^\circ < \eta < 45^\circ$, whereas it causes a positive one for $45^\circ < \eta < 90^\circ$. It can also be seen that the other two effective material constants 'A' and 'C' are symmetric about an angle of 45° , which is due to the fact that the lay-up used is made up of two angles which differ by 90° . This demonstrates that selecting an appropriate ply-angle and lay-up allows a designer to control both the magnitude and direction of the coupling present in order to optimize the beam dynamic behavior depending on application.

Chapter 4: Results and Discussion

In this section, numerical examples are performed for each type of extension-torsion system examined in this report, the helical spring, wire rope, composite box-beam, and circular tube.

The first example is a uniform helical spring. In this problem two different Boundary Conditions (BC) CL/CL and CL/FR are reviewed. The spring natural frequencies are evaluated using different methods, discussed earlier in this report, and are compared to those published in literature. The convergence rates for the DFE and FEM solutions and the effect of the different BCs on them is examined. The natural modes of vibration are also shown, in order to allow for an appreciation of how these vibrating members deform.

The second example is a wire rope. For this application three different BC of interest, namely the CL/CL, CL/SW, and SW/SW are reviewed. The wires coupled frequencies are evaluated and compared to those of literature. The mode shapes for each of the coupled frequencies are also discussed. This demonstrates the importance in including the coupling between extension and torsion when attempting to determine the deformation behavior of these types of systems. The convergence rates for each of the solution methods (DFE and FEM) are compared for the BCs of interest, where the effect of element (member) length is also examined.

In the composite beam examples, two different cross-sections of interest are examined, a rectangular (box) section and a circular section. For each of these geometries, two different cantilevered (CL/FR) examples are studied in order to determine the effect of lay-up on the dynamic behavior of the beams. The first example of each cross-section makes use of a composite laminate which is made up of two different ply-angles. In these examples the stiffness and vibration characteristics of the beams are compared to values from literature in order to verify the stiffness pre-processor tool as well as the frequency solver. The effect that changes in ply-angle have on stiffness, frequency value, and mode shape will also be reviewed in these examples. The second example performed for each cross-section makes use of a laminate which consists of only one ply-angle, with either six layers, as in the box-beam example, or one as in the circular tube example. The effect that ply-angle has on the stiffness parameters and dynamic

characteristics of these types of beams is reviewed, with results being compared to those available in literature. The results of this laminate can be seen to be very different then those of the first, demonstrating the importance of the selection of the appropriate lay-up on the effective dynamic tailoring of a composite beam.

4.1 Uniform Helical Spring Example

First, the spring frequency values were calculated using the methods developed in this report and were then compared those of the literature. The aim was to verify the answers and check the convergence rates of the Matlab programs created. The results obtained from the DSM software created were exactly the same as those presented in reference [10] for the DSM method. The spring element properties are the same as those given in reference [10] and are shown in Table 2.

Geometric Properties	C_{11} (N)	C_{12} (Nm)	C_{22} (Nm ²)	I (kgm)	m (kg/m)	L (m)
	3449.84	-4.4552	2.84778	0.00144399	2.23174	0.3048

Table 2: Geometric and Stiffness Properties; Uniform Spring Example

The results for the all three methods as well as the error associated with each are included in Table 3. As can be seen from the table, the FEM method was executed using both a linear eigenvalue (eigen) solver as well as the Wittrick-Williams (WW) method in order to further verify the WW solver programmed here. Ten elements were used for both the FEM and DFE method in order to compare accuracy with the same number of elements. Any number of elements can be used in the DFE model to obtain acceptable results. The FEM method requires a minimum number of elements, equal to the number of the desired frequency divided by two, for the CL/FR case, and equal to the number of the desired frequency, in the case of the CL/CL frequencies.

It can be seen from Table 3, that all three methods are able to deliver very good approximate results for the low natural frequencies i.e., less than 1% error for the first two

CL/CL and the first four CL/FR modes. It can also be seen that as the frequency number increases the FEM becomes less accurate. This means that in order to obtain these frequencies with adequate accuracy, one should use at least the desired frequency value number of elements in the solution (e.g., in order to obtain adequately accurate approximations for the fourth natural frequency using the FEM method, one should make use of at least four elements in the solution regardless of BC). One should also note that, as expected [31], both the WW and the linear eigen value solvers returned the same values for natural frequency when the FEM model is used. This further helped to verify the validity of the WW solver created and also help to demonstrate the general applicability of the method.

Method	BC	1	(%)	2	(%)	3	(%)	4	(%)	5	(%)	6	(%)
From [10]	CL/CL	64.2		73.2		128.4		146.4		192.6		219.6	
	CL/FR	32.1		36.6		96.3		109.8		160.5		183	
Exact [10]	CL/CL	64.2		73.2		128.4		146.4		192.6		219.6	
	CL/FR	32.1		36.6		96.3		109.8		160.5		183	
DSM	CL/CL	64.2	0	73.2	0	128.4	0	146.4	0	192.6	0	219.6	0
	CL/FR	32.1	0	36.6	0	96.3	0	109.8	0	160.5	0	183	0
FEM eigen 10 elem	CL/CL	64.5	0.41	73.5	0.41	130.5	1.65	148.8	1.65	199.8	3.73	227.8	3.73
	CL/FR	32.1	0.10	36.6	0.10	97.2	0.93	110.8	0.93	164.7	2.59	187.8	2.59
FEM WW 10elem	CL/CL	64.5	0.41	73.5	0.41	130.5	1.65	148.8	1.65	199.8	3.73	227.8	3.73
	CL/FR	32.1	0.10	36.6	0.10	97.2	0.93	110.8	0.93	164.7	2.59	187.8	2.59
DFE 10 elem	CL/CL	64.2	0.01	73.2	0.00	128.4	0.01	146.4	0.00	192.6	0.01	219.6	0.00
	CL/FR	32.1	0.01	36.6	0.00	96.3	0.01	109.8	0.00	160.5	0.01	183	0.00

Table 3: Frequency Results (Hz); Uniform Spring Example

Convergence plots were also drawn for both the FEM and DFE methods in order to obtain a clear picture of how many elements would be required in order to obtain a specified accuracy. It can be seen from Figure 15 and Figure 16 that the FEM is unable to precisely predict the fifth and sixth natural frequencies when using only six elements in the solution, resulting in more than 10% error in the CL/CL solution and 7% error in the CL/FR case. As it can also be seen, the errors associated with the CL/CL natural frequencies are, in general, greater than those obtained for the CL/FR end conditions.

Figure 15: FEM Convergence Rates; CL/CL Uniform Spring Example

Figure 16: FEM Convergence Rates; CL/FR Uniform Spring Example

The convergence plots for the DFE formulation (with the same element range) are given in Figure 17 and Figure 18. These two figures demonstrate the dramatic decrease in error comparing to the FEM method. As a result, fewer DFE elements can be used to obtain the same results as FEM. It also means that higher frequencies can be obtained without greatly increasing the number of elements.

Figure 17: DFE Convergence Rates; CL/CL Uniform Spring Example

Figure 18: DFE Convergence Rates; CL/FR Uniform Spring Example

The CL/CL mode shapes for the first six natural frequencies of vibration are shown in Figures 19 and as can be seen demonstrate the coupled nature of the extension and rotational displacements. It can be seen that all six natural modes are dominated by rotation, with the amount of extensional displacement differing depending on mode. It is worth noting that the third and fourth mode shapes are discontinuous in slope in both rotation and displacement which is consistent with the second order nature of the vibration problem that this represents.

Figures 19 : Mode Plots; CL/CL Uniform Spring Example

These modes are also seen to behave in the predicted way, with the displacement and rotation at both ends of the member being zero for the CL/CL condition. In order to plot the

mode shapes for the different solved frequencies the DSM was used. In order to obtain a smooth shape for the mode plots, 50 elements were used in the solution, giving the smooth curves seen here. The deformations in all CL/CL modes can be seen to be different; however, the general shape of the dominant mode seems to repeat twice (i.e., the same for frequency #1 and #2), with the amplitude/sign changing depending on frequency number. A similar, but more pronounced trend will be seen in the CL/FR modes of this section.

The first six CL/FR natural modes of vibration are shown in Figures 20. As can be seen, these modes deform in the predicted manner, with zero deformations at the root ($\xi = 0$) and the spring tip ($\xi = 1$) free to move. Similar to the CL/CL modes for this example, the CL/FR modes are all torsion dominated where the amount of extension differs depending on mode. These modes are all slope continuous, which is not a requirement of this BC, but is controlled by the differential equations of motion. Like the CL/CL modes, the CL/FR ones seem to repeat dominant deformation for two modes, with only the non-dominant portion changing with frequency number (i.e., the dominant deformation of the 1st and 2nd, the 3rd and 4th, and the 5th and 6th modes, are the same, with only the non-dominant deformation changing). This, as it will be seen later in this report, seems to be an odd property of this example, and does not seem to be a general property of this BC or extension-torsion coupled systems.

Figures 20 : Mode Plots; CL/FR Uniform Spring Example

4.2 Wire Rope Example

The 2nd group of structural elements of interest exhibiting extension-torsion coupled vibrations is that of cables or wire ropes. These cables are used widely as structural members where axial stiffness (C_{11}) is paramount and large bending loads are not encountered [13,10]. In order to validate the methods developed in this project, and to verify their practical applicability, the natural frequencies of a wire rope example are evaluated and compared to those published in reference [13]. The geometric and stiffness properties for this example are the same as those reported in reference [13] (see Table 4).

C_{11} (Lb)	C_{12} (ftLb)	C_{22} (ft ² Lb)	m (Lbs ² /ft ²)	I (Lbs ²)
4440000	19000	99.2	0.0331	0.0000143

Table 4: Geometric and Stiffness Properties; Wire Rope Example

The first 5 natural frequencies of vibration were calculated for both a wire length of 100ft as well as a wire length of 4000ft and are shown in Table 5. The reason why two different lengths were used was to study the effect of wire length on the accuracy of the solution. As it can be seen from Table 5, DSM results are compared to those presented in reference [13], which makes use of an exact algebraic solution to the differential equations of motion for each boundary condition. The difference between the DSM results and those of reference [13] can be attributed to the fact that the values published in [13] were presented in a small log scale graph, and therefore can not be read precisely. The boundary conditions in this problem are easy to visualize if you think of a floating object attached by a single string/wire. The CL/CL boundary condition for this cable corresponds to the situation where the wire is not allowed to rotate at either end due to the attachment method, and does not deform in extension due the floating nature of the application, which maintains constant wire length. The CL/SW and SW/SW conditions correspond to the situations where either one or both ends of the rope/wire are allowed to rotate at their point of attachment. Both of these BCs do not allow for extensional displacement of either end again due to the floating nature of these ropes in service. It should also be noted that there exists a zero natural frequency for the SW/SW boundary condition, which corresponds to the rigid body mode of constant rotation of the entire rope. It can also be

seen in Table 5 that length does not affect the accuracy of the methods derived here with the relative errors being similar for each respective frequency regardless of length.

L = 100 ft								L = 4000 ft							
Freq Number	DSM	DFE 10 elem	% Diff	FEM 10 elem	% Diff	Ref [13]	% Diff	Freq Number	DSM	DFE 10 elem	% Diff	FEM 10 elem	% Diff	Ref [13]	% Diff
1 CL/CL	5.48	5.5	0	5.5	0.4	5	-1.2	1 CL/CL	0.140	0.140	0.4	0.140	0.4	0.147	-5
2 CL/CL	11	11.1	1	11.14	1.7	11	0.2	2 CL/CL	0.270	0.280	1.4	0.280	1.6	0.275	-2
3 CL/CL	16.4	16.93	3	17.04	3.7	16	-3.1	3 CL/CL	0.420	0.420	1.1	0.430	1.7	0.415	1
4 CL/CL	21.9	23.09	5	23.36	6.6	23	5.3	4 CL/CL	0.550	0.580	5.4	0.580	6.6	0.563	-2
5 CL/CL	27.4	29.66	8	30.2	10	28	1.7	5 CL/CL	0.680	0.740	8.4	0.750	10	0.651	4
Freq Number	DSM	DFE 10 elem	% Diff	FEM 10 elem	% Diff	Ref [13]	% Diff	Freq Number	DSM	DFE 10 elem	% Diff	FEM 10 elem	% Diff	Ref [13]	% Diff
1 CL/SW	4.62	4.63	0	4.63	0.3	5	0	1 CL/SW	0.120	0.120	0.3	0.120	0.3	0.114	5
2 CL/SW	9.44	9.54	1	9.56	1.2	9	-3.9	2 CL/SW	0.240	0.240	1.0	0.240	1.2	0.243	-1
3 CL/SW	14.5	14.81	2	14.88	2.8	14	-2.1	3 CL/SW	0.360	0.370	2.3	0.370	2.8	0.352	2
4 CL/SW	19.6	20.46	4	20.65	5.2	18	-11	4 CL/SW	0.490	0.510	4.2	0.520	5.2	0.496	-1
5 CL/SW	24.9	26.49	7	26.87	8.1	26	5.7	5 CL/SW	0.620	0.660	6.5	0.670	8.1	0.612	1
Freq Number	DSM	DFE 10 elem	% Diff	FEM 10 elem	% Diff	Ref [13]	% Diff	Freq Number	DSM	DFE 10 elem	% Diff	FEM 10 elem	% Diff	Ref [13]	% Diff
1 SW/SW	0	0		0				1 SW/SW	0.000	0.000		0			
2 SW/SW	9.24	9.32	1	9.34	1.2	9	0	2 SW/SW	0.230	0.230	1.0	0.230	1.2	0.215	7
3 SW/SW	10.5	10.61	1	10.64	1.4	10	-7.5	3 SW/SW	0.260	0.270	1.1	0.270	1.4	0.254	2
4 SW/SW	18.9	19.64	4	19.82	4.9	19	1.1	4 SW/SW	0.470	0.490	4.0	0.500	4.9	0.466	1
5 SW/SW	20.9	21.83	4	22.04	5.3	20	-4.4	5 SW/SW	0.520	0.550	4.3	0.550	5.3	0.512	2
6 SW/SW	29	31.61	9	32.24	11	30	4.5	6 SW/SW	0.720	0.790	9.2	0.810	11	0.723	-0

Table 5: Frequency Results (Hz); Wire Rope Example

The modes of vibration in this example, exhibit some interesting behavior due to the nature of the boundary conditions applied to the system. The mode shapes corresponding to the first and second CL/CL wire natural frequencies are shown in Figure 21 and Figure 22, respectively. As depicted in both figures, and more clearly in Figure 22, the displacements in this problem satisfy only C^0 continuity conditions which is consistent with the second order nature of the differential equations of motion and the degrees of freedom used in the elements formulation. This can be easily explained by the fact that, unlike bending, slope (i.e., first derivatives of displacement with respect to x) discontinuity with respect to extension and rotation is physically possible.

Figure 21: First Natural Mode; CL/CL Wire Rope Example

Figure 22: Second Natural Mode; CL/CL Wire Rope Example

The CL/CL mode plots for the 3rd to 5th natural frequencies obtained in this example are included in Appendix B for reference.

The first mode of vibration for the CL/SW boundary condition is presented in Figure 23. As depicted in the figure, the first CL/SW mode is dominated by rotation, similar to the first two CL/CL natural modes. The deformations in these modes follow predicted patterns, with the

extension being zero at both ends and the rotation being zero at the clamped one. The higher CL/SW modes are shown in Appendix B and are all seen to be dominated by rotation as well.

Figure 23: First Natural Mode; CL/SW Wire Rope Example

The first mode shape for the SW/SW boundary condition corresponds to the natural frequency equal to zero or a rigid body mode, which represents pure rotation of the entire rope. Due to the triviality and lack of practical applicability of the first rigid body mode in this study, it is not included here. Figure 24 shows the second mode corresponding to the first non zero frequency value. As can be seen, this mode is dominated by torsion, while containing very little extensional displacement.

The mode shapes for the remaining SW/SW frequency values calculated are shown in Appendix B, and again are all dominated by rotation.

Figure 24: Second Natural Mode; SW/SW Wire Rope Example

The DFE convergence rates for the three different boundary conditions of interest, namely, CL/CL, CL/SW and SW/SW, are shown in Figure 25, Figure 26 and Figure 27, respectively. This example demonstrates the fact that the convergence rates depend on the type of boundary conditions of the system. Comparison between Figure 25, Figure 26 and Figure 27 shows that the CL/CL BC requires an increased number of elements in order to obtain comparable accuracy to the CL/SW and SW/SW solutions. This can be attributed to the fact that the CL/CL boundary condition eliminates four degrees of freedom in the problem and therefore reduces the apparent number of elements (i.e., dimension of the eigenvalue system) used in the solution. A similar trend is seen in the CL/SW BC, with its accuracy lying between that of the CL/CL and the SW/SW conditions. The reason why the SW/SW BC appears to have worse convergence rates than the other two conditions may be explained by the fact that the SW/SW frequencies are one mode higher than those of other BCs due to the presence of the rigid body mode (i.e., zero natural frequency).

The convergence rates for the FEM solution for all three boundary conditions are included in Appendix B. These plots demonstrate trends similar to those seen in the other examples, with the convergence of the DFE being generally superior to those obtained from the FEM particularly for higher frequencies and/or when the BC tend to eliminate numerous degrees of freedom from the problem (e.g., CL/CL modes).

Figure 25: DFE Convergence Rates; CL/CL Wire Rope Example

Figure 26: DFE Convergence Rates; CL/SW Wire Rope Example

Figure 27: DFE Convergence Rates; SW/SW Wire Rope Example

4.3 Uniform CUS Composite Box-Beam Example #1

In order to validate the CUS configuration anisotropic beam theory developed in this project, examples using both geometries of interest are undertaken. The first example represents the free vibration analysis of a CUS configuration cantilevered composite box-beam, as discussed in references [2,3,4,5] (refer to Table 6).

Base (b)	Height (h)	Thickness (t)	Length (L)
1.32 in	.66 in	.033 in	100 in

Table 6: Geometry of Uniform CUS Composite Box-Beam Example # 1

The beam is made up of a $[20^\circ/-70^\circ/20^\circ/-70^\circ/-70^\circ/20^\circ]$ anti-symmetric laminate of T300/5208 Graphite/epoxy with material properties as shown in Table 7 (also see reference [3]).

E_1	E_2	G_{12}	ν_{12}	ρ
$21.3 \times 10^6 \text{ lb/in}^2$	$1.6 \times 10^6 \text{ lb/in}^2$	$.93 \times 10^6 \text{ lb/in}^2$	0.38	$1.501 \times 10^{-4} \text{ lbs}^2$

Table 7: Properties of T300/5208 Graphite Epoxy

The stiffness parameters of the box-beam were calculated using the method presented in section 3.3, and then compared to those published in references [3,5] in order to validate the pre-processor created in this project. The calculated stiffness terms are shown in Table 8 and are compared to those available in literature, taking NABSA data [3] as the reference values. NABSA is a finite element beam program based on the work of reference [30], which takes into account all warping, coupling and shear effects and has been confirmed in reference [11] through calculated static displacement comparison with experimental results. The TAIL program is a similar finite element program, except it does not account for all warpings.

Method	Axial Stiff (Lb)	% Diff	Coupling Stiff (Lbin)	% Diff	Torsional Stiff (Lbin ²)	% Diff	Vert. Bending Stiff (Lbin ²)	% Diff	Hor. Bending Stiff (Lbin ²)	% Diff
Present	1,142,000	0.44	-150,890	-1	51,428	-23	61,744	18	176,410	-32
NABSA [3]	1,137,000		-152,200		66,450		52,410		259,700	
TAIL [3]	1,140,000	0.26	-151,400	-1	94,040	42	51,020	-3	260,300	0

Table 8: Stiffness Comparison; Uniform CUS Composite Box-Beam Example # 1

As can be seen, the axial and coupled stiffness terms calculated using the present methodology compare well with published values, while a significant difference occurs in the torsional stiffness terms of the beam. The difference between NABSA and the present method can be attributed to the fact that the present analysis does not include all warping effects, particularly the effects of the end restraint on the beam. As stated in reference [18] this effect tends to increase the torsional rigidity, which is consistent with the findings of this work. The large difference in bending rigidities is due to the fact that both NABSA and TAIL make use of a bending-shear coupled analysis resulting in different bending stiffness integrals than the present method.

The composite beam natural frequencies were calculated using the DSM, FEM and DFE methods. The results were then compared to each other, and also to those published in references [1,3,4] in order to verify their validity. The results of reference [1] make use of the NABSA stiffness values in a DSM solution, similar to the one in this report. Reference [4] is based on the same stiffness method used here, except this author makes use of a different finite element based vibration solution resulting in different frequency values. The TAIL stiffness method frequencies reported in reference [3] also make use of a finite element solution. The first four extension-torsion coupled frequencies for the cantilevered box-beam are shown in Table 9. As it can be seen, the resulting natural frequencies compare well with those presented in references [1,3,4], where the DSM results have been used as reference values. The first three Vertical Bending (VB) and Horizontal Bending (HB) frequencies are also included and compare well with published values (see Table 9). These bending frequencies were calculated using an uncoupled Euler-Bernoulli (EB) analysis as outlined in Appendix C. The present method is able to calculate similar bending frequencies to those of literature, regardless of the large differences in bending stiffness magnitude due to the type of vibration analysis which is used. The frequencies calculated in the references use a coupled bending shear analysis and not the uncoupled EB one which is used here. This demonstrates the fact that the stiffness terms and the vibration solution must make use of similar assumptions and methodology; the effect of using inconsistent stiffness and solution methodologies will be demonstrated later.

Freq Num	DSM Exact	DFE 10 Elem	% Diff	FEM 10 Elem	% Diff	Published Values							
						NABSA [3]	% Diff	Ref [1]	% Diff	Ref [4]	% Diff	TAIL [3]	% Diff
1 ET	178.75	178.82	0.04	178.93	0.1	180.32	-0.87	187.79	-4.81	177.05	0.96	177.23	0.86
2 ET	536.24	538.17	0.36	541.22	0.93	544.47	-1.51	527.69	1.62	531.15	0.96	535.15	0.20
3 ET	631.97	632.23	0.04	632.62	0.1			563.36	-4.81				
4 ET	893.74	902.6	0.10	916.85	2.59			939.95	-42.95				
1 VB	3.1933					3	6.44			2.959	7.94	3.09	3.34
2 VB	20.012					19.04	5.10			18.54	7.94	19.64	1.89
3 VB	56.034					54.65	2.53			51.92	7.92	56.36	-0.58
1 HB	5.3976					5.19	4.00			5.1	5.84	5.74	-5.97
2 HB	33.826					32.88	2.88			31.98	5.77	36.39	-7.05
3 HB	94.715					93.39	1.42			89.55	5.77	103.29	-8.30

Table 9: Frequency Results (Hz); Uniform CUS Composite Box-Beam Example # 1

The first four mode shapes were also found in order to depict the extension-torsion coupling and the way the displacements affect each other. The first natural mode of the composite beam is shown in Figure 28, where the coupled nature causes both rotation and displacement to occur simultaneously. This mode is dominated by torsion effects, with the magnitude of the extensional displacement being significantly less than that of rotation. It can also be seen from Figure 28 that, for this mode, the extensional and torsional displacements oscillate with the same sign (S), therefore this mode will be designated ETS 1 since it is the first extension-torsion mode with the same sign for both displacements.

Figure 28: First Natural Mode; Uniform CUS Composite Box-Beam Example #1

The mode shape for the second natural frequency is shown in Figure 29. As it can be seen, the second mode is dominated by torsion as well. Also, similar to the first mode, this mode oscillates with the displacements having the same sign, meaning a designation of ETS 2.

Figure 29: Second Natural Mode; Uniform CUS Composite Box-Beam Example #1

Figure 30: Third Natural Mode; Uniform CUS Composite Box-Beam Example #1

The third mode (Figure 30) is the first one for which the displacements oscillate with opposite (O) sign displacements meaning a designation of ETO 1. It will be seen later that the ply-angles, which the laminate consists of, play a very important role in determining the order of these frequencies (i.e., ETO 1 corresponding to the third natural frequency, instead of the second or first). This is also the first mode with significant extension; however, it is not overly dominant since a large amount of rotation is also present.

The fourth mode shown in Figure 31 is again dominated by rotation. The primary reason for the dominance of torsion in lower frequency modes can be attributed to the significantly lower value of torsional stiffness when compared to the axial one.

Figure 31: Fourth Natural Mode; Uniform CUS Composite Box-Beam Example # 1

The DFE convergence rates are shown in Figure 32 and follow the same general trend as those obtained for the other extension-torsion coupled applications. The DFE method continues to demonstrate superior convergence rates to those found from the FEM. It can be seen (Figure 32), that, for coupled vibration, unlike uncoupled, the error is not frequency number dependant, with the third frequency showing a better convergence rate in both the FEM and DFE solution than does the second one. This convergence trend is similar to those found in the DFE solution for bending-torsion coupled vibration [22,29].

Figure 32: DFE Convergence Rates; Uniform CUS Composite Box-Beam Example #1

The FEM convergence rates are shown in Figure 33 and follow the same simple monotonic trends as previously seen for other applications. Also, the FEM shows better convergence for the third natural frequency than it does for the second one. This behavior goes against convention and can be attributed to the coupled nature of the problem, a similar trend was also reported in references [22,29] for the FEM vibration analysis of coupled bending-torsion composite beams.

Figure 33: FEM Convergence Rate; Uniform CUS Composite Box-Beam Example #1

In order to completely validate the DSM solver being used, the stiffness terms, mass moment of inertia, and mass per unit length values reported in reference [1] were used in the DSM solver developed in this study in order to duplicate the values published therein. Since a similar (DSM) method is used in this publication, the difference in frequency values must be justified. The differences occur due to the fact that reference [1] makes use of the stiffness terms of NABSA [3] and not those evaluated in the present method. They are also compounded by the fact that reference [1] uses mass per unit length and mass moment of inertia values which are different from those developed here. The effect of these differences in stiffness and geometric properties on the final frequency result will now be investigated.

In order to better understand the effect that differences in stiffness magnitude and geometric properties has on the natural frequencies, all three stiffness methods (NABSA [3], TAIL [3], and present) will be compared for both the geometric properties found in this report as well as those published in reference [1]. The results for the first four coupled ET natural

frequencies using the DSM method are shown in Table 10 . As it can be seen, significant variation occurs when different stiffness and geometric parameters are used.

	Freq Num	$m = 1.90 \text{ e-}5 \text{ (Lbs}^2/\text{ft}^2)$ $I = 5.79 \text{ e-}6 \text{ (Lbs}^2)$	$m = 2.74 \text{ e-}5 \text{ (Lbs}^2/\text{ft}^2)$ [1] $I = 7.60 \text{ e-}6 \text{ (Lbs}^2)$ [1]	Reference Values
Present	1	178.8	155.5	177.1 [4]
	2	536.2	466.6	531.2 [4]
	3	632.0	527.1	N.A.
	4	893.7	777.7	N.A.
NABSA [3]	1	233.1	187.8 [1]	180.3 [3,4]
	2	625.6	527.7 [1]	544.5 [3,4]
	3	699.2	563.4 [1]	N.A.
	4	1165.3	940.0 [1]	N.A.
TAIL [3]	1	272.8	237.2	177.2 [3]
	2	634.1	530.0	535.2 [3]
	3	818.5	711.5	N.A.
	4	1364.1	1185.8	N.A.

Table 10: Effects of Model Variation on Frequency Results

The data presented in Table 10 follow accepted trends, such that, as the magnitude of the extension-torsion coupled stiffness terms (C_{11} , C_{12} , and C_{22}) is increased, the natural frequencies of vibration increase as well. It should however be noted that, for some reason, the reference values seem to diverge from this trend when it comes to calculation of the frequencies using the stiffness parameters derived from the TAIL program. Even though the stiffness terms of TAIL are the largest in magnitude, the reference results using these stiffness terms results in the median frequency values. It should also be noted that the difference in final frequency values calculated is very small within the references; however, using the present geometric properties, the changes in stiffness values used caused significant changes in final frequency results. This demonstrates once again the fact that special care has to be taken to use consistent assumptions and methodology when calculating the stiffness coefficients and vibration frequencies for a given problem.

The effect of changes in ply-angle on the stiffness terms as well as the final frequency values will now be investigated. As it was seen earlier in the ply-angle effects section of this report (Section 3.3.5), the laminate effective material stiffness parameters for the $[\eta/\eta -90/\eta/\eta -90/\eta -90/\eta]$ laminate used in this example vary in a cyclic manner (see Figure 14). Therefore, it

is logical and was found that the cross-sectional stiffness coefficients follow the same pattern, as shown in Figure 34.

Figure 34: Stiffness Terms versus Ply-Angle; Uniform CUS Composite Box-Beam Example # 1

The first two ETS and ETO extension-torsion coupled frequency values are shown Figure 35, and, as it can be seen, these values change significantly with ply-angle. It also can be seen that ETO 1 and ETS 1, as well as, ETO 2 and ETS 2 are symmetric with respect to one another about an angle of 45° . This is different than the behavior noted in reference [3] for a circular tube made from a similar laminate. In reference [3], the frequencies are found to be symmetric with respect to themselves about an angle of ; however, this is believed to be in error since in this investigation the non-dominant displacement was found to change sign with the change in sign of the coupling term which occurred at an angle of 45° . Similar behavior will also be seen and discussed in the first circular tube example (section 4.5) of this report, with an extended explanation about the difference with reference [3].

Figure 35: E-T Coupled Frequency versus Ply-Angle; Uniform CUS Composite Box-Beam Example # 1

An interesting observation is that, even though the sign of the coupling stiffness changes for a $\eta = 20^\circ$ ($45^\circ - 25^\circ$) lay-up versus a $\eta = 70^\circ$ ($45^\circ + 25^\circ$) lay-up, the final frequency values remain the same. The only effect this sign change has on the solution can be seen in the mode of vibration for each lay-up.

Figure 36: First Natural Mode for $\eta = 70^\circ$; Uniform CUS Composite Box-Beam Example # 1

As can be seen with reference to Figure 36, the displacements for the $\eta = 70^\circ$ lay-up are opposite in sign, while those of the $\eta = 20^\circ$ shown in Figure 28 are the same. The dominant deformation remains the same for both angles but the secondary deformation is seen to change

sign. This is related to the sign of the coupling stiffness term, which allows a designer to pick the lay-up angles such that the desired displacements are achieved.

The effect of ply-angle on the Horizontal (HB) and Vertical Bending (VB) frequencies can be seen in Figure 37. As shown, these frequencies decrease when the stiffness decreases, reaching their minimum value at $\eta = 45^\circ$. The modes for these frequencies do not show any changes with ply-angle as they are uncoupled and therefore behave in the same way as an isotropic cantilevered uniform beam. These modes are not included here due to their general availability in vibration literature [17].

Figure 37 : Lateral Bending Frequency versus Ply-Angle; Uniform CUS Composite Box-Beam Example # 1

The change in frequency with ply-angle also allows the designer to modify the vibration characteristics of a beam without altering its physical dimensions. As a result, resonance frequencies may be avoided by tailoring the ply-angle (η). Another important fact is that the frequencies tend to swap. This means that a frequency with displacements of the same sign, e.g., the 4th natural frequency will become the 3rd natural frequency with a significant change in ply-angle. This also means that the lay-up can be tailored in such a way that only the desired type of displacement coupling occurs (e.g., extension and torsion occurring with the same sign).

4.4 Uniform CUS Composite Box-Beam Example # 2

In order to further investigate and verify some of the free vibration characteristics of composite box-beams, another example is studied. This example will further demonstrate the drastic effect of ply-angle on frequency, using an $[\eta]_6$ lay-up. The material and geometric properties of this example the same as those given in reference [4] (see Table 11).

Base (b)	Height (h)	Thickness (t)	Length (L)	
.953 in	.537 in	.03 in	30 in	
E_1	E_2	G_{12}	ν_{12}	ρ
$20.59 \times 10^6 \text{ lb/in}^2$	$1.42 \times 10^6 \text{ lb/in}^2$	$.87 \times 10^6 \text{ lb/in}^2$	0.42	$1.501 \times 10^{-4} \text{ lbs}^2$

Table 11: Geometric and Material Properties; Uniform CUS Composite Box-Beam Example # 2

The stiffness constants were calculated for various angles and are shown in Figure 38. As it can be seen, changes in ply-angle do not cause cyclic variation of stiffness but instead cause a steady decrease. This is due to the fact that the lay-up used in this example is made up of only a single angle and not multiple angles like the lay-up of beam example #1. This means the laminate becomes less stiff as the fiber angle increases and moves away from the principle load direction. It can also be seen that the maximum coupling occurs at a fiber angle of approximately 20° , which means laminates with $\eta = 20^\circ$ will exhibit the most static coupling between extension and torsion deformation.

Figure 38: Stiffness Terms versus Ply-Angle; Uniform CUS Composite Box-Beam Example # 2

The first 15 cantilevered (CL/FR) coupled extension-torsion natural frequencies were calculated using the DSM, FEM and DFE methods, similar to box-beam example #1. The DSM results are included in Table 12 and compared to those presented in reference [4] in order to verify the results. The first two lateral bending frequencies are also included in this table with a similar comparison of results. The results contained in reference [4] are obtained using the same stiffness equations and assumed displacement field as the present study and therefore the similarity in results is expected. The differences occur primarily due to the fact that the frequency values in reference [4] are given graphically and therefore their exact values are not known. Another possible reason for any inconsistencies within the dataset is the composite material pre-processor which was used. Note that in the present research, the simplest classic laminate theory was used; however, the method used in reference [4] is not specified and therefore may be different. The frequencies that have a sign of 'S' in Table 12 are those for which the displacements (extension and rotation) occur with the same sign (ETS), those with a sign of 'O' are those which have opposite sign displacements (ETO).

Freq. Num	Ply Angle = 20°				Ply Angle = 40°				Ply Angle = 60°				Ply Angle = 80°			
	Freq. (Hz)	Sign	Ref [4]	Diff (%)	Freq. (Hz)	Sign	Ref [4]	Diff (%)	Freq. (Hz)	Sign	Ref [4]	Diff (%)	Freq. (Hz)	Sign	Ref [4]	Diff (%)
1 ET	523	S	525	0	552	S	550	0	547	S	550	-1	511	S	500	2
2 ET	1568	S	1550	1	1363	O	1400	-3	922	O	900	2	819	O	800	2
3 ET	2553	O	2550	0	1657	S	1650	0	1642	S	1650	0	1534	S	1500	2
4 ET	2614	S	2600	1	2761	S	2750	0	2736	S	2700	1	2456	O	2450	0
5 ET	3660	S			3865	S			2765	O	2750	1	2556	S	2550	0
6 ET	4705	S			4090	O	4100	0	3831	S			3578	S		
7 ET	5751	S			4970	S			4608	O	4600	0	4093	O	4100	0
8 ET	6796	S			6074	S			4925	S			4601	S		
9 ET	7660	O	7750	-1	6817	O	6800	0	6020	S			5623	S		
10 ET	7842	S			7179	S			6452	O			5730	O		
11 ET	8888	S			8283	S			7114	S			6645	S		
12 ET	9933	S			9387	S			8209	S			7367	O		
13 ET	10979	S			9544	O			8295	O			7668	S		
14 ET	12024	S			10492	S			9303	S			8690	S		
15 ET	12767	O	12750	0	11596	S			10139	O			9004	O		
1 VB	30.1		29.0	4	19.0		18.5	3	15.6		15.0	4	14.5		14.0	4
1 HB	46.4		46.0	1	29.3		28.5	3	24.0		24.0	0	22.4		22.0	2

Table 12: Frequency Results (Hz); Uniform CUS Composite Box-Beam Example # 2

As can be seen, the frequencies which consist of modes with opposite sign displacements swap significantly, with the third one starting as the 15th frequency for the 20° laminate, changing to the 7th one for the 80° laminate. This is illustrated graphically in Figure 39, and it can be seen that the natural frequency values decrease significantly for the opposite sign modes as the stiffnesses decrease. It can also be seen that after an angle of approximately 60° the frequency values begin to level off. This is due to the behavior of the stiffness terms, with their values also leveling out after this angle. It should also be noted that even though the ET coupled frequencies of the same sign (ETS) appear to be symmetric about 45°, they are not, since their values differ slightly.

Figure 39: ET Coupled Frequency versus Ply-Angle; Uniform CUS Composite Box-Beam Example # 2

The behavior of the lateral bending stiffness terms (Figure 38) is similar to that of the ET coupled ones (C_{11} , C_{12} and C_{22}), i.e., they decrease sharply until an angle of approximately 60° when they begin to level out. The variation of the fundamental vertical and horizontal bending frequencies with ply-angle is shown in Figure 40. As can be seen, ply-angle plays as significant a role with them as it does with the opposite sign displacement extension-torsion frequencies (ETO).

Figure 40: Lateral Bending Frequency versus Ply-Angle; Uniform CUS Composite Box-Beam Example # 2

Figures 41: DFE Convergence; Uniform CUS Composite Box-Beam Example # 2

In order to determine the effect of ply-angle on DFE convergence rates, convergence tests were carried out with angles of 20° , 40° , 60° and 80° , respectively (Figures 41). As it can be seen, as ply-angle increases so do the convergence rates. The reason for the increase in convergence rates can be partly attributed to the fact that as the ply-angle increases, the magnitude of the stiffness and therefore the frequency value decreases, resulting in a more accurate solution. Convergence plots for the FEM method are included in Appendix D and demonstrate the same trend with convergence increasing with ply-angle; however, like the other examples, the FEM converges less quickly towards the exact solution than does the DFE.

The effect of changes in ply-angle on the mode shapes of the same and opposite sign displacements will now be examined. As can be seen in Figures 42, the mode shape for the first same sign displacement mode (ETS 1) can be seen to be almost identical for both 20° and 40° , and also for 60° and 80° as well (not shown here). The consistency of same sign mode shape can be seen for all same sign frequencies and is demonstrated further by the mode shapes for ETS 2 and 3, included in Appendix D.

Figures 42 : Mode Plot for ETS 1; Uniform CUS Composite Box-Beam Example # 2

Changes in ply-angle are seen to have a different effect when it comes to the deformation behavior of modes which displace with opposite signs. Figures 43 show the first extension-torsion coupled mode oscillating with opposite sign displacement (ETO 1) and it can be seen that ply-angle has a dramatic effect on the dynamic behavior of this type of mode. The dominant

deformation, in this case extensional displacement, is seen to remain dominant and of constant shape for the different angles. The rotation is seen to change significantly in magnitude, depending strongly on the cross-sections stiffness terms, which are directly related to the ply-angle used. This behavior is illustrated further by the mode shapes of ETO 2 and 3, included in Appendix D. It can be seen that even though for an angle of 20° the coupling term is at its maximum value, this mode does not exhibit the largest non-dominant deformation, which instead occurs at 40° . This demonstrates that it is not simply the magnitude of the coupling stiffness term (C_{12}) which determines the dynamic behavior for these beams, but instead is a combination of all three stiffness terms (C_{11} , C_{12} , C_{22}).

Figures 43 : Mode Plots for ETO 1; Uniform CUS Composite Box-Beam Example # 2

Another very similar example to this one is investigated theoretically in references [4,11,26-28] and experimentally confirmed in references [11,28]. The geometric and material properties of this example are similar to those used previously in this section and are given in reference [28] as (Table 13):

Base (b)	Height (h)	Thickness (t)	Length (L)
.953 in	.537 in	.03 in	33.25 in

E_1	E_2	G_{12}	ν_{12}	ρ
$20.59 \times 10^6 \text{ lb/in}^2$	$1.42 \times 10^6 \text{ lb/in}^2$	$.89 \times 10^6 \text{ lb/in}^2$	0.42	$1.352 \times 10^{-4} \text{ lbs}^2$

Table 13: Geometric and Material Properties; Uniform CUS Composite Box-Beam Example # 2b

In order to further verify the stiffness pre-processor developed, this example is undertaken with the output stiffness values compared to those of the literature as seen in Table 14. The percentage difference shown in this table is for each method versus the present theory. As can be seen, the extension-torsion coupled portion is in good agreement with both literature and experiment. The theory presented in references [2,26-28] corresponds to the stiffness values given in reference [27] and is a contour based displacement analysis similar to the one developed here. This method does, however, include many effects which are neglected here, such as wall thickness, transverse shear coupling and warping restraints. The program NABSA is a finite element based beam model also very similar to the model created here, which is widely used in the literature as a reference tool when deriving new stiffness theories [4,5,38,30] since it includes all warping effects. The VABS program is a cross-sectional finite element method, which uses an asymptotic approach to develop the stiffness constants for the section. The experimental method used for determining the stiffness constants is similar to that outlined in the wire rope section of this report (see ref [14]), with the exact testing given in detail in reference [11]. The large difference in the bending stiffness between the present model and those of the literature [11,27] is due to the incorporation of shear coupling into their bending models, resulting in different bending stiffness integrals. This also means that the stiffness values used in these references go through a coupled analysis similar to that of the extension-torsion coupling studied in this report, but consisting of 4th order (bending) equations of motion.

Stiffness Parameter	Method										
	Present	Theory [27]	%	VABS [11]	%	Exp [11]	%	NABSA [27]	%	VABS [27]	%
Axial Stiffness (Lb)	1,421,300	1,438,000	-1	1,432,000	-1	1,582,000	-10	1,438,000	-1	1,445,000	-2
Coupling Stiffness (Lbin)	-104,860	-106,400	-1	-106,000	-1	-122,000	-14	-107,500	-2	-109,000	-4
Torsional Stiffness (Lbin ²)	16,550	16,620	0	16,890	-2	18,000	-8	16,780	-1	17,190	-4
H. Bending Stiffness (Lbin ²)	43,288	68,730	-37	55,130	-21			66,220	-35	54,620	-21
V. Bending Stiffness (Lbin ²)	102,930	172,800	-40	136,500	-25			172,600	-40	135,200	-24

Table 14 : Stiffness Comparison; Uniform CUS Composite Box-Beam Example # 2b

Fundamental frequency results for this example are shown in Table 15 and are within 11% of literature values, except when compared to those of reference [28], when shear coupling is neglected. It can be seen that the inclusion of shear coupling in the bending model acts to reduce the estimated frequency, which is similar to the trend found from the inclusion of extension-torsion coupling versus pure extension (see reference [13]). The present method is able to adequately predict the lateral bending frequencies for the beams used in this example without incorporating shear coupling. This is due to the fact that, in the present study, the bending stiffness integrals are derived assuming no shear and/or shear coupling from the beginning and therefore end up in different forms than those found in references [2,26-28,30]. This and the uncoupled Euler-Bernoulli analysis (see Appendix C) used in this study, make the present method completely different to that used in references [2,11,26-28,30] and instead similar to that of reference [4]. However, it should be noted that due to the neglecting of the bending-shear coupling in the present model, the mode shapes are not necessarily exact. The reason for this is due to the fact that there would be additional shear deformation which would be caused by the bending of the beam. This deformation could drastically change the mode shape, causing out-of-plane displacement, as well as changes in displacement magnitude, depending on the lay-up used and the geometry of the beam. The first horizontal bending frequency for this beam was found to be 47.7 Hz, and the first three ET coupled frequencies were found to be 468.7 Hz, 1406.2 Hz,

and 2343.7 Hz, respectively. The reason these frequencies are not included in the Table 15 is due to the lack of reference values available for comparison.

Freq	Method										
	Present	[28] Shear	%	[28] No Shear	%	Exp [28]	%	Ref [4]	%	Ref [21]	%
1 VB	30.9 Hz	28.0 Hz	10.4	39.0 Hz	-20.8	29.5 Hz	4.7	28.7 Hz	7.7	30.0 Hz	3.0

Table 15: Frequency Results (Hz); Uniform CUS Composite Box-Beam Example # 2b

It can be seen from this example that the dynamic behavior of a composite beam is strongly related to the type of lay-up of which it consists, with the results of this example being dramatically different than those of beam example #1. The opposite sign coupled extension-torsion frequencies were found to change values as well as mode shape significantly with changes in ply-angle. The bending frequencies were also found to change significantly with ply-angle; However, the shape would remain constant, since this model did not include shear couplings. The same sign extension-torsion modes are found not to change value as significantly as the opposite sign ones; It was also found that the mode shape for these frequencies remains almost constant with ply-angle, which is an interesting observation.

4.5 Uniform CUS Composite Circular Tube Example #1

In order to further validate the numerical methods developed in this study, the free vibration analysis of a circular tube model, with cantilevered end conditions, is performed and the comparison was made between the natural frequencies and those available in the literature [3]. The geometry of this circular tube example is shown in Figure 44.

Figure 44 : Geometry of Uniform CUS Composite Circular Tube Example # 1

The circular tube consists of a $[\eta / \eta -90 / \eta / \eta -90 / \eta -90 / \eta]$ lay-up configuration and has a length $L=100$ in and a total thickness of 0.033 in, with material properties similar to those used in box-beam example #1 (in order to be consistent, see Table 7). The first three coupled as well as three bending CL/FR frequencies of vibration were evaluated using the three DSM/exact, DFE and FEM methods (see Table 16). The DSM/Exact natural frequencies were first compared to those presented in reference [3]. Excellent agreement was found between the DSM coupled extension-torsion modes and the published results (see Table 16). The DFE and FEM natural frequency results cited in the table are all obtained using a 5-element mesh, and, as it can be seen, even using this small number of elements, the DFE method is able to closely approximate all three extension-torsion coupled frequencies. It can be seen that the FEM needs more elements to adequately approximate the third frequency; However, due to the short computational time associated with this method, the number of elements can be increased significantly before reaching that of the DFE. The reference values of [3] were read from a graph and that is why the values are integer (rounded and not exact). The bending frequencies are also adequately approximated, where most of the error occurs due to the accuracy of the source data.

Freq Num	$\eta = 20^\circ$					$\eta = 60^\circ$				
	DSM	DFE	FEM	[3]	% Diff	DSM	DFE	FEM	[3]	% Diff
1 ET	228.80	229.16	229.73	230	-0.52	272.00	273.19	273.56	280	-2.94
2 ET	647.20	648.86	649.86	640	1.11	582.70	584.26	585.96	585	-0.39
3 ET	686.40	696.16	711.97	690	-0.52	816.10	823.02	842.12	820	-0.48
1 B	12.46			14	-12.39	10.46			10	4.44
2 B	78.06			90	-15.29	65.68			75	-14.19
3 B	218.58			250	-14.37	183.62			210	-14.37

Table 16: Frequency Results (Hz); Uniform CUS Composite Circular Tube Example # 1

The stiffness terms of interest for the two angles examined in Table 16 are shown in Table 17.

Angle (Degrees)	Axial Stiffness (Lb)	Coupling Stiffness (Lbin)	Torsional Stiffness (Lbin ²)
20	3,743,000	-2,349,800	3,799,200
60	2,527,400	2,481,100	6,365,600

Table 17: Stiffness Terms; Uniform CUS Composite Circular Tube Example # 1

The cross-sections stiffness terms as well as the corresponding natural frequencies for various angles (η) are presented in Figure 45 and Figure 46, respectively. It can be seen that, similar to beam example #1, the stiffness parameters vary in a periodic fashion, with the coupling stiffness (C_{12}) changing sign at $\eta=45^\circ$. It can also be seen that the other stiffness terms reach their extrema values at this angle.

Figure 45: Stiffness Terms versus Ply-Angle; Uniform CUS Composite Circular Tube Example # 1

Figure 46: ET Coupled Frequency versus Ply-Angle; Uniform CUS Composite Circular Tube Example # 1

As it can be seen from Figure 46, the ET coupled natural frequencies follow the same trend as previously seen in Beam Example # 1, with ETS 1 and ETO 1 being symmetric with respect to one another about $\eta=45^\circ$. The change from ETS to ETO at this point is due to the change in the sign of the coupling stiffness term (C_{12}). This deviates from the results reported in reference [3] where the authors state that ETS 1 is symmetric with itself about $\eta=45^\circ$, instead of being symmetric with ETO 1. Their results would mean that the sign of the coupling stiffness term (C_{12}) plays no role in the problem. It can be seen however from the cross-sections constitutive relations (108), (109), (3), and (4), that a change in the sign of ' C_{12} ' results in a change in the sign of the non-dominant displacement, which is consistent with the findings of this report. The frequency values at different ply-angles compare well with the graphical data given in reference [3], which confirms the validity of the methods proposed here for composite circular tubes.

The effect of the ply-angle on bending frequencies of the system was also studied (see Figure 47). As expected, the same trends as those seen in beam example #1 were obtained for all three bending modes since a similar lay-up was used in both examples. As it can be seen, the bending frequencies also attain their extreme values at $\eta=45^\circ$, with them all being a minimum.

Figure 47: Lateral Bending Frequency versus Ply-Angle; Uniform CUS Composite Circular Tube Example #1

The first extension-torsion coupled mode shapes of the circular tube are illustrated in Figure 48 and Figure 49, respectively, for a $\eta = 20^\circ$ and $\eta = 60^\circ$. It can be seen that the behavior of this mode changes significantly with changes in ply-angle, going from a rotation dominated mode at $\eta = 20^\circ$ to an extension dominated one at $\eta = 60^\circ$.

Figure 48: First Natural Mode (ETS 1) for $\eta = 20^\circ$; Uniform CUS Composite Circular Tube Example # 1

Figure 49: First Natural Mode for $\eta = 60^\circ$ (ETO 1); Uniform CUS Composite Circular Tube Example # 1

As depicted in Figure 46, similar to beam example #1, $\eta = 45^\circ$ represents an axis of symmetry where the $(45^\circ - \delta)$ and $(45^\circ + \delta)$ lead to the same frequency values but have a reversal in sign of the non-dominant degree of freedom. This can be seen by comparing Figure 48 and Figure 50, where the dominant rotational displacement is the same for both and the non-dominant extensional displacement changes the sign, i.e., the mode goes from ETS 1 at $\eta = (45^\circ - 25^\circ) = 20^\circ$ to ETO 1 at $\eta = (45^\circ + 25^\circ) = 70^\circ$.

Figure 50: First Natural Mode for $\eta = 70^\circ$ (ETO 1); Uniform CUS Composite Circular Tube Example # 1

The second and third natural mode plots for angles of 20, 60 and 70 degrees are included in Appendix E, for reference. Also included in this Appendix are mode shapes for ETS 1, ETS 2, and ETO 1 for an angle of $\eta=40^\circ$, in order to allow for a direct comparison to similar mode plots presented in reference [3]. These graphical results are in excellent agreement with the reference data and confirm the correctness of the calculated mode shapes and the coupled behavior of the system.

The DFE convergence test results for $\eta = 20^\circ, 60^\circ$ and 70° are shown in Figures 51. It can be seen that the convergence rates for angles of 20° and 70° are identical, due to the fact that the magnitude of the stiffness terms as well as frequency values are the same for both angles (i.e., 45-25 and 45+25 degrees). It can also be seen that unlike Beam Example # 2, there is no major change in convergence rate with changes in ply-angle (η). The FEM convergence rates are included in Appendix E for reference and demonstrate the same trends as the DFE, with the rates being the same for angles of 20° and 70° and ply-angle not playing a significant role in the convergence rates of the solution. When compared to the FEM, the DFE method demonstrates superior convergence rates, which is similar to the trend seen for all the other uniform examples.

Generally speaking, the applicability of the vibrational analysis method developed in this research was validated for multiple geometries. It is seen that the laminate configuration (lay-up and stacking sequence) plays the most significant role in determining the dynamic behavior of the flexible composite beams. Even though the numerical results of the last two examples are different due to geometric effects, one can appreciate the fact that, from a dynamic point of view (i.e., the way changes in ply-angle affect the beam stiffness parameters and frequency values), composite circular tubes and box-beams are similar.

Figures 51: DFE Convergence Rates; Uniform CUS Composite Circular Tube Example #1

4.6 Uniform CUS Composite Circular Tube Example # 2

To further verify the applicability of the circular tube model created here, the natural frequencies for a cantilevered (CL/FR) $[\eta]$ CUS lay-up of geometry shown in Figure 52 and the material properties of beam example # 1 (see Table 7), were calculated and compared to those presented in reference [3].

Figure 52: Geometry of Uniform CUS Composite Circular Tube Example #2

The stiffness terms for angles of 20 degrees and 60 degrees are shown in Table 18 and as it can be seen, these values are significantly lower than those of the other CUS beam examples in this report due to the fact that the laminate is made of only a single lamina.

Angle (Degree)	Axial Stiffness (Lb)	Coupling Stiffness (Lbin)	Torsional Stiffness (Lbin ²)
20	965,290	-575,410	641,980
60	132,060	-496,950	346,780

Table 18: Stiffness Terms; Uniform CUS Composite Circular Tube Example # 2

The variation of the circular tube cross-sectional stiffness coefficients with ply-angle is shown in Figure 53. As it can be observed, their behavior is similar to that of box-beam example #2. It is also worth noting that, due to geometric effects, the torsional rigidity of the composite circular tube changes far more than that for the box-beam.

Figure 53: Stiffness Terms versus Ply-Angle; Uniform CUS Composite Circular Tube Example # 2

The first three coupled extension-torsion and three bending natural frequencies of the system were calculated using the DSM/exact, and 5-element DFE and FEM models. The frequencies for $\eta = 20^\circ$ as well as $\eta = 60^\circ$ lay-ups are shown in Table 19. The DSM/exact values are also compared to those available in reference [3] in order to validate the results. One should note that the results taken from reference [3] were read from a graph and therefore are not overly accurate. This may account for some of the discrepancy between them and those obtained from DSM/exact, DFE and FEM models.

Freq Num	Alpha = 20 degrees					Alpha = 60 degrees				
	DSM	DFE	FEM	[3]	% Diff	DSM	DFE	FEM	[3]	% Diff
1 ET	202.70	203.18	203.57	205	-1.12	213.65	213.70	214.53	215	-0.63
2 ET	608.20	620.15	630.89	605	0.53	294.09	294.16	295.31	300	-1.97
3 ET	798.20	800.10	801.50	810	-1.46	640.96	642.29	664.87	630	1.74
1 B	13.19			14	-5.79	6.80			6.50	4.57
2 B	82.66			90	-8.16	42.60			40	6.49
3 B	231.44			260	-10.98	119.27			140	-14.81

Table 19: Frequency Results (Hz); Uniform CUS Composite Circular Tube Example # 2

The variation of the first three extension-torsion coupled frequencies with ply angle (η) is shown in Figure 54, and, as it can be seen, their behavior is similar to that of beam example #2 due to the similarity in lay-up used. This figure also agrees very well with a similar figure

published in reference [3] which allows for the verification of the model across numerous ply angles.

Figure 54: ET Coupled Frequency versus Ply-Angle; Uniform CUS Composite Circular Tube Example # 2

The variation of the first three bending frequencies with ply angle (η) is shown in Figure 55, revealing the same decreasing trend seen in beam example #2. This graph also compares well to the one published in reference [3] allowing for further verification of the bending model used for a wide range of ply-angles ($0^\circ < \eta < 90^\circ$).

Figure 55: Lateral Bending Frequency versus Ply-Angle; Uniform CUS Composite Circular Tube Example # 2

The DFE and FEM convergence tests results follow the same trends as those obtained for the other examples and are presented in Appendix F for reference. The first natural mode for $\eta=20^\circ$ and $\eta=60^\circ$ are shown in Figure 56 and Figure 57, respectively. As it can be seen, the ply orientation has a dramatic impact on the amount of coupling present in the problem. This greatly affects the natural frequency of vibration and demonstrates the importance of accurately accounting for this coupling within the analysis.

Figure 56: First Natural Mode for $\eta = 20^\circ$ (ETS 1); Uniform CUS Composite Circular Tube Example # 2

Figure 57: First Natural Mode for $\eta = 60^\circ$ (ETS 1); Uniform CUS Composite Circular Tube Example # 2

The second natural mode of vibration for natural mode for $\eta=20^\circ$ and $\eta=60^\circ$ are shown in Figure 58 and Figure 59, respectively.

Figure 58: Second Natural Mode for $\eta = 20^\circ$ (ETS 2); Uniform CUS Composite Circular Tube Example # 2

Figure 59: Second Natural Mode for $\eta = 60^\circ$ (ETO 1); Uniform CUS Composite Circular Tube Example # 2

It can be seen that the change in ply-angle has a significant effect on this mode, causing it to change from a largely coupled mode of the same sign displacements to a mode with smaller coupling of opposite sign displacements. This is similar to the results seen in beam example #2 where the opposite sign modes tend to shift position significantly with changes in ply angle. The third natural mode shape of the system is also included in Appendix F for reference.

4.7 Linearly Tapered CUS Composite Box-Beam Example

The theory developed in this study can be easily expanded to include more real world effects. To demonstrate, an example with changing geometry will now be undertaken. To simplify the analysis, the averaging method (section 3.3.4) developed in this report is used in a linearly tapered example similar to beam example #1 in order to determine the effects of tapering on both the stiffness, as well as the dynamic behavior of the beam. The root geometry of the beam is the same as that of beam example #1 with a base (b) of 1.32 in and a height (h) of .66 in. The taper coefficient (c) used for both dimensions in this example is -0.5 which causes the dimensions to become half of their root values at the tip. In order to fully understand the effect which tapering in each direction plays in the dynamic behavior of a composite box beam, three separate examples were undertaken, each with a different set of taper coefficients used for the dimensions: ' Nb '=1 ' Nh '=0, ' Nb '=1 ' Nh '=1 and ' Nb '=0 ' Nh '=1. The lay-up used in this example was a [20/-70/20/-70/-70/20] ($[\eta / \eta -90/ \eta / \eta -90/ \eta -90/ \eta]$) with the material properties of beam example # 1 (see Table 7).

The variation of stiffness along the length of the beam for the three sets of different taper coefficients are shown in Figures 60, and, as can be seen, variation in the base dimension causes the largest change in stiffness. This makes perfect sense due to the fact that this dimension changes most in magnitude.

Figures 60 : Stiffness Terms versus Ply-Angle; Linearly Tapered CUS Composite Box-Beam

The frequency values found using these stiffness variations are shown in Table 20, and, as it can be seen, the taper causes a significant change in the dynamic characteristics of the box-beam. When both the height and base dimension are varied the dynamic characteristics are the most different from those of the uniform example. It should also be noted that varying the height dimension only does not drastically affect the dynamic characteristics. This is due to the fact that the height value does not change as much as the base one does due to the approximating function used. The small effect that the height has on the dynamic behavior is further demonstrated by comparing the dually tapered beam to the base only taper beam. Results are similar to the case where uniform and height-tapered beams are compared. The DSM, DFE and FEM solution values quoted are based on a 10 element mesh.

Freq	$N_b = 1$		$N_h = 0$		Freq	$N_b = 1$		$N_h = 1$		Freq	$N_b = 0$		$N_h = 1$		$N_b = 0$
#	DSM	DFE	FEM	EXACT	#	DSM	DFE	FEM	EXACT	#	DSM	DFE	FEM	EXACT	$N_h = 0$
1	234.3	234.4	234.7	234.7	1	255.8	256.0	256.8	256.8	1	184.2	184.3	184.5	184.4	178.7
2	592.2	594.5	598.1	593.6	2	569.3	571.7	575.3	572.4	2	474.6	476.4	479.4	475.5	536.2
3	697.9	698.2	698.8	698.5	3	740.3	740.8	741.6	741.5	3	657.8	658.2	658.7	658.1	632
4	967.7	977.5	993.3	970.4	4	906.6	919.0	934.5	915.2	4	780.8	789.0	802.0	788.9	893.7

Table 20 : Frequency Results (Hz); Linearly Tapered CUS Composite Box-Beam

The natural frequency values listed as EXACT in Table 20 are from a 100 element DSM solution for which all of the frequencies examined had converged to more than 4 decimal points. These frequencies were then used as a reference in order to determine the convergence rates for the DSM, DFE and FEM methods. An interesting trend observed from Table 20 is that the DSM method seems to underestimate the results, i.e., it converges on the natural frequencies from below and not from above, as is typical in uniform DFE and FEM solutions. The DFE method behaves in the same manner with some frequencies, but not all. The FEM method can be seen to behave in its typical manner, i.e., over-estimating the results or approaching the exact values from above. The DFE results make sense because it is an intermediate method between the DSM and FEM and therefore represents a mixture of their behaviors.

The convergence rates for the DSM, DFE and FEM methods, for the three different tapers, are shown in Figures 61, Figures 62 and Figures 63, respectively. It can be seen that the DSM method has the best convergence rates for the single dimension tapers, with the DFE being the next most accurate and the FEM method requiring the most elements. The DFE method is seen to be more accurate in the double tapered example than the DSM method; However, the difference between the two methods is less than 2% for all cases. It can also be seen that, comparing to the FEM, the DSM and DFE methods, both maintain their superiority with regard to the determination of the higher frequencies, similar to the uniform cases.

Figures 61 : DSM, DFE and FEM Convergence Rates $Nb=0$ $Nh=1$; Linearly Tapered CUS Composite Box-Beam

Figures 62 : DSM, DFE and FEM Convergence Rates $Nb=1$ $Nh=0$; Linearly Tapered CUS Composite Box-Beam

Figures 63 : DSM, DFE and FEM Convergence Rates $Nb=1$ $Nh=1$; Linearly Tapered CUS Composite Box-Beam

The mode shapes for these tapers are interesting due to the way the stiffness terms vary along the length. The first natural mode for the 3 different tapers used is shown in Figures 64. As it can be seen, the taper does not affect significantly the mode shape. When the tapered first mode shape is compared to that of the uniform beam example #1 (Figure 28), it is seen that it generally follows the same pattern with same sign displacements for extension and torsion. It can also be seen that the exact shape of the mode is slightly different for different tapers and when compared to that of the uniform example. This can be explained by the fact that the stiffness values at each point along the beam are different for each of the examples, leading to a difference in displacement and rotation at each point.

Figures 64 : First Natural Mode; Linearly Tapered CUS Composite Box-Beam

The second mode (Figures 65) shows more dependence on taper than does the first one but, as it can be seen, the shape of the mode does not change significantly. It can be realized that all of these plots are similar in shape to that of the uniform example (section 4.3). It can be seen that, like the first frequency, the exact shape of the modes is slightly different due to the changes in stiffness along the length for each case.

Figures 65 : Second Natural Mode; Linearly Tapered CUS Composite Box-Beam

The third and fourth natural modes are shown in Appendix G and demonstrate the same trends seen in the first two frequencies.

The effect of ply-angle on the dynamic behavior of a tapered composite box-beam is shown in Figures 66. It can be seen that ply-angle changes have a similar effect on a tapered box-beam as they do on a uniform one, with not only the frequency values but also the associated modes changing significantly. Also, changes in ply-angle change the modes from torsion to extension domination with increasing angle up to 45° . The modes for this example also illustrate the reversal of the non-dominant deformation for angles which are symmetric about 45° , meaning same sign modes become opposite sign but still share the same frequency value.

Figures 66 : ET Coupled Frequency versus Ply-Angle; Linearly Tapered CUS Composite Box-Beam

The present tapering methodology could also be used for higher order tapers and tapers using dimensions of different order number, as long as a sufficiently dense mesh is used in order to closely approximate the actual geometry of the beam.

Chapter 5: Concluding Remarks

Of primary concern in this project is the Extension-Torsion (ET) coupled vibration analysis for various structural members. In order to evaluate the natural frequencies and modes of vibrations of these systems, three solution methods were derived. The first formulation, being the Dynamic Stiffness Matrix (DSM), makes use of algebraic inversion of the dynamic (frequency dependant) stiffness matrix and results in a nonlinear eigenvalue problem that in turn yields the exact solution to the differential equations of motion. Then, the Finite Element Method (FEM) was derived which exploits well-known established algorithms and is widely used in the engineering field to solve numerous types of differential equations. Based on the application of a Galerkin weighted residual method to the differential equations of motion for the system, together with linear approximations for the field variables, the FEM leads to a linear eigenvalue problem, and consequently an approximate solution. The final formulation developed in this project is the so-called Dynamic Finite Element (DFE) method. This method makes use of a Galerkin weighted residual method, similar to the FEM, while trigonometric interpolation functions are used to approximate the unknown variables. Illustrative examples of this report allowed for the validation, verification of the correctness and the practical applicability of all the three methods, based on a direct comparison of the results obtained.

In order to make use of the coupled vibration theory developed here, certain structural elements of interest with real engineering applications were identified. The first of these systems was a helical spring, exhibiting geometrically coupled extension-torsion (ET) behavior. The CL/CL and CL/FR boundary conditions were examined since both find practical application, with the solved frequency results showing excellent agreement with those of reference [10]. The spring natural mode shapes were also depicted in order to gain an appreciation of the coupling between the extension and rotation displacements. The spring natural modes, all torsion dominated, demonstrated the C^0 continuity requirement of these displacements. The DFE and FEM convergence rates, when compared to the exact DSM results, demonstrated the fact that the DFE is able to converge faster to the exact solution, particularly for higher natural frequencies. The convergence rates for both methods also demonstrated the fact that the boundary conditions

play a role in the convergence rates of the solution, with the CL/CL boundary condition requiring more elements to gain the same level of accuracy as the CL/FR solution.

The next system studied here was a braided wire rope/cable exhibiting ET coupling due to geometric effects, similar to helical springs. In the wire rope/cable example, the CL/CL boundary condition was first studied. The conventional CL/CL end conditions, in this case, were accompanied by a set of new CL/SW and SW/SW ones, unique to this type of system, resulting from the practical engineering applications of these ropes. The natural frequencies for all three boundary conditions were calculated and compared to those available in literature in order to verify the evaluated results. The mode shapes for all three boundary conditions are also included, and once again help to illustrate both the coupled nature of the problem as well as the C^0 continuity requirement. The effect that rope length had on the problem was also examined, with the frequency results showing excellent agreement to published results over a large range of lengths. Length was also found to have no impact on the accuracy of either the FEM or DFE solution, with both methods showing similar error rates for each frequency (using the same number of elements) regardless of overall system length. The effect of the CL/CL, CL/SW and SW/SW boundary conditions on the FEM and DFE convergence rates is also examined, with the CL/CL once again demonstrating the highest error, followed by the CL/SW and then finally the most accurate the SW/SW condition. The reason for this is due to the number of degrees-of-freedom (DOF) removed from the system when implementing each of these boundary conditions, with the CL and SW boundary conditions eliminating two and one DOF, respectively. Eliminating DOF functions to reduce the dimension of the eigenvalue system, resulting in the reduced accuracy.

The final systems examined in this project are thin-walled closed-section laminated composite (anisotropic) beams. Two particular geometries of interest were examined, a rectangular section, as well as a circular one. This allows one to gain a better understanding of the effects of geometry on the vibration behavior of this type of systems. Each of these cross-sections was examined using two very different lay-ups; this allows one to better understand the role of the material in determining the dynamic behavior of flexible composite beams. In the

first beam and tube examples, the frequency results are seen to compare well with those of literature. It is seen that the trends in the stiffness terms and frequency results of these two geometries is similar, since they make use of similar laminates. This shows the fact that the laminate composition controls the trends in the dynamic behavior of composite beams while the geometry effects only the frequency magnitude. A similar trend is seen when comparing the second box-beam and tube example, both making use of a similar laminate. The frequency results for these two examples are also seen to agree well with those in the literature for various ply-angles. The effect of ply-angle on convergence rates was also examined for all the examples with box and tube examples #1 showing almost steady convergence regardless of angle, and box-beam and tube #2 showing a steady increase in convergence with play-angle. The difference in convergence behavior is due to the difference in frequency behavior, with the first examples being cyclic and the second resulting in a steady decrease. The effect of the bending-shear coupling on the beam vibration behavior is also examined proving that the inclusion of shear tends to reduce the frequency values. The effect of the sign of the coupling stiffness term is also shown in the first beam and tube examples, demonstrating a reversal of sign of the non-dominate deformation mode with a change in coupling stiffness sign. This sign change was found to have no effect on the natural frequencies of the problem.

In order to expand the thin-walled closed-section anisotropic theory to more realistic geometries, a simple tapering method was developed and used in order to illustrate the effects of taper ratio on the frequency values and mode shapes for these types of beams. It was found that the general trends remain the same as those of a uniform beam made out of the same laminate, except that the mode shapes and frequency values change depending on the direction and the taper ratio. It was found that the DSM offers the fastest convergence to reference (exact) values obtained from a 100-element DSM model, with the DFE being second and the FEM being last.

This report was able to successfully derive and implement an ET coupled vibration solution on numerous types of problems, demonstrating the general applicability of three DSM, FEM and DFE methods to these systems. The results were verified at all levels, when published results permitted, allowing one to see the differences and similarities between the available models. This allows one to choose a model of sufficient accuracy and ease of implementation depending on the situation, making the designer better able to understand and make use of the geometric/material displacement couplings present in the structural members studied in this project. The proposed DFE could also be extended to the vibration analysis of more complex structural elements.

Chapter 6: Future Work

In order to allow the current theory to be applied to more "real" world applications it may need to be expanded in order to include some of the effects which were neglected here. Since CUS composite beams find their primary application in helicopter rotor and wind turbine blades, the effect of rotation may need to be included. The effect of wall thickness may also need to be included, particularly when determining the dynamic behavior of a helicopter tail rotor, which due to its small cord can behave in a thick-walled manner. The inclusion of rotation and wall thickness was examined in [26-28], with the results reviewed below. Since the aeroelastic behavior of helicopter rotor blades is also of great importance, the current theory may also need to be expanded to include shear coupling in the bending model. The inclusion of this effect would allow for a more accurate determination of the natural mode shapes for the bending frequencies, which are used in most aeroelastic analyses.

The effect of in-plane rotation speed on the stiffness and dynamic behavior of a composite beam is examined in references [26-28] with the natural frequencies reported in Table 21. It can be seen that, as expected, the inclusion of rotation results in the increasing of the fundamental frequency of vibration. It is also noted in reference [28] that rotation up to 1000 rpm has no effect on the mode shape for this type of beam. The increase in natural frequency due to rotation can be significant (14.5%). Therefore the fact that CUS configuration composite beams find their primary application in rotating structures (rotor/wind turbine blades) may warrant the inclusion of its effects in the model if rotational speed is large.

Freq	Method		
	[28] Shear	[28] No Shear	Exp [28]
1 VB No Rotation	28.0	39.0	29.5
1 VB Rotation = 1000rpm	32.5	44.0	34.0

Table 21 : Effect of Rotation on Frequency

The effect of wall thickness on the dynamic behavior of these types of composite beams is also reviewed in references [26-28]. In the thin-walled assumption, which is used in this report, the shell wall is treated as a membrane. This allows for easy determination of the shell energy functional, as well as simplifying its form. Dynamic results for numerous beams with different lay-ups are shown in reference [26] using both a thin and thick-wall model in order to allow for a comparison. It can be seen that the effect seems to be dominated by the slenderness ratio of the beam (thickness/width), with the thin-walled approximation being within 10% of the thick-walled value for slenderness ratios less than about 0.4. References [26-28] also examined the effect of this approximation on higher order frequencies, and it was shown that the percentage error associated with each frequency due to this approximation remains constant up to the third vibration mode. This means that when one is examining a beam for which the slenderness ratio is greater than 0.4, one can no longer make use of a thin-walled assumption and must exploit a more advanced model for the bending behavior of the shell wall due to its increasing importance in the overall behavior of the system [27].

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Appendix A : Terms C_{ij}

The terms C_{ij} given in (137) in general integral form are given in references [4,5] as:

$$C_{11} = \oint \left(A - \frac{B^2}{C} \right) ds + \frac{\left[\oint (B/C) ds \right]^2}{\oint (1/C) ds} \quad (\text{A1})$$

$$C_{12} = \frac{\oint (B/C) ds}{\oint (1/C) ds} A_e \quad (\text{A2})$$

$$C_{13} = -\oint \left(A - \frac{B^2}{C} \right) z ds - \frac{\oint (B/C) ds \oint (B/C) z ds}{\oint (1/C) ds} \quad (\text{A3})$$

$$C_{14} = -\oint \left(A - \frac{B^2}{C} \right) y ds - \frac{\oint (B/C) ds \oint (B/C) y ds}{\oint (1/C) ds} \quad (\text{A4})$$

$$C_{22} = \frac{1}{\oint (1/C) ds} A_e^2 \quad (\text{A5})$$

$$C_{23} = -\frac{\oint (B/C) z ds}{\oint (1/C) ds} A_e \quad (\text{A6})$$

$$C_{24} = -\frac{\oint (B/C) y ds}{\oint (1/C) ds} A_e \quad (\text{A7})$$

$$C_{33} = \oint \left(A - \frac{B^2}{C} \right) z^2 ds + \frac{\left[\oint (B/C) z ds \right]^2}{\oint (1/C) ds} \quad (\text{A8})$$

$$C_{34} = -\oint \left(A - \frac{B^2}{C} \right) y z ds - \frac{\oint (B/C) y ds \oint (B/C) z ds}{\oint (1/C) ds} \quad (\text{A9})$$

$$C_{44} = \oint \left(A - \frac{B^2}{C} \right) y^2 ds + \frac{\left[\oint (B/C) y ds \right]^2}{\oint (1/C) ds} \quad (\text{A10})$$

Appendix B : Wire Rope Example

The mode shapes for the third to the fifth CL/CL ET coupled natural frequencies are shown in Figures 67.

Figures 67 : Natural Modes; CL/CL Wire Rope Example

Figures 68 : Natural Modes; CL/SW Wire Rope Example

The mode shapes for the second to the fifth frequency for the CL/SW boundary condition are shown in Figures 68.

Figures 69 : Natural Modes; SW-SW Wire Rope Example

The mode plots for the SW/SW boundary condition are shown in Figures 69 for the third to the sixth.

Figures 70 : FEM Convergence Rates; Wire Rope Example

The FEM convergence rates for the CL/CL, CL/SW, and SW/SW boundary conditions are shown in Figures 70.

Appendix C : Vertical and Horizontal Uncoupled Bending Vibration

In order to evaluate the vertical and horizontal bending frequencies for the anisotropic members of interest a simple Euler-Bernoulli beam formulation is used. This is valid within the context of these applications since as it was seen in references [4,5], they exhibit uncoupled vertical and horizontal bending displacements. The derivation is carried out with reference to Figure 71 which shows the forces and moments acting in the y-direction on the cross-section. This same model can be used to find the natural frequencies in the z-direction as long as the correct stiffness value is used. For the anisotropic circular tube, the direction of the derivation does not matter since the tube exhibits equal bending stiffness in all directions because of the shape of the cross-section.

Figure 71 : Forces And Moments For Bending [17]

Summing the forces in the y-direction gives:

$$dV - p(x)dx = 0 \quad (\text{B1})$$

Summing moments about any point on the right face of the element gives,

$$dM - Vdx - \frac{1}{2} p(x)(dx)^2 = 0 \quad (\text{B2})$$

rearranging (B1) and substituting into (B2) gives the important results:

$$\frac{dV}{dx} = p(x) \quad \text{and} \quad \frac{dM}{dx} = V$$

Using the classical Euler-Bernoulli assumption that transverse shear plays no role in the curvature of the member; the flexure equation to relate the bending moment to the curvature can be used as,

$$M = C_{33} \frac{d^2 y}{dx^2} \quad (\text{B3})$$

substituting the relations above one is able to right the flexure equation as:

$$\frac{d^2}{dx^2} (C_{33} \frac{d^2 y}{dx^2}) = p(x) \quad (\text{B4})$$

Since for free vibration there are no applied loads, the load per unit length on the beam has to be due to its own interial force and therefore can be written as:

$$\frac{d^2}{dx^2} (C_{33} \frac{d^2 y}{dx^2}) - m\omega^2 y = 0 \quad (\text{B5})$$

Assuming that C_{33} is constant along the length of the beam one can combine the derivative operators in order to obtain a final fourth order differential equation of the form:

$$\frac{d^4 y}{dx^4} - \beta^4 y = 0 \quad \text{with} \quad \beta^4 = m \frac{\omega^2}{C_{33}} \quad (\text{B6})$$

the general solution to this equation is given in many reference [17] and takes the form:

$$y = A \cosh(\beta x) + B \sinh(\beta x) + C \cos(\beta x) + D \sin(\beta x) \quad (\text{B7})$$

Since the boundary condition of interest for the anisotropic beam is cantilevered (clamped-free). Therefore, one can establish the values of constants A-D giving the transcendental equation [17,4],

$$\cos(\beta_n L) \cosh(\beta_n L) = -1 \quad (\text{B8})$$

for which the first four roots are given in reference [4] as:

$$\beta_1 L = 1.87510 \quad \beta_2 L = 4.69409 \quad \beta_3 L = 7.85476 \quad \beta_4 L = 10.99554$$

These values can be used to determine the natural frequencies of vibration using:

$$\omega_n = (\beta_n L)^2 \sqrt{\frac{C_{33}}{mL^4}} \quad (\text{B9})$$

It should be noted that since this is an explicit, closed form solution assuming constant beam properties, it will not be used in the linearly tapered beam example and therefore this example will contain no horizontal and/or vertical bending data.

Appendix D : Uniform CUS Composite Box-Beam Example #2

The convergence rates for the FEM method for angles (η) of 20, 40, 60 and 80 degrees can be seen in Figures 72.

Figures 72 : FEM Convergence Rates; Uniform CUS Composite Box-Beam Example # 2

The mode shapes for frequency ETS 1 for an angle η of 60° and 80° are shown in Figures 73.

Figures 73 : Natural Modes for ETS 1; Uniform CUS Composite Box-Beam Example # 2

Figures 74 : Natural Modes for ETS 2; Uniform CUS Composite Box-Beam Example #2

The mode shapes for ETS 2 for an angle (η) of 20° , 40° , 60° , and 80° respectively are shown in Figures 74.

The mode shapes for ETS 3 for an angle (η) of 20° , 40° , 60° , and 80° respectively are shown in Figures 75.

Figures 75 : Natural Modes for ETS 3; Uniform CUS Composite Box-Beam Example # 2

Mode shapes for frequency ETO 2 for angles (η) of 20° , 40° , 60° and 80° are shown in Figures 76.

Figures 76 : Natural Modes for ETO 2; Uniform CUS Composite Box-Beam Example # 2

The mode shapes for frequency ETO 3 for angles (η) of 20° , 40° , 60° and 80° are shown in Figures 77.

Figures 77 : Natural Modes for ETO 3; Uniform CUS Composite Box-Beam Example # 2

Appendix E : Uniform CUS Composite Circular Tube Example #1

The mode shapes for the second frequency for angles (η) of 20° , 60° and 70° are shown in Figures 78.

Figures 78 : Second Natural Mode; Uniform CUS Composite Circular Tube Example # 1

The mode shapes for the third coupled ET frequency for angles (η) of 20° , 60° , and 70° are shown in Figures 79.

Figures 79 : Third Natural Mode; Uniform CUS Composite Circular Tube Example # 1

The mode shapes for the first three extension-torsion coupled frequencies for an angle (η) of 40° are shown in Figures 80.

Figures 80 : Natural Modes for η of 40° ; Uniform CUS Composite Circular Tube Example # 1

The convergence rates for the FEM method for angles (η) of 20° , 60° , and 70° respectively are shown in Figures 81.

Figures 81 : FEM Convergence Rates; Uniform CUS Composite Circular Tube Example # 1

Appendix F : Uniform CUS Composite Circular Tube Example #2

The convergence rates for the DFE method for angles (η) of 20° , and 60° are shown in Figures 82.

Figures 82 : DFE Convergence Rates; Uniform CUS Composite Circular Tube Example # 2

The convergence rates for the FEM method for angles (η) of 20° and 60° are shown in Figures 83.

Figures 83 : FEM Convergence Rates; Uniform CUS Composite Circular Tube Example # 2

The mode shapes for the third natural frequency are shown for angles (η) of 20° and 60° in Figures 84.

Figures 84 : Third Natural Modes; Uniform Circular Tube Example # 2

Appendix G : Linearly Tapered CUS Composite Box-Beam Example

The mode shapes for the third coupled frequency for all three tapers are shown in Figures 85.

Figures 85 : Third Natural Mode; Linearly Tapered CUS Composite Box-Beam

Figures 86 : Fourth Natural Mode; Linearly Tapered CUS Composite Box-Beam

The mode shapes for the Fourth coupled frequency for all three tapers are shown in Figures 86.

Appendix H : Symmetric Laminate Circular Tube Example

In this example, a lay-up of $[45^\circ / -45^\circ / -45^\circ / 45^\circ]$ graphite epoxy with the same material properties as those used in Box-Beam Example # 1 (see Table 7) was used to model a circular tube with radius (r) of 2 in, a length (L) of 100 in and a total thickness of .022 in. The stiffness terms for this tube are shown in Table 22, and as can be seen the coupling stiffness term ' B ' is zero for this type of laminate.

Axial Stiffness	Coupling Stiffness	Torsional Stiffness
1635700 (Lb)	0 (Lbin)	48484000 (Lbin2)

Table 22: Stiffness Terms; Symmetric Laminate Circular Tube Example

The frequency results for this example are included in Table 23 and show good agreement with reference [3]. The DSM is unable to obtain results for this type of laminate, due to the fact that the stiffness matrix is undefined if the material coupling term is set to zero (refer to section 2.2 for equations of DSM). These results are instead calculated with the DFE using 5 elements, and are verified using the exact frequency equations found in references [10,14].

Frequency	Present	Ref [3]	% Diff
1E	351	365	-4.3
2E	1052.9	1100	-4.3
1T	477.7	473	1.0
2T	1433.1	1420	1.0

Table 23: Frequency Results (Hz); Symmetric Laminate Circular Tube Example

Appendix I : Cross-Ply Laminate Circular Tube Example

The example included in this appendix makes use of a $[0^\circ/0^\circ/0^\circ/90^\circ]$ laminate, made of the graphite epoxy used in Box-Beam Example # 1 (see Table 7). The radius (' r ') of the tube was taken to be 2 in and the length (' L ') 100 in, the total thickness of the laminate used was .022 in. The stiffness terms calculated for this cross-section are given in Table 24, and as can be seen for this type of laminate there is no material coupling between extension and torsion (i.e. Coupling Stiffness, C_{12} is zero).

Axial Stiffness	Coupling Stiffness	Torsional Stiffness
9121700 (Lb)	0 (Lbin)	8227500(Lbin2)

Table 24: Stiffness Terms; Cross-Ply Circular Tube Example

The calculated uncoupled frequency values are shown in Table 25, and as can be seen are excellent agreement with those of reference [3]. The DSM once again was unable to calculate these uncoupled frequency results since the stiffness matrix for this method is undefined when there is no extension-torsion coupling in the problem. The DFE method using 5 elements is instead used to calculate the results comparative results shown in the table.

Frequency	Present	REF [3]	% Diff
1E	828.812	825	0.5
1T	196.78	200	-1.6
2T	590.3538	600	-1.6

Table 25: Frequency Results (Hz); Cross-Ply Circular Tube Example

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