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FREE VIBRATION ANALYSIS OF SANDWICH BEAMS USING DYNAMIC
FINITE ELEMENT (DFE) AND FEM FORMULATIONS

By

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Abstract

Free vibration analysis of sandwich beams using dynamic finite element (DFE) and FEM formulations

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The Dynamic Finite Element (DFE) theory is employed to calculate the natural frequencies and mode shapes of three-layered sandwich beams. Three formulations are developed and tested using a number of different numerical cases. The first two theories pertain to the analysis of straight beams while the third one is developed for the analysis of curved beams. The results from all three derivations agree well with published data that uses Classical Finite Element (FEM) theory and the Dynamic Stiffness Method (DSM) in the analysis of the free vibration behaviour of sandwich beams. For all test cases, the DFE results agree well with the published results as well as the FEM developed as an additional benchmark for the DFE. The results of the research are very encouraging and demonstrate that DFE is a valuable structural analysis tool that can be used in conjunction with FEM and DSM.

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Chapter 1: Introduction

1.1 Introduction to Sandwich Construction

Applications of sandwich construction and composites continue to expand. They are used in a number of industries such as the aerospace, automotive, marine and even sports equipment. Sandwich construction offers designers high strength to weight ratios which is of extremely high importance in aerospace applications as well as good buckling resistance, formability to complex shapes and easy reparability. Due to the many advantages sandwich construction offers over traditional aerospace materials, the analysis of sandwich beams has been investigated by a large number of authors for more than four decades. Sandwich beam construction can also offer energy and vibration damping when a visco-elastic core layer is used. However, such non-conservative systems are not the focus of the current research.

The most common sandwich structure is composed of two thin face sheets with a thicker lightweight low-stiffness core. Common materials used for the face layers are metals and composite while the core is often made of foam or a honeycomb structure made of metal [19]. It is very important that the core, although weaker than the face layers, be strong enough to resist being crushed by the face layers. The current trend in the aerospace industry of using composites and sandwich material to lighten aircraft in an attempt to make them more fuel efficient has caused a recent resurgence into research to develop a reliable method of predicting the vibration behaviour of sandwich structures.

1.2 Literature Survey

In the late 1960s, pioneering work in the field of viscously damped sandwich beams was performed by Di Taranto [20] and Mead and Marcus [38] using classical methods to solve the governing differential equations of motion leading to the natural frequencies and mode shapes of the system. In the early 1970s, Ahmed [1,2] applied the finite element method (FEM) to a curved sandwich beam with an elastic core and tested several different formulations in order to compare their performances in determining the natural frequencies and mode shapes for several different beam configurations. Interest in the vibration behaviour of sandwich beams has seen resurgence in the past decade with the availability of more powerful computing systems. This has allowed for more complex finite element models to be developed. Sainsbury and Zhang [40], Fasana and Marchesiello [21] and Baber, Maddox and Orozco [3] are just some of the researchers who have employed FEM in the analysis of visco-elastically damped sandwich

beams. The additional computing power has also led to the development of the Dynamic Stiffness Method (DSM) which employs symbolic computation to combine the coupled governing differential equations of motion into a single higher order ordinary differential equation for which the solutions are the natural frequencies of the system. Banerjee [4-14] and Howson and Zare [33] have published many papers on DSM showing its successful application to several sandwich beam and composite formulations and beam configurations with a number of papers focusing on sandwich beams with an elastic core.

It is important to note that in all the sandwich element models listed above the beam is assumed to exhibit coupled bending-axial motion only, and there is no twisting or out of plane motion. Also, the assumption is made that the layers are perfectly and rigidly joined together and the interaction of the different materials at the interfaces is ignored. Although it is known that bonding such vastly different materials will cause stress at the interfaces, the study of their interactions and behaviour at the bonding sites is another research topic altogether.

Another important factor that has a large effect on the results of the sandwich beam analysis is the assumed vibration behaviour of the layers chosen. The simplest sandwich beam model exploits the Euler-Bernoulli bending theory for the face layers and allows the core to deform only in shear. This assumption was used in several different DSM and FEM studies such as those by Banerjee [8], Ahmed [1, 2], Mead and Markus [38], Fasana and Marchesiello [21] and Baber *et al.* [3]. In more recent papers, Banerjee derives two new DSM models which increase the complexity of the displacement fields [9, 10]. The first and simpler of the two [10] applies Timoshenko theory to the core which adds shear deformation and rotary inertia to Euler-Bernoulli beam theory; the face plates, however, are assumed to behave like Rayleigh beams which adds shear deformation to Euler-Bernoulli beam theory. The most complex sandwich beam model, developed and used by Banerjee [9], assumes all three layers to be Timoshenko beams. However, increasing the complexity of the model also significantly increases the amount of numerical and symbolic computation in order to achieve the final formulation. All these factors were taken into account when determining the course of action and the end goals of the current research.

1.3 Overview of Finite Element Method Theory (FEM)

Classical FEM theory has a proven track record and is the most commonly used method for structural analysis. FEM is a systematic approach to formulate element stiffness and mass matrices that is easily adaptable to a wide range of systems. The conventional FEM uses polynomial shape functions to approximate the displacement fields which lead to linear eigenvalue problem. Solving the resulting

eigenvalue yields the natural frequencies of the system. All structural analysis software commercially available are based on FEM theory and recently have begun to offer multi-layered elements that can be used to model layered composite materials and sandwich construction (e.g., ANSYS® and MSC NASTRAN/PATRAN®). The main advantages of using FEM theory are [18]:

- There is no limitation to the geometry meaning that the model can be of any shape or size.
- Boundary conditions and loading have no restrictions meaning that forces and displacements can be applied to any point in the model.
- Material properties are not restricted meaning that the material can be anisotropic and that adjacent elements can be made of different materials.
- Components within the model can have different behaviours and different mathematical descriptions can be combined meaning that an element with beam behaviour properties and another element with plate behaviour properties can be used within the same model.
- The accuracy of the model can be easily increased by increasing the number of elements used for the model.

The versatility of the FEM theory, however, does have some downsides such as it is a numerical approximation and the accuracy of its results depends on the number of elements used in the model. This weakness is the most evident when FEM is used to predict system behaviour at higher frequencies because a large number of elements are needed in order to achieve more accurate results.

1.4 Overview of Dynamic Stiffness Method Theory (DSM)

The dynamic stiffness method (DSM) provides an analytical solution to the free vibration problem. This is achieved by combining the coupled governing differential equations of motion of the system into one ordinary differential equation and the most general closed form solution is sought. Applying the boundary conditions forms the dynamic stiffness matrix. The DSM formulation results in a non-linear eigenvalue problem and the Wittrick-Williams algorithm [43] can then be employed as a solution technique. DSM is capable of providing exact results for any of the natural frequencies of the beam with the use of a single element since the continuous element has an infinite number of degrees of freedom. One major disadvantage of DSM methods is that the closed form solution of the governing differential equation must be known, which only occurs for a limited number of special cases such as systems with constant geometric and material properties and only a certain number of boundary conditions.

1.5 Introduction to Dynamic Finite Element (DFE) Theory

Dynamic Finite Element (DFE) theory is a hybrid method that blends the well-established classical FEM theory with the recently developed DSM theory in order to achieve a model that possesses all the best traits of the FEM and DSM while trying to minimize the effects of their limitations; the adaptability of classical FEM with the accuracy of DSM. DFE defines the approximation space using frequency dependent trigonometric basis functions to obtain the appropriate interpolation functions which assume constant parameters over the length of the element. DFE theory was first developed by Hashemi [22] and has since been exploited by Hashemi and his coauthors [23-31] in the free vibration analysis of homogeneous and laminated composite beam configurations exhibiting diverse geometric and material couplings. In the case of a three-layered sandwich beam, the solutions to the uncoupled parts of the equations of motion are used as the basis functions of the approximation space, leading to the Dynamic Trigonometric Shape Functions (DTSFs) which are, in turn, used to express the field variables. The DTSFs are then introduced into the integral form of the equations of motion to derive the element dynamic stiffness matrix. DFE follows a very similar procedure as FEM by first applying the weighted residual method to the differential equations of motion. Next, the element stiffness matrices are derived by discretizing the weak integral form of the equations of motion. For FEM, the polynomial shape functions are applied and the integrations are carried out and evaluated in order to obtain the element matrices. At this point, DFE applies an additional set of integration by parts to the discretized (element) integral equations of motion, substitutes the DTSFs to the newly formed equations, and then carries out the integrations to form the element stiffness matrices. The assembly of the global stiffness matrix from the element matrices follows the same procedure for both FEM and DFE. In the case of FEM, the result is a linear eigenvalue problem (in terms of ω^2); while the DFE methodology results in a non-linear eigenvalue problem. The general DFE formulation procedure is illustrated in Figure 1-1, below. It is important to note that DFE is not limited to uniform and stepped geometry and can be readily extended to applications with variable material and geometric parameters (see, for example, [24, 25, 27, 28, 30]).

Figure 1-1: Flowchart for general DFE formulation procedure.

1.6 Scope of Research

The end goal of this research is to derive and test several DFE formulations for the free vibration analysis of sandwich beams and validate them against FEM and DSM and show that DFE is another viable tool that can be used in structural analysis. The first model is based on the most simple sandwich beam configuration which is a straight symmetric beam where the face layers are assumed to follow Euler-Bernoulli theory and core is allowed to deform in shear only. Although a very simple model, the first test case is meant as a familiarization to the DFE methodology and application. The second model is a more general straight sandwich beam model that can be used to model a wide range of sandwich beams. The model developed assumes that the core layer of the beam behaves according to Timoshenko theory while the faces adhere to Rayleigh beam theory. This derivation can analyze beams where all three layers possess widely different material and geometric properties. The final model moves away from the straight beam case and analyzes a symmetric three-layered curved sandwich beam. The ability to analyze a curved beam is important since in many aerospace applications structural members are curved meaning

that the previous straight sandwich beam formulations are not directly applicable to them. The DFE formulation for a symmetric sandwich beam with a curvature assumes the face layers behave according to Euler-Bernoulli theory and the core deforms in shear only. The formulations and test cases presented are intended to show that DFE theory can be successfully applied to wide gamut of different applications.

Chapter 2: Euler-Bernoulli/Shear Formulation

2.1 Introduction

The simplest representation of a sandwich beam models the stiff face layers as Euler-Bernoulli beam and allows the core only to deform in shear. This model has been previously studied by different researchers [8, 38] and provides an important basis for further development of sandwich beam models.

2.2 Theory

Figure 2-1, below, shows the co-ordinate system and notation used for a symmetric sandwich beam of length L , width b and with three distinct layers. The thicknesses of layer 1 and layer 3 (the top and bottom layer, respectively) are h_f and the thickness of layer 2 (the core) is h_c .

Figure 2-1: The co-ordinate system and notation for a symmetric three-layered sandwich beam.

In the co-ordinate system shown, the origin is set at the left-hand side of the beam with the x -axis coincident with the axis of the beam. The y -axis of the beam is set upward perpendicular to the x -axis, and the z -axis comes out of page, following the right-hand rule. The beam only deflects in the x - y plane. The top and bottom faces, in this case, are modelled as Euler-Bernoulli beams having only axial and bending rigidity and the core is assumed to have only shear rigidity (e.g., the stresses in the core in the longitudinal direction are zero). All three layers are considered to share the same flexural displacement w . It should be noted that w' is the bending slope of the beam which is defined by the first derivative of the flexural displacement, w , with respect to x . The centreline displacements of the layer 1 and layer 3 in the axial direction are u_1 and u_3 , respectively. It should be observed that u_2 (the centreline longitudinal

displacement of layer 2) is no longer used as it can be derived using u_1 and u_3 . Also, for this model, $u_1 = -u_3 = u(x, t)$.

For the beam model developed, there were several assumptions made [8]:

- All displacements and strains are small so that the theory of linear elasticity applies.
- The faces and core of the sandwich beam are made of homogeneous and isotropic materials and the variation of strain within them is linear.
- Transverse normal strain in the faces and core are negligible.
- Transverse shear within the face layers is negligible.
- There is no slippage or delamination between the layers during deformation.

Using the model described above, Banerjee [8] calculates the strain and kinetic energy, forms the Lagrangian and applies Hamilton's principle to obtain the differential equations of motion and corresponding boundary conditions. Appendix A3 provides a more in-depth explanation to the development of equations of motion for the symmetric straight sandwich beam. For free vibration analysis, the assumption of simple harmonic motion produces the following differential equations of motion for the three-layered sandwich beam [8]:

$$(E_1 A_1)u'' - (2k_2 A_2 G_2 / h_c^2)u - (k_2 A_2 G_2 / h_c)(1 + h_f / h_c)w' + m\omega^2 u = 0 \quad (2-1)$$

$$(E_1 I_1)w'''' - (k_2 A_2 G_2 / 2)(1 + h_f / h_c)^2 w'' - (k_2 A_2 G_2 / h_c)(1 + h_f / h_c)u' - m\omega^2 w = 0 \quad (2-2)$$

In the equations above, $u(x)$ and $w(x)$ are the amplitudes of the sinusoidally varying axial and vertical displacements, respectively, and primes denote differentiation with respect to x . $E_1 A_1$ and $E_1 I_1$ are, respectively, the extensional and bending rigidity of the face plates, G_2 is the shear modulus of the core and k_2 is the shape factor (also called the shear correlation factor) for the core cross-section, and m is the mass per unit length ($m = \rho A$).

The appropriate boundary conditions are imposed at $x = 0$ and $x = L$. For example: Clamped at $x = 0$; $u = w = w' = 0$, and Free at $x = L$; $F = M = S = 0$, where the resultant axial force (F), shear force (S) and bending moment (M) are the load or natural boundary conditions defined as:

$$F(x) = -2(E_1 A_1)u' \quad (2-3)$$

$$M(x) = -2(E_1 I_1)w'' \quad (2-4)$$

$$S(x) = 2(E_1 I_1)w''' - (k_2 A_2 G_2)(1 + h_f/h_c)\{(1 + h_f/h_c)w' + (2/h_c)u\} \quad (2-5)$$

A Galerkin integral formulation, based on the weighted-residual method, is used to derive the three-layered sandwich beam model. The method involves the use of integration by parts on different elements of the governing differential equations and then the discretization of the beam length into a number of two-node beam elements (Figure 2-2, below). The discretized integral form of the governing equations is then written as:

$$W_u^k = \int_0^l \delta u'(E_1 A_1)u'dx - \int_0^l \delta u(m\omega^2)udx + \int_0^l \delta u(2k_2 A_2 G_2/h_c^2)udx \\ + \int_0^l \delta u(k_2 A_2 G_2/h_c)(1 + h_f/h_c)w'dx \quad (2-6)$$

$$W_w^k = \int_0^l \delta w''(E_1 I_1)w''dx + \int_0^l \delta w'(k_2 A_2 G_2/2)(1 + h_f/h_c)^2 w'dx - \int_0^l \delta w(m\omega^2)w dx \\ + \int_0^l \delta w'(k_2 A_2 G_2/h_c)(1 + h_f/h_c)udx \quad (2-7)$$

The above equations are also known as the *element* Galerkin-type weak form associated to the discretized equations (2-1) and (2-2) and also satisfy the principle of virtual work written as:

$$W = W_{INT} - W_{EXT} = (W_u + W_w) - W_{EXT} = 0 \quad (2-8)$$

where for free vibration analysis, $W_{EXT} = 0$, and

$$W_{INT} = \sum_{k=1}^{NET} W^k \quad \text{where } W^k = W_u^k + W_w^k \quad (2-9)$$

In the equations above, δu and δw are test (weighting) functions, defined in the same approximation space as u and w , respectively. Each element is defined by nodes j and $j+1$ with corresponding co-ordinates ($l = x_{j+1} - x_j$). The admissibility condition for finite element approximation is controlled by the undiscretized forms of equations (2-6) and (2-7).

Figure 2-2: Domain discretized by N number of 2-noded elements.

2.3 Finite Element Method (FEM) Derivation

The classical FEM model is derived by the use of linear and cubic Hermite type polynomial approximations for the axial and bending displacements, respectively.

$$u(x) = \langle N(x)_u \rangle \{u_j \ u_{j+1}\} = N_{1u}(x)u_j + N_{2u}(x)u_{j+1} \quad (2-10)$$

$$\begin{aligned} w(x) &= \langle N(x)_w \rangle \{w_1 \ w'_1 \ w_2 \ w'_2\} \\ &= N_{1w}(x)w_j + N_{2w}(x)w'_j + N_{3w}(x)w_{j+1} + N_{4w}(x)w'_{j+1} \end{aligned} \quad (2-11)$$

In the equations above, u_j , u_{j+1} , w_j and w_{j+1} are the nodal values at nodes j and $j+1$ corresponding to the axial and flexural displacements, and w'_j and w'_{j+1} stand for the nodal bending slopes. Also, $\langle N(x)_u \rangle$ and $\langle N(x)_w \rangle$ are the shape functions for u and w , respectively, which were derived using the polynomial shape functions described above. The same approximations are used for δu and δw , respectively. The FEM formulation is achieved when equations (2-10) and (2-11) are applied to simplify equations (2-6) and (2-7) and the necessary integrations are applied. This leads to a classical linear (in ω^2) eigenvalue problem as a function of constant mass and stiffness matrices, which can be solved using solvers available in programs such as Matlab[®]. It should be noted that for the present research, the FEM formulation was programmed in Matlab[®]. A flowchart illustrating the process employed in the Matlab[®] programming is included in Appendix A1. All FEM programs in later formulations also follow a very similar process.

2.4 Dynamic Finite Element (DFE) Formulation

In order to obtain the DFE formulation, a set of additional integration by parts must be applied to the element equations (2-6) and (2-7) leading to:

$$\begin{aligned}
W_u^k = & \underbrace{- \int_0^l ((\delta u' E_1 A_1)' + \delta u m \omega^2) u dx}_{(*)} + \underbrace{\int_0^l \delta u (2k_2 A_2 G_2 / h_c^2) u dx + [\delta u' (E_1 A_1) u]_0^l}_{[k]_u^k \text{ Uncoupled}} \\
& + \underbrace{\int_0^l \delta u (k_2 A_2 G_2 / h_c) (1 + h_f / h_c) w' dx}_{[K_{uw}]_{2 \times 4} \text{ Coupling}}
\end{aligned} \tag{2-12}$$

$$\begin{aligned}
W_w^k = & \underbrace{\int_0^l \left(\delta w'''' (E_1 I_1) - \delta w'' (k_2 A_2 G_2 / 2) (1 + h_f / h_c)^2 - \delta w (m \omega^2) \right) w dx}_{(**)} \\
& + \underbrace{[\delta w'' (E_1 I_1) w']_0^l - [\delta w''' (E_1 I_1) w]_0^l + [\delta w' (k_2 A_2 G_2 / 2) (1 + h_f / h_c)^2 w]_0^l}_{[k]_w^k \text{ Uncoupled}} \\
& + \underbrace{\int_0^l \delta w' (k_2 A_2 G_2 / h_c) (1 + h_f / h_c) u dx}_{[K_{wu}]_{4 \times 2} \text{ Coupling}}
\end{aligned} \tag{2-13}$$

Equations (2 – 12) and (2 – 13) are simply another way of evaluating equations (2 – 6) and (2 – 7) at the element level. The non-nodal approximations below, were made for δu , u , δw and w so that the integral terms (*) and (**) in the above equations (2 – 12) and (2 – 13) become zero. (*) has a second order differential characteristic equation in the form of $A_1 D^2 + B_1 \omega^2 = 0$, where A_1 and B_1 are constants and D^2 is the second partial derivative with respect to x . Secondly, (**) has a fourth order differential characteristic equation in the form of $A_2 D^4 - B_2 D^2 - C_2 \omega^2 = 0$, where A_2 , B_2 and C_2 are constants and D^4 and D^2 are the fourth and second partial derivatives with respect to x , respectively. Solving (*) and (**) provide the uncoupled solutions to (2 – 12) and (2 – 13) which are then used as the basis functions of approximation space in order to determine the Dynamic Trigonometric Shape Functions (DTSFs).

It should be noted that in equation (2 – 12), the second integral terms is not included in (*). This term represents the effect of the shear from the core in the face layers. If the additional integral term is included in (*), the prescribed trigonometric shape functions for the δu and u , equation (2 – 16) below, cannot be used because it is no longer a solution to (*). In that case, the solution moves from being in terms of trigonometric functions and instead is written purely in terms of hyperbolic functions. However, investigation showed that due to the non-oscillatory nature of hyperbolic functions, they alone are not suitable shape functions for the free vibration problem. In order to relieve this problem, the second integral term is excluded from (*) and is included in the derivation of the DFE matrix. Therefore, the original basis functions based on trigonometric functions are used, and the non-nodal approximations are:

$$\delta u = \langle P(x) \rangle_u \{\delta a\}; \quad u = \langle P(x) \rangle_u \{a\} \quad (2-14)$$

$$\delta w = \langle P(x) \rangle_w \{\delta b\}; \quad w = \langle P(x) \rangle_w \{b\} \quad (2-15)$$

where $\{a\}$ and $\{b\}$ are the generalized coordinates for u and w , respectively

$$\langle P(x) \rangle_u = \left\langle \cos(\gamma x) \quad \frac{\sin(\gamma x)}{\gamma} \right\rangle \quad (2-16)$$

$$\langle P(x) \rangle_w = \left\langle \cos(\alpha x) \quad \frac{\sin(\alpha x)}{\alpha} \quad \frac{\cosh(\beta x) - \cos(\alpha x)}{\alpha^2 + \beta^2} \quad \frac{\sinh(\beta x) - \sin(\alpha x)}{\alpha^3 + \beta^3} \right\rangle \quad (2-17)$$

In equations (2-16) and (2-17), α , β and γ are calculated based on the characteristic equations (*) and (**) in expressions (2-12) and (2-13) being reduced to zero. The equations for α , β and γ for this specific DFE formulation are shown in (2-18) and (2-19) below.

$$\gamma = \sqrt{\frac{m\omega^2 - (2k_2 A_2 G_2 / h_c^2)}{E_1 A_1}} \quad (2-18)$$

$$\alpha, \beta = \sqrt{\frac{\left\{ (k_2 A_2 G_2 / 2) (1 + h_f / h_c)^2 \right\} \pm \sqrt{\left\{ (k_2 A_2 G_2 / 2) (1 + h_f / h_c)^2 \right\}^2 - 4(E_1 I_1) m \omega^2}}{2(E_1 I_1)}} \quad (2-19)$$

The nodal approximations for the element variables $u(x)$ and $w(x)$ are then rewritten as:

$$u(x) = \langle P(x) \rangle_u [P_n]_u^{-1} \{u_n\}_u = \langle N(x) \rangle_u \{u_1 \ u_2\} \quad (2-20)$$

$$w(x) = \langle P(x) \rangle_w [P_n]_w^{-1} \{u_n\}_w = \langle N(x) \rangle_w \{w_1 \ w'_1 \ w_2 \ w'_2\} \quad (2-21)$$

Where, $\langle N(x) \rangle_u$ and $\langle N(x) \rangle_w$ are the dynamic (frequency-dependent) shape functions of the approximation space. Similar expressions are also written for the test (weighting) functions, $\delta u(x)$ and $\delta w(x)$. Substituting the above nodal approximations into equations (2-12) and (2-13) and performing the integrations and evaluations within the equations leads to the matrix form of the equations:

$$W^k = ([k]_u^k \text{ Uncoupled} + [k]_w^k \text{ Uncoupled} + [k]_{\text{Coupling}}^k) \{u_n\} = [k(\omega)]^k \{u_n\} \quad (2-22)$$

where $[k(\omega)]^k$ represents the frequency dependent element dynamic stiffness matrix for coupled bending-axial vibrations of a symmetric sandwich beam element k . Appendix A6 provides a more detailed explanation on the formation of the element stiffness matrix. The standard assembly is then used to assume the global equation:

$$W = \sum_{elements} W^k = \langle \delta U_n \rangle [K(\omega)] \{U_n\} = 0 \quad (2 - 23)$$

It should be noted that Matlab[®] was used in the calculation of the integral terms for the element dynamic stiffness matrix. Matlab[®] performs the parametric calculations using complex arithmetic and as a result some of the elements in the matrix $[K]_{Coupling}^k$ are complex. However, the resulting dynamic stiffness matrix $[K(\omega)]$, is symmetric and real, with the imaginary parts of each element being zero. A flowchart outlining the development of the Matlab[®] used in analyzing the DFE formulation is included in Appendix A2; the programming process is also applied to all subsequent DFE programming.

2.5 Numerical Tests and Results

The newly obtained DFE can now be used to compute the natural frequencies and mode shapes of sandwich beam structures. The solution to the problem lies in finding the eigenvalues (natural frequencies), ω , and eigenvectors (natural modes), $\{U_n\}$ that make W zero for any arbitrary $\langle \delta U \rangle$ while satisfying the appropriate boundary conditions. A reliable method of calculating the natural frequencies is the use of the Wittrick-Williams algorithm (see [43] for an example) in conjunction with the DFE. However, in this study a less elaborate determinant search method was used to compute the natural frequencies of a symmetric cantilever sandwich beam taken from literature [8]. The following beam data was used in the analysis: $E_1 A_1 = 31,500\text{N}$; $E_1 I_1 = 1.362 \text{ Nm}^2$; $k_2 A_2 G_2 = 1050 \text{ N}$; $\rho_1 = 2680 \text{ kg}$; $\rho_c = 32.8 \text{ kg}$; $m = 0.001225 \text{ kg/m}$; $h_f = 0.4527 \text{ mm}$; $h_c = 12.7 \text{ mm}$ and $L = 0.7112 \text{ m}$.

The first four coupled natural frequencies of the beam were calculated using the developed DFE and FEM theory and compared to those published by Banerjee [8] obtained using the Stodola method, DSM and other FEM models, which are shown in the last column of Table 2-1. The results are summarized in Table 2-1 below:

Table 2-1: Coupled natural frequencies for a symmetric three-layered sandwich beam

ω_n (rad/s)	DSM Taken from [8]	FEM 20 Elements	FEM 40 Elements	DFE 20 Elements	Error	Stadola Taken from [8]	FEM results Taken from [9]
ω_1	197.67	197.71	197.68	197.75	0.04 %	214.45	213.44
ω_2	1217.05	1218.8	1217.49	1219	0.15 %	1269.83	1259.78
ω_3	3325.06	3334.1	3327.26	3334.56	0.29 %	3313.75	3248.41
ω_4	6320.88	6349.2	6329.68	6349.6	0.45 %	5980.96	5767.96

The FEM model used to obtain the results in columns three and four of Table 2-1 was created based on a linear approximation for the axial displacement, $u(x)$, and a cubic Hermite shape function for $w(x)$. Column three in the table shows the results for a 20-element model, while the fourth column shows the results for a 40-element model. The fifth column of the table shows the results for 20-element DFE model using the DFE element derived earlier. In the second column of Table 2-1, the results reported by Banerjee [8] are listed and used as the benchmark for the developed FEM and DFE. It can be seen that the first four sets of results agree very well with one another with the DFE error being smaller than 0.5% for a 20-element model (average of the error < 0.25%).

The first four mode shapes were also obtained using the DFE theory and agree with those reported by Banerjee [8] and are shown in the figures below. The frequencies used to find the mode shapes were 99.99% of the values found for the natural frequencies since using the actual values for natural frequencies leads to singular global dynamic stiffness matrix, which cannot be reversed to find the corresponding natural modes of the system.

Figure 2-3: Normalized mode 1 for cantilever symmetrical three-layered sandwich beam (197.75 rad/s).

Figure 2-4: Normalized mode 2 for cantilever symmetrical three-layered sandwich beam (1219 rad/s).

Figure 2-5: Normalized mode 3 for cantilever symmetrical three-layered sandwich beam (3334.56 rad/s).

Figure 2-6: Normalized mode 4 for cantilever symmetrical three-layered sandwich beam (6349.6 rad/s).

It can be seen that the first four natural modes are dominated by bending deformation, $w(x)$, also confirmed by Banerjee [8]. This can be attributed the fact that the bending rigidity of the top and bottom faces (layers 1 and 3), $E_1 I_1$, are very small when compared to the axial rigidity $E_1 A_1$.

2.6 Concluding Remarks

A dynamic finite element formulation has been developed for a symmetrical three-layered sandwich beam based on the Galerkin-type weak integral form of the differential equations of motion. The DFE formulation exploits FEM methodology in conjunction with dynamic trigonometric shape functions. The application of the DFE theory has led to numerical results that agree well with exact DSM, FEM and other published results. The theory and methodology presented can be advantageously extended to more complex sandwich structural elements which can be of variable geometry and/or material properties.

Once again, it should be noted that the DFE formulation presented above applies only to symmetrical sandwich beams in which the axial and bending loads are assumed to be carried solely by the top and bottom (face) layers and the core carries only shear loads. The core is also assumed to be much less dense than the face layers and the core inertia effects are ignored. A more general theory accounting for the asymmetry of the cross-section, as well as the axial and bending properties of the face layers and the axial, bending and shear properties of the core layer, is developed in Chapter 3.

 

Chapter 3: Rayleigh/Timoshenko Formulation

3.1 Introduction

The dynamic finite element formulation developed in the previous chapter is only applicable to the free vibration analysis of symmetric three-layered straight sandwich beams in which the top and bottom (face) layers have common cross-sections and material properties. However, this is rarely the case in many real applications of and the need for a more general formulation is required which allows for variations in the cross-sections and material properties of the layers. The formulation developed in the present chapter models the top and bottom face layers as Rayleigh beams which adds the effects of rotary inertia to Euler-Bernoulli theory; whereas the core layer is modelled as a Timoshenko beam which adds the effects of rotary inertia and shear deformation to Euler-Bernoulli theory. Banerjee uses the same beam configuration [10] as a basis for the derivation of a DSM model.

3.2 Theory

Figure 3-1, below, shows the notation and Cartesian co-ordinate system used for a three-layered sandwich beam with three distinct layers and length L .

Figure 3-1: The co-ordinate system and notation for a three-layered sandwich beam.

In the co-ordinate system shown, the origin is set at the left-hand side of the beam with the x-axis coincident with the axis of the beam. The y-axis of the beam is set upward perpendicular to the x-axis, and the z-axis comes out of page, following the right-hand rule. The beam only deflects in the x-y plane. The way the beam layers are modelled (Rayleigh and Timoshenko) lead to a specific system of

deflections; all three layers share a common flexure, w . Face layers 1 and 3, governed by Rayleigh beam theory, and are assumed to bend in such a way that the cross-section rotates to be normal to the mid-plane flexure. The top and bottom layers have linearly varying longitudinal displacements and rotary inertia is taken into account. Core layer 2, governed by Timoshenko beam theory, also possesses a linearly varying axial displacement; however, the cross-section of the core does not rotate in order to be normal to the common flexure, but shears as necessary. Following the displacement field described above, it can be observed that the axial displacements vary linearly for each layer. The axial displacement for layer 1 vary linearly from $u_1-h_1w'/2$ on the top to $u_1+h_1w'/2$ at the interface between layers 1 and 2, and from $u_3-h_3w'/2$ at the interface between layers 2 and 3 to $u_3+h_3w'/2$ at the bottom of layer 3. Primes denote partial differentiation with respect to x . The middle layer has consequential shear as its imposed motion is determined by the continuity of the two face layers and the assumed variation across its thickness (thus, the core mid-layer axial displacement u_2 is no longer used).

Each layer has its own unique geometric and material properties which use a subscript to distinguish the layer number. For example, the top layer (layer 1) has thickness h_1 , width b_1 , Young's modulus E_1 , density ρ_1 , cross-sectional area $A_1 (=b_1h_1)$ and second moment of inertia I_1 . Layer 2 and layer 3 possess the same types of material and geometric properties as layer 1 with layer 2 having the additional material property of shear modulus G_2 .

For the beam model developed, several assumptions are made [10]:

- All displacements and strains are small so that the theory of linear elasticity applies.
- The faces and core of the sandwich beam are made of homogeneous and isotropic materials and the variation of strain within them is linear.
- Transverse normal strain in the faces and core are negligible.
- Transverse shear within the face layers is negligible.
- There is no slippage or delamination between the layers during deformation.

Using the model and assumptions above, Banerjee [10] calculates the strain and kinetic energy of the system in order to form the Lagrangian and applies Hamilton's principle to produce the differential equations of motion and boundary conditions. Appendix A4 provides further explanation in the development of the equations of motion for the straight sandwich beam. For free vibration analysis, the assumption of simple harmonic motion is used, producing the following differential equations of motion for the three-layered sandwich beam [10]:

$$\begin{aligned} & \left(-B_1 \frac{\partial^2}{\partial t^2} + C_1 \frac{\partial^2}{\partial x^2} - C_7 \right) u_1 + \left(-B_6 \frac{\partial^2}{\partial t^2} + C_6 \frac{\partial^2}{\partial x^2} - C_{12} \right) u_3 \\ & + \left(-B_5 \frac{\partial^2}{\partial t^2} + C_5 \frac{\partial^2}{\partial x^2} - C_{11} \right) \frac{\partial w}{\partial x} = 0 \end{aligned} \quad (3-1)$$

$$\begin{aligned} & \left(-B_6 \frac{\partial^2}{\partial t^2} + C_6 \frac{\partial^2}{\partial x^2} - C_{12} \right) u_1 + \left(-B_2 \frac{\partial^2}{\partial t^2} + C_2 \frac{\partial^2}{\partial x^2} - C_8 \right) u_3 \\ & + \left(-B_4 \frac{\partial^2}{\partial t^2} + C_4 \frac{\partial^2}{\partial x^2} - C_{10} \right) \frac{\partial w}{\partial x} = 0 \end{aligned} \quad (3-2)$$

$$\begin{aligned} & \left(-B_5 \frac{\partial^2}{\partial t^2} + C_5 \frac{\partial^2}{\partial x^2} - C_{11} \right) \frac{\partial u_1}{\partial x} + \left(-B_4 \frac{\partial^2}{\partial t^2} + C_4 \frac{\partial^2}{\partial x^2} - C_{10} \right) \frac{\partial u_3}{\partial x} \\ & + \left(-B_3 \frac{\partial^2}{\partial x^3 \partial t^2} + C_3 \frac{\partial^4}{\partial x^4} + C_9 \frac{\partial^2}{\partial x^2} - M \frac{\partial^2}{\partial t^2} \right) = 0 \end{aligned} \quad (3-3)$$

where $u_1(x)$, $u_3(x)$ and $w(x)$ are the amplitudes of the sinusoidally varying axial displacements in the face layers and vertical displacements, respectively, and:

$$\begin{aligned} B_1 &= m_1 + m_2/3; \quad B_2 = m_3 + m_2/3; \quad B_3 = \rho_1 I_1 + m_2(h_1^2 - h_1 h_3 + h_3^2)/12 + \rho_3 I_3; \\ B_4 &= m_1 \left(\frac{h_1}{2} - h_3 \right)/6; \quad B_5 = m_2 \left(h_1 - \frac{h_3}{2} \right)/6; \quad B_6 = m_2/6; \quad M = m_1 + m_2 + m_3 \end{aligned} \quad (3-4)$$

where m_i is the mass per unit length of each layer calculated by $m_i = \rho_i A_i$ and:

$$\begin{aligned} C_1 &= E_1 A_1 + E_2 A_2/3; \quad C_2 = E_2 A_2/3 + E_3 A_3; \quad C_3 = E_1 I_1 + E_2 A_2(h_1^2 - h_1 h_3 + h_3^2)/12 + E_3 I_3; \\ C_4 &= E_2 A_2 \left(\frac{h_1}{2} - h_3 \right)/6; \quad C_5 = E_2 A_2 \left(h_1 - \frac{h_3}{2} \right)/6; \quad C_6 = E_2 A_2/6; \quad C_7 = k_2 A_2 G_2/h_2^2; \\ C_8 &= C_7; \quad C_9 = a^2 C_7; \quad C_{10} = -a C_7; \quad C_{11} = C_{10}; \quad C_{12} = -C_7; \end{aligned} \quad (3-5)$$

The appropriate boundary conditions are imposed at $x = 0$ and $x = L$. For example: clamped at $x = 0$; $u_1 = u_3 = w = w' = 0$, Free at $x = L$; $F_1 = F_3 = M = S = 0$, where the resulting axial forces in layers 1 and 3 (F_1 and F_3) are:

$$F_1 = -C_1 \frac{\partial u_1}{\partial x} - C_6 \frac{\partial u_3}{\partial x} - C_5 \frac{\partial^2 w}{\partial x^2} \quad \text{and} \quad F_3 = -C_6 \frac{\partial u_1}{\partial x} - C_2 \frac{\partial u_3}{\partial x} - C_4 \frac{\partial^2 w}{\partial x^2} \quad (3-6)$$

Note that each of the two axial forces, F_1 and F_2 , includes a contribution from core layer 2. The total transverse shear force, $S(x)$, in the y -direction, is given by:

$$S = \left(-B_5 \frac{\partial^2}{\partial t^2} + C_5 \frac{\partial^2}{\partial x^2} - C_{11} \right) u_1 + \left(-B_4 \frac{\partial^2}{\partial t^2} + C_4 \frac{\partial^2}{\partial x^2} - C_{10} \right) u_3 + \left(-B_3 \frac{\partial^2}{\partial t^2} + C_3 \frac{\partial^2}{\partial x^2} - C_9 \right) \quad (3-7)$$

and the total bending moment, $M(x)$, acting on the beam is:

$$M = -C_5 \frac{\partial u_1}{\partial x} - C_4 \frac{\partial u_3}{\partial x} - C_3 \frac{\partial^2 w}{\partial x^2} \quad (3-8)$$

For harmonic oscillations, the integral form of the governing equations (3-1), (3-2) and (3-3) are obtained by applying the weighted residual method. Subsequently, application of the appropriate number of integration by parts leads to the weak integral form of governing equations written as:

$$W_{U1} = \int_0^L \delta u_1' C_1 u_1' dx - \int_0^L \delta u_1 (\omega^2 B_1 - C_7) u_1 dx + \int_0^L \delta u_1' C_6 u_3' dx - \int_0^L \delta u_1 (\omega^2 B_6 - C_{12}) u_3 dx + \int_0^L \delta u_1' C_5 w'' dx - \int_0^L \delta u_1 (\omega^2 B_5 - C_{11}) w' dx = 0 \quad (3-9)$$

$$W_{U3} = \int_0^L \delta u_3' C_2 u_3' dx - \int_0^L \delta u_3 (\omega^2 B_2 - C_8) u_3 dx + \int_0^L \delta u_3' C_6 u_1' dx - \int_0^L \delta u_3 (\omega^2 B_6 - C_{12}) u_1 dx + \int_0^L \delta u_3' C_4 w'' dx - \int_0^L \delta u_3 (\omega^2 B_4 - C_{10}) w' dx = 0 \quad (3-10)$$

$$\begin{aligned}
W_W = & \int_0^L \delta w'' C_3 w'' dx - \int_0^L \delta w' (\omega^2 B_3 - C_9) w' dx - \int_0^L \delta w \omega^2 M w dx + \int_0^L \delta w'' C_5 u_1' dx \\
& - \int_0^L \delta w' (\omega^2 B_5 - C_{11}) u_1 dx + \int_0^L \delta w'' C_4 u_3 dx \\
& - \int_0^L \delta w' (\omega^2 B_4 - C_{10}) u_3 dx = 0
\end{aligned} \tag{3-11}$$

All of the resulting global boundary terms in the equations above are equal to zero. Reducing the order of the governing equations relaxes the continuity requirements for the field variables. Equations (3 – 9), (3 – 10) and (3 – 11) also satisfy the principle of Virtual Work (VW), which states that $W = W_{INT} + W_{EXT}$; where, $W_{INT} = W_{U1} + W_{U3} + W_W$ and $W_{EXT} = 0$, for free vibration. The domain is then discretized into 2-noded elements (see Figure 3-2, below), and the W_{INT} for the *element* domain (*i*) is:

$$W_{INT} = \sum_{i=1}^{\text{Number of Elements}} W_{INT}^i; \quad W_{INT}^i = W_{u1}^i + W_{u3}^i + W_w^i \tag{3-12}$$

W_{u1}^i , W_{u3}^i and W_w^i , are the first and second extensional and flexural components of the element VW, respectively. In the equations above, δu_1 , δu_3 and δw are test functions, both defined in the same approximation spaces as their corresponding field variables, u_1 , u_3 and w , respectively.

Figure 3-2: Domain discretized by N number of 2-noded elements.

3.3 Finite Element Method (FEM) Derivation

The conventional finite element formulation is obtained by applying the standard FEM process on the following discretized forms of equations (3 – 9), (3 – 10) and (3 – 11), written at the *element* level:

$$\begin{aligned}
W_{u1}^i = & \int_0^l \delta u_1' C_1 u_1' dx - \int_0^l u_1 (\omega^2 B_1 - C_7) u_1 dx + \int_0^l \delta u_1' C_6 u_3' dx - \int_0^l \delta u_1 (\omega^2 B_6 - C_{12}) u_3 dx \\
& + \int_0^l \delta u_1' C_5 w'' dx - \int_0^l \delta u_1 (\omega^2 B_5 - C_{11}) w' dx
\end{aligned} \quad (3-13)$$

$$\begin{aligned}
W_{u3}^i = & \int_0^l \delta u_3' C_2 u_3' dx - \int_0^l \delta u_3 (\omega^2 B_2 - C_8) u_3 dx + \int_0^l \delta u_3' C_6 u_1' dx - \int_0^l \delta u_3 (\omega^2 B_6 - C_{12}) u_1 dx \\
& + \int_0^l \delta u_3' C_4 w'' dx - \int_0^l \delta u_3 (\omega^2 B_4 - C_{10}) w' dx
\end{aligned} \quad (3-14)$$

$$\begin{aligned}
W_w^i = & \int_0^l \delta w'' C_3 w'' dx - \int_0^l \delta w' (\omega^2 B_3 - C_9) w' dx - \int_0^l \delta w \omega^2 M w dx + \int_0^l \delta w'' C_5 u_1' dx \\
& - \int_0^l \delta w' (\omega^2 B_5 - C_{11}) u_1 dx + \int_0^l \delta w'' C_4 u_3 dx \\
& - \int_0^l \delta w' (\omega^2 B_4 - C_{10}) u_3 dx
\end{aligned} \quad (3-15)$$

The next step in FEM formulation (after the discretization) is the introduction of appropriate shape functions to approximate the field variables. In the present study, linear polynomial and cubic Hermite functions are used for extensional displacements (u_1 and u_3) and flexural displacement (w), respectively.

$$u_1(x) = \langle N(x)_{u1} \rangle \{u_{j,u1} \ u_{j+1,u1}\} = N_{1u1}(x)u_{j,u1} + N_{2u1}(x)u_{j+1,u1} \quad (3-16)$$

$$u_3(x) = \langle N(x)_{u3} \rangle \{u_{j,u3} \ u_{j+1,u3}\} = N_{1u3}(x)u_{j,u3} + N_{2u3}(x)u_{j+1,u3} \quad (3-17)$$

$$\begin{aligned}
w(x) = & \langle N(x)_w \rangle \{w_1 \ w_1' \ w_2 \ w_2'\} \\
= & N_{1w}(x)w_j + N_{2w}(x)w_j' + N_{3w}(x)w_{j+1} + N_{4w}(x)w_{j+1}'
\end{aligned} \quad (3-18)$$

In equations (3-16) and (3-17) above, $u_{j,u1}$, $u_{j+1,u1}$, $u_{j,u3}$, and $u_{j+1,u3}$ are the nodal values at nodes j and $j+1$ corresponding to the axial displacements for the top and bottom faces of the sandwich beam, respectively. For equation (3-18), w_j , w_j' , w_{j+1} and w_{j+1}' are the nodal values for the flexural displacements and the bending slopes at nodes j and $j+1$, respectively. $\langle N(x)_{u1} \rangle$, $\langle N(x)_{u3} \rangle$ and $\langle N(x)_w \rangle$ are the shape functions for u_1 , u_3 and w , respectively which were found using the polynomial shape functions described above. The FEM formulation is completed when equations (3-16), (3-17) and (3-18) are employed to simplify equations (3-13), (3-14) and (3-15). Performing the integrations for equations (3-13), (3-14) and (3-15) achieves the classical linear (in ω^2) eigenvalue problem as a

function of constant mass and stiffness matrices, which can be solved using a program such as Matlab®. It should be noted that for current research, the FEM formulation was also programmed in Matlab®. The programming flowchart for the FEM program can be found in Appendix A1.

3.4 Dynamic Finite Element (DFE) Derivation

In order to obtain to the DFE formulation, the FEM element equations must undergo additional integration by parts giving leading to a new form of equations (3 – 13), (3 – 14) and (3 – 15):

$$\begin{aligned}
 W_{u1}^i = & - \underbrace{\int_0^l u_1 (C_1 \delta u_1'' + \omega^2 B_1 \delta u_1) dx}_{(***)} + \underbrace{\int_0^l \delta u_1 C_7 u_1 dx}_{[K]_{u1}^k \text{ Uncoupled Shear Term}} + \underbrace{[\delta u_1' C_1 u_1]_0^l}_{[K]_{u1}^k \text{ Uncoupled}} \\
 & + \underbrace{\int_0^l \delta u_1' C_6 u_3' dx - \int_0^l \delta u_1 (\omega^2 B_6 - C_{12}) u_3 dx}_{[K_{u1u3}]_{2 \times 2} \text{ Coupling}} \\
 & + \underbrace{\int_0^l \delta u_1' C_5 w'' dx - \int_0^l \delta u_1 (\omega^2 B_5 - C_{11}) w' dx}_{[K_{u1w}]_{2 \times 4} \text{ Coupling}}
 \end{aligned} \tag{3-19}$$

$$\begin{aligned}
 W_{u3}^i = & - \underbrace{\int_0^l u_3 (C_2 \delta u_3'' + \omega^2 B_2 \delta u_3) dx}_{(****)} + \underbrace{\int_0^l \delta u_3 C_8 u_3 dx}_{[K]_{u3}^k \text{ Uncoupled Shear Term}} + \underbrace{[\delta u_3' C_2 u_3]_0^l}_{[K]_{u3}^k \text{ Uncoupled}} \\
 & + \underbrace{\int_0^l \delta u_3' C_6 u_1' dx - \int_0^l \delta u_3 (\omega^2 B_6 - C_{12}) u_1 dx}_{[K_{u3u1}]_{2 \times 2} \text{ Coupling}} \\
 & + \underbrace{\int_0^l \delta u_3' C_4 w'' dx - \int_0^l \delta u_3 (\omega^2 B_4 - C_{10}) w' dx}_{[K_{u3w}]_{2 \times 4} \text{ Coupling}}
 \end{aligned} \tag{3-20}$$

$$\begin{aligned}
W_w^i = & \underbrace{\int_0^l w(C_3 \delta w'''' - (C_9 - \omega^2 B_3) \delta w'' - \omega^2 M \delta w) dx}_{\text{****}} \\
& - \underbrace{[\delta w''' C_3 w]_0^l + [\partial w'' C_3 w']_0^l - [\delta w'(\omega^2 B_3 - C_9) w]_0^l}_{[K]_w^k \text{ Uncoupled}} \\
& + \underbrace{\int_0^l \delta w'' C_5 u_1' dx - \int_0^l \delta w'(\omega^2 B_5 - C_{11}) u_1 dx}_{[K_{wu1}]_{4 \times 2} \text{ Coupling}} \\
& + \underbrace{\int_0^l \delta w'' C_4 u_3' dx - \int_0^l \delta w'(\omega^2 B_4 - C_{10}) u_3 dx}_{[K_{wu3}]_{4 \times 2} \text{ Coupling}} \quad (3-21)
\end{aligned}$$

Expressions (3 – 19), (3 – 20) and (3 – 21) are simply another way of evaluating equations (3 – 13), (3 – 14) and (3 – 15) at the element level. The following non-nodal approximation for δu_1 , u_1 , δu_3 , u_3 , δw and w are made in order to make the non-coupled integral terms in the above expressions zero:

$$\delta u_1 = \langle P(x) \rangle_{u1} \{\delta a\}; \quad u_1 = \langle P(x) \rangle_{u1} \{a\} \quad (3-22)$$

$$\delta u_3 = \langle P(x) \rangle_{u3} \{\delta b\}; \quad u_3 = \langle P(x) \rangle_{u3} \{b\} \quad (3-23)$$

$$\delta w = \langle P(x) \rangle_w \{\delta c\}; \quad w = \langle P(x) \rangle_w \{c\} \quad (3-24)$$

where $\{a\}$, $\{b\}$ and $\{c\}$ are the generalized coordinates for u_1 , u_3 and w , respectively and:

$$\langle P(x) \rangle_{u1} = \left\langle \cos(\gamma x) \quad \frac{\sin(\gamma x)}{\gamma} \right\rangle \quad (3-25)$$

$$\langle P(x) \rangle_{u3} = \left\langle \cos(\mu x) \quad \frac{\sin(\mu x)}{\mu} \right\rangle \quad (3-26)$$

$$\langle P(x) \rangle_w = \left\langle \cos(\alpha x) \quad \frac{\sin(\alpha x)}{\alpha} \quad \frac{\cosh(\beta x) - \cos(\alpha x)}{\alpha^2 + \beta^2} \quad \frac{\sinh(\beta x) - \sin(\alpha x)}{\alpha^3 + \beta^3} \right\rangle \quad (3-27)$$

In equations (3 – 25), (3 – 26) and (3 – 27) above, α , β , γ and μ are based on the characteristic equations (***), (****) and (*****) in (3 – 19), (3 – 20) and (3 – 21), respectively, being reduced to zero. For (3 – 19) and (3 – 20), the characteristic equations (***), (****) and (*****) are second order differential equations having the form $A_1 D^2 + B_1 \omega^2 = 0$, where A_1 and B_1 are constants and D^2 is the second partial derivative with respect to x . The characteristic equation (*****) found in (3 – 21) is a fourth order differential equation having the form $A_2 D^4 - B_2 D^2 - C_2 \omega^2 = 0$, where A_2 , B_2 and C_2 are constants and D^4 and D^2 are

the fourth and second partial derivatives with respect to x , respectively. Solving (***), (****) and (*****) gives the solution to the uncoupled parts of equations (3 – 19), (3 – 20) and (3 – 21) which are used as the basis functions to find the DTSFs.

It is important to note that there are integral terms that have been left out of (***), (****) and (*****) in equations (3 – 19) and (3 – 20), respectively. The integral terms not included in (***), (****) and (*****) represent the effects of shear from the core layer on the face layers. The inclusion of these terms in (***), (****) and (*****) change the equation so that equations (3 – 25) and (3 – 26) are no longer suitable solutions to the integral. As explained in Section 2.4 earlier, the inclusion of the shear terms move the solutions to the integrals for the face layers from being in terms of trigonometric functions to hyperbolic functions. Previous experience showed that a shape function composed solely of hyperbolic functions does not provide a solution to the free vibration problem. As in Section 2.4, the shear term is left out of the main integral and added to the DFE matrix. It should be noted that there is no changeover frequency for equation (3 – 21) because it is a higher order equation than (3 – 19) and (3 – 20) and the basis function used is a combination of trigonometric and hyperbolic functions.

During numerical testing, a special case was found involving sandwich beams with a very soft central (core) layer, such as rubber. For the previously tested cases, it was determined that there is a “changeover frequency” for each of the face layers. Above that calculated frequency (which can be different for each face layer), the omitted shear term can be included in the integral used to calculate the shape functions while keeping the solution in terms of trigonometric functions. In earlier cases, the changeover frequency was well above the frequencies of the fundamental modes being studied, but for sandwich beams with very soft cores, the changeover frequency becomes very low and can be within the range being studied. For an asymmetric beam, or where the top and bottom layers are constructed from different materials, the changeover frequency can be different for each face layer and that fact must be taken into account while developing and programming the DFE of such a case. For the rubber core beam, the changeover frequencies for the top and bottom faces are 869.73 rad/s and 746.81 rad/s, respectively. In comparison, the changeover frequencies for a lead-core beam are 45,208.96 rad/s and 43,351.79 rad/s for the top and bottom faces, respectively.

For frequencies below the changeover frequencies of the face layers α , β , γ , and μ are:

$$\gamma = \sqrt{\frac{\omega^2 B_1}{C_1}} \quad (3 - 28)$$

$$\mu = \sqrt{\frac{\omega^2 B_2}{C_2}} \quad (3 - 29)$$

$$\alpha, \beta = \sqrt{\frac{(C_9 - \omega^2 B_3) \pm \sqrt{(C_9 - \omega^2 B_3)^2 \pm 4C_3 \omega^2 M}}{2C_3}} \quad (3 - 30)$$

For frequencies higher than the changeover frequencies of the face layers, equations (3 - 28) and (3 - 29) have become (3 - 28a) and (3 - 29a).

$$\gamma = \sqrt{\frac{\omega^2 B_1 - C_7}{C_1}} \quad (3 - 28a)$$

$$\mu = \sqrt{\frac{\omega^2 B_2 - C_8}{C_2}} \quad (3 - 29a)$$

It should be noted that if the beam is not symmetrical or the faces are made of different materials, they can have different changeover frequencies. For example, it is possible that for some sections of the analysis equation (3 - 28) is used with (3 - 29a) when the changeover frequency for equation (3 - 20) is lower than (3 - 19) before both frequencies switch over, or vice versa when the changeover frequency for equation (3 - 19) is lower than (3 - 20).

Once α , β , γ , and μ are calculated, the nodal approximations for the element variables $u_1(x)$, $u_3(x)$ and $w(x)$ are then rewritten as:

$$u_1(x) = \langle P(x)_{u1} \rangle [P_n]_{u1}^{-1} \{u_n\}_{u1} = \langle N(x)_{u1} \rangle \{u_{1u1} \ u_{2u1}\} \quad (3 - 30)$$

$$u_3(x) = \langle P(x)_{u3} \rangle [P_n]_{u3}^{-1} \{u_n\}_{u3} = \langle N(x)_{u3} \rangle \{u_{1u3} \ u_{2u3}\} \quad (3 - 31)$$

$$w(x) = \langle P(x)_w \rangle [P_n]_w^{-1} \{u_n\}_w = \langle N(x)_w \rangle \{w_1 \ w'_1 \ w_2 \ w'_2\} \quad (3 - 32)$$

In equations (3 - 30), (3 - 31) and (3 - 32) above, $\langle N(x)_{u1} \rangle$, $\langle N(x)_{u3} \rangle$ and $\langle N(x)_w \rangle$ are the dynamic (frequency-dependent) shape functions of the approximation space. Similar expressions are also written for the test (weighting) functions δu_1 , δu_3 and δw . By substituting the above nodal approximations into equations (3 - 19), (3 - 20) and (3 - 21), calculating the integrals and evaluating the boundary conditions, the matrix form of the equations is formed:

$$\begin{aligned}
W^k &= ([k]_{u1}^k \text{ Uncoupled} + [k]_{u3}^k \text{ Uncoupled} + [k]_w^k \text{ Uncoupled} + [k]_{\text{Coupling}}^k) \{u_n\} \\
&= [k(\omega)]^k \{u_n\}
\end{aligned} \tag{3 - 33}$$

where $[k(\omega)]^k$ represents the frequency dependent element dynamic stiffness matrix for coupled bending-axial vibrations of a three-layered sandwich beam element k . Appendix A7 contains a more detailed description of the process used to obtain the element stiffness matrix. The standard assembly method is then used to assume the global equation:

$$W = \sum_{\text{elements}} W^k = \langle \delta U_n \rangle [K(\omega)] \{U_n\} = 0 \tag{3 - 34}$$

It should be noted that Matlab[®] was used in the calculation of the integral terms for the element dynamic stiffness matrix. Matlab[®] performs the parametric calculations using complex arithmetic and as a result some of the elements in the matrix $[K]_{\text{Coupling}}^k$ are complex. However, the resulting dynamic stiffness matrix $[K]$ is symmetric and real, with the imaginary parts of each element being zero. The process flowchart for the DFE program can be found in Appendix A2.

The completed formulation is used to test an array of beam configurations, including the degenerate symmetric straight sandwich beam. However, with the current formulation, when the top and bottom faces are made identical, the system decouples. Banerjee [10] also encountered the same problem and explains that for a symmetric sandwich beam, the general case split into two separate systems. In the first system, the beam has purely flexural displacement but no longitudinal displacement; while in the second system, the beam has only longitudinal displacement and no flexural displacement. The only time that the modes interact with each other in the symmetric case is when both systems share a common eigenvalue. Banerjee [10] also presented a very simple and effective solution to the problem of analyzing a symmetric sandwich beam system using the general case; the use of limiting analysis. For limiting analysis, instead of the two faces being exactly the same they are related to each other by ε (for example: $E_3 A_3 = E_1 A_1 (1 + \varepsilon)$) and as ε approaches zero, the two face layers become identical. For the following numerical analysis of a symmetric sandwich beam ε was taken to be 10^{-4} .

3.5 Numerical Tests and Results

The newly obtained DFE can now be used to compute the natural frequencies and mode shapes of sandwich beam structures. The solution to the problem lies in finding the eigenvalues (natural

frequencies), ω , and eigenvectors (natural modes), $\{U_n\}$ that make W zero for any arbitrary $\langle \delta U \rangle$ while satisfying the appropriate boundary conditions. A reliable method of calculating the natural frequencies is the use of the Wittrick-Williams algorithm (see, for example, [43]) in conjunction with the DFE. However, in the present study a simpler determinant search method was used to compute the natural frequencies of the test cases. The reason for this is that in order to employ the Wittrick-Williams algorithm, the clamped-clamped natural frequencies of all the beam elements forming the overall beam structure must be known in order to determine the natural frequencies of the system. In practice, the element natural frequencies with both ends clamped are unknown and must be found as well.

The first example is a cantilevered sandwich beam with a rubber core and steel faceplates taken from Banerjee [10]. The beam under consideration is a three-layered beam of rectangular cross-section and length 0.5 m. The top and bottom layers are constructed of steel and have thicknesses of 2 mm and 3 mm, respectively, whereas the middle layer is made of rubber and has a thickness of 20 mm. Each of the three layers has the same width of 40 mm. The material properties used for steel (denoted with the subscript s) and rubber (denoted with the subscript r) are $E_s = 210$ GPa, $G_s = 80$ GPa, $\rho_s = 7850$ kg/m³ and $E_r = 1.5$ MPa, $G_r = 0.5$ MPa, $\rho_r = 950$ kg/m³, respectively. The shape factor (shear correction factor) is taken to be 2/3. The first four natural frequencies of the beam are listed in Table 3-1 where the DFE results are compared with those obtained from DSM and conventional FEM methods.

Table 3-1: Coupled natural frequencies for a three-layered sandwich beam with rubber core and steel face plates.

ω_n (rad/s)	Exact DSM [10]	DFE 1 Element	DFE 2 Elements	FEM 5 Elements	FEM 10 Elements	FEM 15 Elements
ω_1	67.50	67.53	67.50	67.51	67.50	67.50
ω_2	316.6	316.6	316.6	316.8	316.6	316.6
ω_3	827.7	827.7	827.7	830.6	827.9	827.7
ω_4	1594.	1594	1594.	1613	1596	1594.

The second column in Table 3-1 shows the results found by Banerjee [10]. Banerjee's results are used as the benchmark when comparing the DFE and FEM results. The third and fourth column in Table 3-1 shows the results of the DFE formulation using only 1 and 2-elements. Columns five through seven in Table 3-1 show the FEM results which were created based on a linear approximation for the axial displacements, $u_1(x)$ and $u_3(x)$, and cubic Hermite shape functions for $w(x)$. Column five in the table shows the results for a 5-element model, while the sixth and seventh columns show the results for 10- and 15-element models, respectively. It can be seen in seen that all the results agree very well with one another, but the DFE converges much faster than FEM. The DFE formulation provides results that match exactly with the DSM with the use of only two elements while using 1 element has a maximum error of

0.044% for the first natural frequency and matches the DFE for the remaining frequencies. On the other hand, the FEM formulation requires 15 elements in order to match the DSM results for up to the fourth frequency.

The mode shapes of the first four natural frequencies are shown below. In order to increase the smoothness of the graphs 20 DFE elements were used to find and plot the mode shapes. As explained before, the mode shapes were found using values 99.99% of the actual natural frequencies of the system since using the actual values for natural frequencies leads to singular global dynamic stiffness matrix, which cannot be reversed to find the corresponding natural modes of the system.

Figure 3-3: Normalized mode 1 for cantilever rubber-core sandwich beam (67.5 rad/s).

Figure 3-4: Normalized mode 2 for cantilever rubber-core sandwich beam (316.6 rad/s).

Figure 3-5: Normalized mode 3 for cantilever rubber-core sandwich beam (827.7 rad/s).

Figure 3-6: Normalized mode 4 for cantilever rubber-core sandwich beam (1594 rad/s).

As can be seen from Figures 3-3 to 3-6 above, the mode shapes are dominated by large flexural deformations w while almost no deformation can be seen in the axial directions u_1 and u_3 (u_1 and u_3 cannot be differentiated in Figures 3-3 to 3-6, above). This was expected because of the extremely large extensional stiffness of the two face plates when compared to the core. The low density of the core compared to the face layers also minimizes the inertia effects of the core allowing the face layers to dictate the free vibration behaviour of the sandwich beam. The mode shapes were also found to agree very well with those presented by Banerjee [10].

In the second example, the rubber core of the previous example is replaced by lead. The cantilever sandwich beam has a length of 0.5 m and width of 40 mm. The faces are constructed of steel and have thicknesses of 2 mm and 3 mm for the top and bottom faces, respectively and the lead core has a thickness of 20 mm. The material properties used for steel (denoted with the subscript s) and lead (denoted with the subscript l) are $E_s = 210$ GPa, $G_s = 80$ GPa, $\rho_s = 7850$ kg/m³ and $E_l = 16$ GPa, $G_l = 5.5$ GPa, $\rho_l = 11100$ kg/m³, respectively. The shape factor (shear correction factor) is taken to be $2/3$. The first four natural frequencies of the beam are listed in Table 3-2 along with DSM and FEM results.

Table 3-2: Coupled natural frequencies for a three-layered sandwich beam with lead core and steel face plates.

ω_n (rad/s)	Exact DSM [10]	DFE 5 Elements	DFE 10 Elements	DFE 20 Elements	FEM 5 Elements	FEM 10 Elements	FEM 20 Elements	% Error 20-Element DFE and DSM
ω_1	321.4	388.8	336.5	324	323.98	322.05	321.56	0.809
ω_2	1864	2272.5	1958	1882	1963.14	1889.26	1870.69	0.966
ω_3	4718	5806	4972.1	4769.6	5226.33	4842.96	4750.07	1.094
ω_4	7193	7195.5	7193.7	7193.15	7223.22	7200.58	7194.85	0.002

Columns 3 though 5 in Table 3-2 show the natural frequencies found using the DFE derivation using 5, 10 and 20 elements, respectively. Columns 6 though 8 show the FEM results again using 5, 10 and 20 elements, respectively. Column 2 shows the DSM results found by Banerjee [10], which was used as the benchmark to compare the other results against and column 9 shows the percent difference between the 20-element DFE and the DSM. It can be seen from Table 3-2 that the FEM has slightly better results for the first two natural frequencies when a small number of elements are used. However, as the number of elements increases, the DFE begins to converge very quickly to the DSM results. For the third and fourth natural frequencies, the DFE and FEM both perform very well; with the FEM performing slightly better than the DFE for the third natural frequency while the DFE outperforms the FEM for the fourth natural frequency. The maximum error for the 20-element DFE is 1.094% for the third natural frequency with an average error of 0.718% for all four frequencies. The first four modes are shown in the Figures below. Once again, the frequencies used to find the mode shapes were 99.99% of the values found for the natural frequencies since, as it was stated before, using the actual values for natural frequencies result in singular global dynamic stiffness matrix, which cannot be reversed to find the corresponding natural modes of the system.

Figure 3-7: Normalized mode 1 for cantilever lead-core sandwich beam (324 rad/s).

Figure 3-8: Normalized mode 2 for cantilever lead-core sandwich beam (1882 rad/s).

Figure 3-9: Normalized mode 3 for cantilever lead-core sandwich beam (4769.6 rad/s).

Figure 3-10: Normalized mode 4 for cantilever lead-core sandwich beam (7193.15 rad/s).

The figures above show that the first three modes of the lead-core sandwich beam are dominated by bending while the fourth mode is predominantly an axial mode. Although the first three modes are mainly flexural, there is also quite a bit of participation by membrane displacement showing some coupling which was not seen with the rubber-core sandwich beam. Also, this beam possesses an axial mode, which again was not seen with the rubber-core beam. The mode shapes were found to match extremely well with those calculated by Banerjee [10]. The differences in the free vibration behaviour of the rubber-core and lead-core sandwich beams can most likely be explained by the fact that contrary to the former case, the lead-core bending and axial rigidities and densities are comparable with the face layers' parameters, leading to a much larger effect and interaction with the faceplates. Also, with the beam's core being constructed with lead, the assumption that the core is low-density when compared to the faceplates is no longer valid and the inertia terms associated with the core have a much larger effect on the beam behaviour than a beam possessing a light-weight core.

The third and final beam example taken from Banerjee [10] studies a symmetric cantilevered sandwich beam and the effect of the inclusion of the core's flexural and membrane rigidities. The beam has a length $L = 0.7112$ m, with a width $b = 25.4$ mm, core thickness $h_2 = 12.7$ mm and face thicknesses $h_1 = h_3 = 0.4572$ mm. The mechanical properties of the face layers are: Young's modulus $E_1 = E_3 = 69$ GPa and density $\rho_1 = \rho_3 = 2680$ kg/m³; the mechanical properties of the core layer are Young's modulus $E_2 = 0.215$ GPa, shear modulus $G_2 = 82.8$ MPa and density $\rho_2 = 32.8$ kg/m³. Table 3-3, below, lists the results of the DFE and FEM and compares them to the published results for the sandwich beam case when $E_2 = 0$, which makes the bending and axial stiffness of the core zero.

Table 3-3: Natural frequencies for symmetric three-layered sandwich beam with no core bending and axial stiffness.

ω_n (rad/s)	Exact DSM [10]	DFE 10 Elements	DFE 20 Elements	DFE 30 Elements	FEM 10 Elements	FEM 20 Elements	FEM 30 Elements
ω_1	211.99	221.86	213.50	212.06	211.49	211.18	211.12
ω_2	1249.1	1273.41	1224.91	1216.17	1227.17	1214.98	1212.47
ω_3	3213.2	3185.07	3061.92	3037.86	3118.78	3047.16	3035.09
ω_4	5686.9	5491.38	5276.24	5236.97	5410.14	5266.31	5236.60

It can be seen in the table above that for 10 elements, the FEM performs better than the DFE formulation for the first natural frequency and is much closer to the value found by Banerjee. However, when the number of elements is increased to 20 and 30 elements, both models converge to nearly the same number, 212 rad/s for the 30-element DFE and 211 rad/s for the 30-element FEM, which agree very well with the value found by Banerjee. For the remaining natural frequencies it can be seen that with the 10-element model there is very little agreement between the FEM and DFE model, with the maximum error being

between the two models being 4.67% for the first natural frequency. By increasing the number of elements, the FEM and DFE frequencies begin to converge very quickly and for the 30-element models there is good agreement between the DFE and FEM with the maximum error of 0.44% (for the first natural frequency). Even though the agreement between the FEM and DFE models is very good for the listed frequencies, the FEM and DFE frequencies for the second, third and fourth frequencies are lower than those given by Banerjee [10]. The 30-element DFE results for the second, third and fourth natural frequencies are 2.64%, 5.46% and 7.91% lower than those reported by Banerjee.

Table 3-4, below, lists the results of the case when $E_2 = 0.215$ GPa and the bending and axial stiffness of the core are taken into account.

Table 3-4: frequencies for symmetric three-layered sandwich beam with core bending and axial stiffness taken into account.

ω_n (rad/s)	Exact DSM [10]	DFE 10 Elements	DFE 20 Elements	DFE 30 Elements	FEM 10 Elements	FEM 20 Elements	FEM 30 Elements
ω_1	230.84	223.12	214.88	213.38	212.94	212.56	212.43
ω_2	1358.42	1297.26	1231.25	1222.58	1233.77	1221.70	1218.94
ω_3	3491.57	3196.38	3074.30	3052.12	3118.78	3060.04	3047.85
ω_4	6173.86	5505.58	5292.45	5253.62	5426.91	5283.22	5253.43

As with the previous case, there is a large difference between the results for the 10-element DFE and FEM with the largest difference of 4.89% for the second natural frequency. However, as the number of elements is increased to 20 and 30 the agreement between the frequencies of the DFE and FEM models begin to converge and agree very well with each other for the 30-element models; with errors of 0.45%, 0.30%, 0.14% and 0.004% for the first four modes, respectively. As expected, the sandwich beam models with the core's axial and bending stiffness term ($E_2 \neq 0$) have higher values for all four natural frequencies than those with $E_2 = 0$. The first natural frequency had the largest increase of 0.62%, while the second, third and fourth natural frequencies had increases of 0.53%, 0.47% and 0.32%, respectively. The inclusion of the core's bending and axial stiffness increases the overall stiffness of the beam causing the increase in the natural frequencies of the beam.

The mode shapes of the two beam models are compared in the figures below. For each natural frequency, the mode shape for the model that ignores the core's bending and axial stiffness is shown on the left, while the mode shape on the right is for the model that takes the core's bending and axial stiffnesses is on the right. The mode shapes are generated using a 30-element DFE model for both beam cases. The mode shapes were again found using values 99.99% of the actual natural frequencies of the system.

Figure 3-11: Normalized mode 1 for symmetric sandwich beam; ignoring core bending and axial stiffness – 212.06 rad/s (left) and taking the core bending and axial stiffness – 213.38 (right).

Figure 3-12: Normalized mode 2 for symmetric sandwich beam; ignoring core bending and axial stiffness – 1216.17 rad/s (left) and taking the core bending and axial stiffness – 1222.58 (right).

Figure 3-13: Normalized mode 3 for symmetric sandwich beam; ignoring core bending and axial stiffness – 3037.86 rad/s (left) and taking the core bending and axial stiffness – 3052.12 (right).

Figure 3-14: Normalized mode 4 for symmetric sandwich beam; ignoring core bending and axial stiffness – 5236.97 rad/s (left) and taking the core bending and axial stiffness – 5253.62 (right).

The mode shapes for both beam models agree very well with each other. All four modes are dominated by bending with very little axial displacement involved. The addition of the bending and axial stiffnesses to the model has very little effect on the mode shapes. This can be attributed to the fact that the core's bending and axial stiffnesses, in the present case, are small compared to the face layers' rigidities, meaning the mode shapes are still dominated by the free vibration behaviour of the face layers.

Additional numerical tests were also performed for the Rayleigh-Timoshenko sandwich beam formulation based on the work of Howson and Zare [33]. Howson and Zare, too, employed a DSM approach to solve the free vibration problem for a straight sandwich beam governed by assumptions similar to those made in the present development. They analyzed a variety of sandwich beam configurations, but the ones of interest were the cantilever, doubly clamped and the doubly simply supported end conditions. The results of the DFE and the DSM matched well for low frequencies for the cantilever and clamped configurations; however, there are large discrepancies for the simply supported case and higher frequencies for the other configurations. It is to be noticed that the explicit expressions for the coupled equations of motion and boundary condition assumptions used by Howson and Zare were not available and therefore could not be compared to the DFE formulation to see if the cause of the divergence in results could be identified and repaired. For completeness, the results of the DFE and FEM analysis of the Howson and Zare beam cases are listed in Appendix A9.

3.6 Concluding Remarks

A dynamic finite element formulation has been developed for a non-symmetrical three-layered sandwich beam based on the weak integral form of the differential equations of motion. The current DFE formulation improves on the one developed in the previous chapter as it is more flexible and applicable to a wider range of applications since the beam being studied no longer needs to be symmetrical. All three layers of the sandwich beam can have different geometric and material properties and still be successfully analyzed. The DFE formulation presented in this chapter models the face layers as Rayleigh beams taking the rotary inertia effects of the faces into account. The core layer is modelled as a Timoshenko beam with axial, flexural and shear deformation and takes rotary inertia into account as opposed to the previous model presented in Chapter 2, where the core had only shear deformation.

After completing the DFE formulation, it was used to find the natural frequencies and mode shapes of a variety of different sandwich beam configurations reported in the literature. The first two test cases involved an asymmetric beam with steel face layers, but with two dissimilar core materials; rubber and lead. DFE models using only 1 and 2-elements performed very well for the soft-core (rubber) beam matching the DSM and FEM natural frequency results. For the lead-core beam, the DFE also performed well, but needed 20 elements in order to provide satisfactory results compared to the DSM and FEM. The slow convergence of the DFE is most likely the result of core and face layers possessing similar material properties. The material and mechanical properties of lead are similar to those of steel and therefore the lead core has a much larger effect on overall behaviour of the beam than a rubber core. Lastly, the DFE was used for the symmetric sandwich beam case and the effects of including the axial and bending stiffness of the core was studied. The calculated result showed that the inclusion of the core's axial and bending stiffness had a minor effect on the natural frequencies of the beam driving them slightly higher than the case where they were not taken into account.

The next sandwich beam formulation studied by Banerjee *et al.* [9] models all three faces as Timoshenko beams. After development and testing performed by Banerjee *et al.* [9], the conclusion was made that the improvements in performance compared to the current model do not justify the additional complexity, difficulty and computing power necessary in developing and implementing the model presented in [9]. Instead, the focus of the following chapter is shifted from straight beams to the free vibration analysis of curved sandwich beams.

Chapter 4: Curved Beam Formulation

4.1 Introduction

In the previous chapters, a general DFE formulation for straight sandwich beams was successfully developed. However, in the aerospace industry and many other applications, various curved beam and other structural elements are commonly used. Thus, it is important to derive a DFE formulation that has the ability to analyze the free vibration characteristics of such members. The element under consideration is a symmetrical three-layered sandwich beam with a constant curvature first studied by Ahmed [1, 2].

4.2 Theory

Figure 4-1 below shows the notation and corresponding co-ordinate system used for a symmetrical curved three-layered sandwich beam with a length of S and radius R at the mid-plane of the beam. The thicknesses of the inner and outer face layers are t while the thickness of the core is t_c .

Figure 4-1: The co-ordinate system and notation for a curved symmetric three-layered sandwich beam.

In the co-ordinate system shown, the z-axis is the normal co-ordinate measured from the centre of each layer and the y-axis is the circumferential co-ordinate and is coincident with the centreline of the beam. The beam is only deflects in the y-z plane. The top and bottom faces, in this case, are modelled as Euler-Bernoulli beams while the core is assumed to have only shear rigidity (e.g., the stresses in the core in the longitudinal direction are zero). The centreline displacements of layers 1 and 3 are v_1 and v_2 , respectively. The main focus of the model is flexural vibration, w and is common among all three layers, which leads to the assumption $v_1 = -v_2 = -v$.

For the beam model developed, the following assumptions made [1]:

- All displacements and strains are so small that the theory of linear elasticity still applies.
- The face materials are homogeneous and elastic, while the core material is assumed to be homogeneous, orthotropic and rigid in the z-direction.
- The transverse displacement w does not vary throughout the thickness of the beam.
- The shear within the faces is negligible.
- The bending strain within the core is negligible.
- There is no slippage or delamination between the layers during deformation.

Using the model and assumptions described above, Ahmed [1] used the principle of minimum potential energy to obtain the differential equations of motion and corresponding boundary conditions. Appendix A5 has a more detailed explanation of the procedure used to calculate the differential equations of motion. For free vibration analysis, the assumption of simple harmonic motion is used, leading to the following form of the differential equations of motion for a curved symmetrical sandwich beam [1]:

$$\frac{\partial^2 v}{\partial y^2} + \frac{1}{\alpha^2} (\omega^2 Q_1 - 4\beta^2) v - \frac{2h\beta^2}{\alpha^2} \cdot \frac{\partial w}{\partial y} = 0 \quad (4-1)$$

$$\frac{\partial^4 w}{\partial y^4} - \frac{h^2 \beta^2}{\gamma^2} \cdot \frac{\partial^2 w}{\partial y^2} + \frac{1}{\gamma^2} \left(\frac{\alpha^2}{R^2} - \omega^2 Q_2 \right) w - \frac{2h\beta^2}{\gamma^2} \cdot \frac{\partial v}{\partial w} = 0 \quad (4-2)$$

where:

$$\begin{aligned} \alpha^2 &= 2Et; & \beta^2 &= \left(\frac{1}{t_c} + \frac{t_c}{4R^2} \right) \cdot G_c; & \gamma^2 &= \frac{Et^3}{6}; & h &= t + t_c; \\ Q_1 &= 2t\rho_f + \frac{1}{3}t_c\rho_c; & Q_2 &= 2t\rho_f + t_c\rho_c \end{aligned} \quad (4-3)$$

In the equations above, $v(y)$ and $w(y)$ are the amplitudes of the sinusoidally varying circumferential and radial displacements, respectively. E is the Young's modulus of the face layers, G_c is the shear modulus of the core layer while ρ and ρ_c are the densities of the face and core materials, respectively.

The appropriate boundary conditions are imposed at $y = 0$ and $y = S$.

For example: clamped at $y = 0$ and $y = S$; $v = w = \frac{\partial w}{\partial y} = 0$.

For simply supported at $y = 0$ and $y = S$; $\frac{\partial v}{\partial y} = w = \frac{\partial^2 w}{\partial y^2} = 0$.

For a cantilever configuration; at $y = 0$: $v = w = \frac{\partial w}{\partial y} = 0$; and at $y = S$: $\frac{\partial v}{\partial y} = \frac{\partial^2 w}{\partial y^2} = 0$ and a resultant

force term of $\left[2\gamma^2 \cdot \frac{\partial^3 w}{\partial y^3} + 2\beta^2 h \left(2v + h \cdot \frac{\partial w}{\partial y} \right) \right] = 0$.

For harmonic oscillation, the weak form of the governing equations (4 – 1) and (4 – 2) are obtained by applying a Galerkin-type integral formulation, based on the weighted-residual method. The method involves the use of integration by parts on different elements of the governing differential equations and then the discretization of the beam length into a number of two-noded beam elements (Figure 4-2 below). Applying the appropriate number of integration by parts to the governing equations and discretization lead to the following form (in the equations below, primes denote integration with respect to y):

$$W_v^k = \int_0^l \delta v' \alpha^2 v' dy - \int_0^l \delta v (\omega^2 Q_1 - 4\beta^2) v dy + \int_0^l \delta v 2h\beta^2 w' dy \quad (4 - 4)$$

$$W_w^k = \int_0^l \delta w'' \gamma^2 w'' dy + \int_0^l \delta w' h^2 \beta^2 w' dy + \int_0^l \delta w \left(\frac{\alpha^2}{R^2} - \omega^2 Q_2 \right) w dy + \int_0^l \delta w' 2\beta^2 h v dy \quad (4 - 5)$$

All of the resulting global boundary terms produced by integration by parts before discretization in the equations above are equal to zero. The above equations are known as the *element* Galerkin-type weak form associated to the discretized equations (4 – 4) and (4 – 5) and also satisfy the principle of virtual work:

$$W = W_{INT} - W_{EXT} = (W_u + W_w) - W_{EXT} = 0 \quad (4 - 6)$$

For the free vibration analysis, $W_{EXT} = 0$, and

$$W_{INT} = \sum_{k=1}^{NET} W^k \quad \text{where } W^k = W_u^k + W_w^k \quad (4-7)$$

In the equations above, δv and δw are the test functions, both defined in the same approximation spaces as v and w . Each element is defined by nodes j and $j+1$ with the corresponding co-ordinates ($l = x_{j+1} - x_j$). The admissibility condition for finite element approximation is controlled by the undiscretized forms of equations (4-4) and (4-5).

Figure 4-2: Domain discretized by N number of 2-noded elements.

4.3 Finite Elements Method (FEM) Derivations

Two different FEM models were derived in for the curved beam model. The first has three degrees of freedom (DOFs) per node and uses a linear approximation for the axial displacement and a Hermite type polynomial approximation for the bending displacement.

$$v(x) = \langle N(y)_v \rangle \{v_j \ v_{j+1}\} = N_{1v}(x)v_j + N_{2v}(x)v_{j+1} \quad (4-8)$$

$$\begin{aligned} w(x) &= \langle N(y)_w \rangle \{w_j \ w'_j \ w_{j+1} \ w'_{j+1}\} \\ &= N_{1w}(x)w_j + N_{2w}(x)w'_j + N_{3w}(x)w_{j+1} + N_{4w}(x)w'_{j+1} \end{aligned} \quad (4-9)$$

In the equations above, v_j , v_{j+1} , w_j and w_{j+1} are the nodal values at j and $j+1$ corresponding to the circumferential and radial displacements, respectively (these can be likened to the axial and flexural displacements for a straight beam). w'_j and w'_{j+1} represent the nodal values of the rate of change of the radial displacements with respect to x (which can be likened to the bending slope for a straight beam). The same approximations were also used for δv and δw , respectively. The first FEM formulation is completed when the nodal approximations expressed by equations (4-8) and (4-9) are applied to simplify equations (4-4) and (4-5). Similar approximations are also used for the corresponding test functions, δv and δw , and the integrations are performed to achieve the classical linear (in ω^2) eigenvalue problem as functions of constant mass and stiffness matrices, which can be solved using programs such as Matlab®.

In the second FEM model the number of DOFs per node is increased to four and Hermite type polynomial approximations are used for both the axial and bending displacements.

$$\begin{aligned} v(x) &= \langle N(y)_v \rangle \{v_j \ v'_j \ v_{j+1} \ v'_{j+1}\} \\ &= N_{1v}(x)v_1 + N_{2v}(x)v'_1 + N_{3v}(x)v_2 + N_{4v}(x)v'_2 \end{aligned} \quad (4-10)$$

$$\begin{aligned} w(x) &= \langle N(y)_w \rangle \{w_j \ w'_j \ w_{j+1} \ w'_{j+1}\} \\ &= N_{1w}(x)w_1 + N_{2w}(x)w'_1 + N_{3w}(x)w_2 + N_{4w}(x)w'_2 \end{aligned} \quad (4-11)$$

In the equations above, v_j , v_{j+1} , w_j and w_{j+1} are the nodal values at j and $j+1$ corresponding to the circumferential and radial displacements, respectively. v'_j , v'_{j+1} , w'_j and w'_{j+1} are the nodal values at j and $j+1$ for the rate of change with respect to y for the circumferential and radial displacements, respectively. The same approximations are also used for δv and δw . The second FEM formulation applies equations (4-10) and (4-11) to simplify equations (4-4) and (4-5) instead of equations (4-8) and (4-9) to produce the linear (in ω^2) eigenvalue problem as a function of constant mass and stiffness matrices, which can again be solved using programs such as Matlab®. For the current research, both FEM models were programmed using Matlab®.

4.4 Dynamic Finite Element (DFE) Formulation

In order to obtain the DFE formulation, an additional set of integration by parts must be applied to the element equations (4-4) and (4-5) leading to:

$$W_v^k = - \underbrace{\int_0^l (\delta v'' \alpha^2 + \delta v \omega^2 Q_1) v dy}_{(*****)} + \underbrace{\int_0^l \delta v 4\beta^2 v dy + [\delta v' \alpha^2 v]_0^l}_{[k]_v^k \text{ Uncoupled}} + \underbrace{\int_0^l \delta v 2h\beta^2 w' dy}_{[k_{vw}]_{2 \times 4} \text{ Coupling}} \quad (4-12)$$

$$\begin{aligned} W_w^k &= \underbrace{\int_0^l \left(\delta w'''' \gamma^2 - \delta w'' h^2 \beta^2 + \delta w \left(\frac{\alpha^2}{R^2} - \omega^2 Q_2 \right) \right) w dy}_{(*****)} \\ &\quad + \underbrace{[\delta w'' \gamma^2 w']_0^l - [\delta w'''' \gamma^2 w]_0^l + [\delta w' h^2 \beta^2 w]_0^l}_{[k]_w^k \text{ Uncoupled}} + \underbrace{\int_0^l \delta w' 2\beta^2 h v dy}_{[k_{wv}]_{4 \times 2} \text{ Coupling}} \end{aligned} \quad (4-13)$$

Equation (4 – 12) and (4 – 13) are another way of evaluating equations (4 – 4) and (4 – 5) at the element level. The non-nodal approximations

$$\delta v = \langle P(y) \rangle_v \{\delta a\}; \quad v = \langle P(y) \rangle_v \{a\} \quad (4 - 14)$$

$$\delta w = \langle P(y) \rangle_w \{\delta b\}; \quad w = \langle P(y) \rangle_w \{b\} \quad (4 - 15)$$

where $\{a\}$ and $\{b\}$ are the generalized coordinates for v and w , respectively with the basis functions of approximation space expressed as:

$$\langle P(y) \rangle_u = \left\langle \cos(\varepsilon y) \quad \frac{\sin(\varepsilon y)}{\varepsilon} \right\rangle \quad (4 - 16)$$

$$\langle P(y) \rangle_w = \left\langle \cos(\sigma y) \quad \frac{\sin(\sigma y)}{\sigma} \quad \frac{\cosh(\tau x) - \cos(\sigma x)}{\sigma^2 + \tau^2} \quad \frac{\sinh(\tau x) - \sin(\sigma x)}{\sigma^3 + \tau^3} \right\rangle \quad (4 - 17)$$

The non-nodal approximations (4-14) and (4-15) are made for δv , v , δw and w so that the integral terms (*****) and (*****) in equations (4 – 12) and (4 – 13) become zero. (*****) has a second order differential characteristic equation of the form $A_1 D^2 + B_1 \omega^2 = 0$, where A_1 and B_1 are constants and D^2 is the second partial derivative with respect to x . (*****) has a fourth order differential characteristic equation of the form $A_2 D^4 - B_2 D^2 + C_2 \omega^2 = 0$, where A_2 , B_2 and C_2 are constants and D^4 and D^2 are the fourth and second partial derivatives with respect to x , respectively. Solving (*****) and (*****) yields the solution to the uncoupled parts of equations (4 – 12) and (4 – 13), which are then used in the basis functions to find the DTSEs.

It should be pointed out that in equation (4 – 12) an integral term $(\int_0^l \delta v 4\beta^2 v dy)$ was purposely left out of (*****). This term represents the effect of the shear from the core on the face layers. The inclusion of this term in (*****) would change the solution to (*****) from being in terms trigonometric functions to being in terms of purely hyperbolic functions making it impossible to find the solution to the free vibration problem. Above a given frequency, the excluded integral term can be included in the (*****) term, however, for the test cases being studied the frequency is much higher than the range being studied. The problem is solved by simply excluding the additional integral term from (*****) and evaluating it separately and using the originally proposed basis functions (4 – 16) and (4 – 17).

In equations (4 – 16) and (4 – 17), ε , σ and τ (shown below) are calculated based on the characteristic equations (*****) and (*****) in expressions (4 – 12) and (4 – 13) being reduced to zero.

$$\varepsilon = \sqrt{\frac{\omega^2 Q_1}{\alpha^2}} \quad (4 - 18)$$

$$\sigma, \tau = \sqrt{\frac{(h^2 \beta^2) \pm \sqrt{(h^2 \beta^2)^2 - 4\gamma^2 \left(\frac{\alpha^2}{R^2} - \omega^2 Q_2 \right)}}{2\gamma^2}} \quad (4 - 19)$$

The nodal approximations for the element variables $v(y)$ and $w(y)$ are then written as:

$$v(y) = \langle P(y)_v \rangle [P_n]_v^{-1} \{u_n\}_v = \langle N(y)_v \rangle \{v_1 \ v_2\} \quad (4 - 20)$$

$$w(y) = \langle P(y)_w \rangle [P_n]_w^{-1} \{u_n\}_w = \langle N(y)_w \rangle \{w_1 \ w'_1 \ w_2 \ w'_2\} \quad (4 - 21)$$

where $\langle N(y)_v \rangle$ and $\langle N(y)_w \rangle$ are the dynamic (frequency-dependent) shape functions of the approximation space. Similar expressions are also written for the test (weighting) functions, $\delta v(y)$ and $\delta w(y)$. Substituting the above nodal approximations into equations (4 – 12) and (4 – 13) and carrying out the integrations and term evaluations leads to the matrix form of the equations:

$$W^k = ([k]_v^k \text{ Uncoupled} + [k]_w^k \text{ Uncoupled} + [k]_{Coupling}^k) \{u_n\} = [k(\omega)]^k \{u_n\} \quad (4 - 22)$$

where $[k(\omega)]^k$ represents the frequency dependent *element* dynamic stiffness matrix for coupled bending-axial vibrations of a curved symmetric sandwich beam element k . Appendix A8 provides a more in-depth description of the process used to obtain the element stiffness matrix. The standard assembly method is used to obtain the global equation:

$$W = \sum_{\text{elements}} W^k = \langle \delta U_n \rangle [K(\omega)] \{U_n\} = 0 \quad (4 - 23)$$

It should be noted that Matlab[®] was used in the calculation of the integral terms for the element dynamic stiffness matrix. Matlab[®] performs the calculations using complex arithmetic and as a result some of the elements in the matrix $[K]_{Coupling}^k$ are complex. However, the resulting dynamic stiffness matrix $[K]$ is real and symmetric, with the imaginary parts of each element being zero.

4.5 Numerical Tests and Results

The newly developed DFE can now be used can be used to compute the natural frequencies and mode shapes of curved symmetrical sandwich beams. The solution to the problem lies in finding the eigenvalues (natural frequencies), ω , and eigenvectors (natural modes), $\{U_n\}$, that satisfy expression (4 – 23) and the appropriate boundary conditions for any arbitrary $\langle \delta U \rangle$ while satisfying the appropriate boundary conditions. The Wittrick-Williams algorithm [43] is one of the most reliable methods employed in conjunction with DFE to calculate natural frequencies; however, in this study a simple determinant search method is utilized to compute the natural frequencies of the system.

The same curved symmetrical sandwich beam is used for first three test cases. The beam considered has a span, $S = 0.7112$ m, with the top and bottom faces having thicknesses, $t = 0.4572$ mm and a core thickness, $t_c = 12.7$ mm with a radius of curvature, $R = 4.225$ m. The material properties of the face layers are: $E = 68.9$ GPa and $\rho_f = 2680$ kg/m³; while the core has properties of $G_c = 82.68$ MPa and $\rho_c = 32.8$ kg/m³. The first test case studies the natural frequencies of the beam described above while in a cantilever configuration. The DFE and FEM results are presented and compared with those reported by Ahmed [1] in Table 4-1 below. Ahmed [1] uses a 10-element FEM model which has four DOFs per node. The model developed by Ahmed employs similar polynomial Hermite shape functions such as those found in equations (4 – 10) and (4 – 11) for the approximation space of the field variables v , v' , w and w' , respectively.

Table 4-1: Natural frequencies of a curved symmetric sandwich beam in a cantilever configuration.

ω_n (rad/s)	Ahmed FEM 10 Element [1]	DFE 20 Elements	DFE 30 Elements	DFE 40 Elements	FEM – 3 DOF 20 Elements	FEM – 3 DOF 40 Elements	FEM - 4 DOF 20 Elements	FEM - 4 DOF 40 Elements
ω_1	1124.69	1124.69	1121.93	1121.8	1121.67	1121.61	1121.61	1121.61
ω_2	1671.33	1678.87	1671.89	1668.37	1668.25	1665.67	1665.48	1664.98
ω_3	3430.62	3451.98	3420.38	3408.88	3420.32	3402.97	3402.41	3398.51
ω_4	5868.50	5901.80	5838.65	5817.10	5860.33	5811.82	5811.07	5799.69
ω_5	8664.51	8695.93	8600.30	8567.37	8659.42	8566.24	8564.74	8524.02

The formulated results for the FEM and DFE models agree very well with one another with the maximum percent difference being 1.53% for the fifth natural frequency for 20-element models when comparing the DFE and the 4-DOF FEM. For the 40-element models, the largest difference is 0.51% again for the fifth mode when comparing the DFE and 4-DOF FEM. Comparing the convergence rates of the three models, it can be seen that the 4-DOF FEM formulation converges faster than the DFE and the 3-DOF FEM. This

is due to the extra degree of freedom in the 4-DOF formulation; however, the extra degree of freedom also seems to have the effect of reducing the stiffness of the beam model which can be seen in the lower values of the calculated natural frequencies from the 4-DOF FEM formulation. When comparing the calculated results to those reported by Ahmed it is observed that the majority of the calculated results are smaller. However, it should be considered that Ahmed used only 10 elements in the model and the model may not have fully converged. The use of additional elements in the model will cause the reported frequencies to move closer to the calculated results.

Finally, the normalized mode shapes are computed using a 40-element DFE model for the cantilevered curved sandwich beam and are shown in the figures below. Unfortunately, Ahmed does not provide the mode shapes for curved sandwich beams. However, the curved beam studied has a large radius of curvature compared to its span so the calculated mode shapes of the test cases in the previous chapter can be used as a rough guideline to gauge the acceptability of the current mode shapes. The mode shapes were taken at 99.99% of the natural frequencies because using the actual values for natural frequencies leads to singular global dynamic stiffness matrix, which cannot be reversed to find the corresponding natural modes of the system.

Figure 4-3: Normalized mode 1 for curved symmetric sandwich beam with cantilever end conditions (1121.8 rad/s).

Figure 4-4: Normalized mode 2 for curved symmetric sandwich beam with cantilever end conditions (1668.37 rad/s).

Figure 4-5: Normalized mode 3 for curved symmetric sandwich beam with cantilever end conditions (3408.88 rad/s).

Figure 4-6: Normalized mode 4 for curved symmetric sandwich beam with cantilever end conditions (5817.10 rad/s).

Figure 4-7: Normalized mode 5 for curved symmetric sandwich beam with cantilever end conditions (8567.37 rad/s).

As can be seen in Figures 4-3 to 4-7 above, all five mode shapes calculated are dominated by radial displacements. This was expected as the bending stiffness of the beam is much smaller than its axial stiffness and the primary concern of the equations derived by Ahmed was to study the flexural behaviour of the beam. The undeformed shape of the beam was not included in the figures above because the beam's short length (0.7112) with respect to its large radius of curvature (4.225 m) would make the beam appear nearly straight.

The next test case uses the same beam properties as the previous example, but both ends of the beam are clamped. The results of the DFE, 3- and 4-DOF FEM formulations along with those reported by Ahmed [1, 2] are listed in Table 4-2 below. The first set of results from Ahmed [2] shown in the second column of Table 4-2, below, are from a 10-element FEM model where each node has 6 DOFs. The DOFs are associated with circumferential displacement (v and v'), radial displacement (w and w') and transverse shear in the x-y plane (ϕ and ϕ' , which none of the derived models takes into account). For each of the displacements, a Hermite polynomial shape function similar to equations (4 – 10) and (4 – 11) was used to define the approximation space for the field variables. For the 10-element model results from Ahmed [1] shown in the third column of Table 4-2, below, each node has 4 DOFs. The FEM model developed again employs similar polynomial Hermite shape functions such as those found in equations (4 – 10) and (4 – 11) for the approximation space of the field variables v , v' , w and w' , respectively.

Table 4-2: Natural frequencies of a curved symmetric sandwich beam with both ends clamped.

ω_n (rad/s)	Ahmed FEM 6- DOF 10 Element [2]	Ahmed FEM 4- DOF 10 Element [1]	DFE 20 Elements	DFE 30 Elements	DFE 40 Elements	FEM – 3 DOF 20 Elements	FEM – 3 DOF 40 Elements	FEM - 4 DOF 20 Elements	FEM - 4 DOF 40 Elements
ω_1	1507.96	1658.76	1665.67	1655.62	1652.23	1653.73	1649.96	1649.84	1648.96
ω_2	2978.23	3279.82	3295.53	3263.30	3252.30	3272.97	3249.60	3250.92	3244.20
ω_3	5296.73	5585.75	5580.10	5520.47	5499.99	5563.57	5502.19	5508.34	5488.74
ω_4	7872.83	8243.54	8203.96	8112.91	8081.62	8208.29	8093.94	8107.70	8069.37
ω_5	10662.6	11102.4	11020.1	10896.0	10853.0	11054.8	10878.2	10900.1	10839.1

Table 4-2 above, shows that for the first two natural frequencies, the DFE results are slightly larger than those obtained from both FEM formulations, but for the third through fifth natural frequencies, the DFE values are smaller than those found by the 3-DOF FEM formulation but larger than the 4-DOF FEM formulation. For 20-elements FEM models, the largest percent difference is 1.4% seen between the 3-

DOF and 4-DOF FEM formulations (in the fifth natural frequencies), but when the number of elements is increased to 40, the difference reduces to difference becomes 0.36%, which is still the largest when comparing all 3 models.

The largest difference when comparing the 40-element DFE and 3-DOF FEM models is 0.23% for the fifth natural frequency with the rest of the error being smaller. When comparing the 40-element DFE and 4-DOF FEM models, the largest error is 0.25% for the second natural frequency. The dramatic decrease in the discrepancies of the three models indicates that they are all converging to nearly the same values for the natural frequencies. When comparing the results to those of Ahmed, it can be seen that they agree very well with the 4-DOF model, although, they are smaller in value. The main reason for this is that Ahmed only used 10 elements and an increase in the number of elements used would give lower values. Looking at Ahmed's results for the 6-DOF model, it can be seen that they are considerably lower than all the calculated values. When comparing the DFE to Ahmed's 6-DOF formulation, the largest differences can be seen for the first two natural frequencies with a difference of 9.56% and 9.20%, respectively. For the third, fourth and fifth natural frequencies, the difference between the DFE and Ahmed 6-DOF formulation is 3.84%, 2.65% and 1.79%, respectively. Ahmed [1] states that the difference in values is most likely due to the differences in formulations between the two models. The equations of motion upon which the DFE is based on ignores the shear of the face layers and also ignores the bending and axial stiffness of the core while the 6-DOF formulation takes all of these factors into account. Lastly, the five natural modes of the curved symmetrical sandwich beam are determined. The frequency values used in the calculations of the mode shapes of the beam are 99.99% of the natural frequencies because the displacements cannot be evaluated as the true value of the natural frequency is approached. They are shown in the figures below generated using a 40-element DFE model.

Figure 4-8: Normalized mode 1 for curved symmetric sandwich beam with clamped end conditions (1652.23 rad/s).

Figure 4-9: Normalized mode 2 for curved symmetric sandwich beam with clamped end conditions (3252.30 rad/s).

Figure 4-10: Normalized mode 3 for curved symmetric sandwich beam with clamped end conditions (5499.99 rad/s).

Figure 4-11: Normalized mode 4 for curved symmetric sandwich beam with clamped end conditions (8081.62 rad/s).

Figure 4-12: Normalized mode 5 for curved symmetric sandwich beam with clamped end conditions (10853.0 rad/s).

As expected, the mode shapes for the curved symmetrical sandwich beam with clamped-clamped end conditions shown above exhibit mainly radial displacement. Some circumferential displacement is also observed but is small when compared the magnitude of the radial displacement. This can be explained by the fact that the beam's axial stiffness is much higher than its bending stiffness. Also, the mode shapes conform to the clamped-clamped boundary conditions applied to the beam; the radial and circumferential displacements are zero at the end points as is the slope.

The third numerical case uses the beam described earlier in the section with both ends simply supported. The DFE, 3- and 4-DOF FEM formulations are used to calculate the beam's natural frequencies and mode shapes. The results of these models are listed along with those reported by Ahmed [1], obtained using a 10-element FEM model with 4-DOFs per node (Table 4-3). The FEM model developed by Ahmed [1] uses polynomial Hermite shape functions similar to equations (4 – 10) and (4 – 11) for the approximation space of the field variables v , v' , w and w' , respectively.

Table 4-3: Natural frequencies of a curved symmetric sandwich beam with both ends simply supported.

ω_n (rad/s)	Ahmed FEM 4- DOF 10 Element [1]	DFE 20 Elements	DFE 30 Elements	DFE 40 Elements	FEM – 3 DOF 20 Elements	FEM – 3 DOF 40 Elements	FEM - 4 DOF 20 Elements	FEM - 4 DOF 40 Elements
ω_1	1253.5	1253.50	1250.35	1249.47	1248.60	1248.34	1248.34	1248.34
ω_2	2475.58	2501.96	2480.60	2472.87	2471.74	2466.65	2464.89	2464.89
ω_3	4687.26	4746.95	4697.94	4680.97	4690.84	4669.22	4662.06	4662.06
ω_4	7382.74	7478.88	7397.82	7370.11	7405.49	7354.72	7337.82	7337.82
ω_5	10298.1	10433.86	10318.88	10278.98	10351.3	10261.38	10231.41	10231.35

As it can be seen from the 2nd row in Table 4-3, there is good agreement between all the 20-element models, with the biggest discrepancy being between the DFE and the 4-DOF FEM formulations; the FEM first natural frequency, ω_1 , is 0.41% smaller than that obtained from the DFE. Although the agreement for the first frequency is good, when the remaining frequencies are examined, the growing difference in the frequencies can be observed in the higher modes. When comparing the 20-element DFE and the 20-element 3-DOF FEM formulations, the largest difference in the values calculated is the second natural frequency, with the FEM value being 1.21% smaller than the DFE result. The difference between the DFE and 3-DOF FEM decreases with the higher natural frequencies. Increasing the number of elements from 20 to 40, reduces the difference between the two models for the second frequency to 0.25% remaining the maximum and the difference for the other frequencies decreasing with the increase in mode number.

Comparing the 20-element DFE and the 4-DOF FEM models, the trend is reversed; the two values are closest for the first natural frequency and increase with the higher frequencies with the largest difference being for the fifth natural frequency where the FEM is 1.94% smaller than the DFE. When the number of elements used in the model is increased to 40, the agreement between the two formulations becomes much better with the maximum relative error being 0.46% for the fifth frequency. Increasing the number of elements from 20 to 40 considerably reduces the relative error between all the models suggesting they are converging. Looking at Ahmed's results, it can be seen that for the 20-element DFE model, the first natural frequency match perfectly. But with the increase in the mode, the difference between the DFE and Ahmed's results grow to a maximum of 1.32% for the fifth natural frequency.

As seen in Table 4-3 above, increasing the number of elements in the DFE to 40 reduces the values of all the DFE frequencies lower than those reported by Ahmed; the maximum difference now in the first mode with the DFE frequency 0.32% smaller than the value reported by Ahmed. Although increasing the number of elements seems to have gone in the opposite direction of what it was intended, it should be

noted that Ahmed only used 10 elements in the reported FEM results [1] and based on the trend observed; increasing the number of elements will lower the values of the frequencies better matching the DFE results. Using the 40-element DF model, the mode shapes are calculated and are shown in the figures below. The mode shapes were found using values 99.99% of the actual natural frequencies of the system because displacements of the system become impossible to evaluate as the values near the natural frequencies.

Figure 4-13: Normalized mode 1 for curved symmetric sandwich beam with simply supported end conditions (1249.47 rad/s).

Figure 4-14: Normalized mode 2 for curved symmetric sandwich beam with simply supported end conditions (2472.87 rad/s).

Figure 4-15: Normalized mode 3 for curved symmetric sandwich beam with simply supported end conditions (4680.97 rad/s).

Figure 4-16: Normalized mode 4 for curved symmetric sandwich beam with simply supported end conditions (7370.11 rad/s).

Figure 4-17: Normalized mode 5 for curved symmetric sandwich beam with simply supported end conditions (10278.98 rad/s).

As can be seen from Figures 4-13 to 4-17, the mode shapes for the curved symmetric sandwich beam with simply supported end conditions are dominated by radial displacement which is the expected result due to the beam's high axial stiffness in comparison to its bending stiffness. It is worth noting that at the end points there is some axial displacement. This is in accordance with the fact that for the simply supported end condition, the circumferential displacement is not forced to zero giving the possibility of a non-zero value for displacement at the end points.

In the final numerical test uses the curved symmetrical sandwich beam formulation is applied to a straight beam case. The beam has a length of $S = 0.9144$ m, radius $R = \infty$, with face thickness $t = 0.4572$ mm and core thickness $t_c = 12.7$ mm. The mechanical properties of the face layers are: $E = 68.9$ GPa and $\rho_f = 2680$ kg/m³; while the core has properties of $G_c = 82.68$ MPa and $\rho_c = 32.8$ kg/m³. The natural frequencies of the beam are calculated using the DFE method as well as the 3-DOF and 4-DOF FEM formulations and compared to the results published by Ahmed [1] (see Table 4-4). In the case of a straight beam, the radial displacement and circumferential displacements directly translate into the flexural and axial displacements, respectively.

Table 4-4: Natural frequencies of a straight symmetric sandwich beam with both ends simply supported.

ω_n (rad/s)	Ahmed FEM 4- DOF 15 Element [1]	DFE 20 Elements	DFE 30 Elements	DFE 40 Elements	FEM – 3 DOF 20 Elements	FEM – 3 DOF 40 Elements	FEM - 4 DOF 20 Elements	FEM - 4 DOF 40 Elements
ω_1	361.35	370.02	363.55	361.41	359.27	359.02	358.9	358.9
ω_3	2938.58	3012.35	2958.63	2952.72	2940.53	2924.32	2918.92	2918.92
ω_5	6980.62	7169.24	69870.47	6993.5	7044.65	6966.04	6939.9	6939.84
ω_7	11573.63	11885.02	11667	11590.97	11739.57	11558.74	11498.35	11497.91
ω_9	16298.58	16728.98	16422.99	16315.99	16582.33	16284.01	16183.66	16181.65

The values in Table 4-4, above, shows good agreement between the calculated values for the natural frequencies for all DFE and FEM models, with the agreement between formulations getting better with the use of more elements. The maximum difference in the results when comparing the 20-element DFE to the 3-DOF FEM models is seen for the first natural frequency with the 3-DOF FEM value being 2.91% smaller. The largest between the 20-element DFE and 4-DOF FEM difference is for the ninth natural frequency where the FEM is 3.26% smaller.

Upon increasing the number of elements used in the three models to 40, the differences in the frequencies become much smaller. The largest difference between the DFE and the FEM models is the third natural frequency; with 3- and 4-DOF FEM values being 0.96% and 1.14% smaller than the DFE formulation, respectively. The results for the 40-element models suggest that the models are converging towards a similar value and better agreement between the models will be seen by increasing the number of elements further.

It should be noted that the results for the 4-DOF FEM formulation as well as Ahmed's are smaller than those calculated by the DFE, which is derived from a 3-DOF per node assumption. Ahmed [1] encountered a similar trend when comparing the 3- and 4-DOF FEM formulations and found the reason to be the additional degree of freedom at the node softens the beam's stiffness matrix that lowers the calculated natural frequencies, which is also why the DFE agrees more readily with the 3-DOF FEM models. The mode shapes were also calculated for the listed natural frequencies and are shown in the figures below. Due to the higher mode numbers, a 40-element DFE model was used for smoothness. The graphs were taken for values 99.99% of the natural frequencies since, as stated before, using the actual values for natural frequencies leads to singular global dynamic stiffness matrix, which cannot be reversed to find the corresponding natural modes of the system.

Figure 4-18: Normalized mode 1 for straight symmetric sandwich beam with simply supported end conditions (361.41 rad/s).

Figure 4-19: Normalized mode 3 for straight symmetric sandwich beam with simply supported end conditions (2952.72 rad/s).

Figure 4-20: Normalized mode 5 for straight symmetric sandwich beam with simply supported end conditions (6993.5 rad/s).

Figure 4-21: Normalized mode 7 for straight symmetric sandwich beam with simply supported end conditions (11590.97rad/s).

Figure 4-22: Normalized mode 9 for straight symmetric sandwich beam with simply supported end conditions (16315.99 rad/s).

As with the other cases, the straight simply supported symmetric sandwich beam mode shapes are dominated by the radial (flexural) displacement, with little to no involvement from the circumferential (axial) displacement. This behaviour is due to the bending stiffness of the beam being much lower than its axial stiffness. Like the previous case above, the simply-supported end conditions do not force the axial displacement to zero, hence, the possibility of non-zero axial displacements at the end points.

4.6 Concluding Remarks

Based on the theory developed by Ahmed [1, 2] and the weak integral form of the differential equations of motion, a dynamic finite element (DFE) formulation for the free vibration analysis of symmetric curved sandwich beams has been accomplished. The current formulation models the face layer as Euler-Bernoulli beams and allows the core to deform in shear only. The DFE formulation is used to calculate the natural frequencies and mode shapes for four separate test cases. The first three used the same curved beam with different end conditions: cantilever, both ends clamped and lastly, both ends simply supported. The final test case used the DFE formulation to determine the natural frequencies of a simply supported straight sandwich beam

All the numerical tests show satisfactory agreement between the results for the developed DFE, FEM and those published in literature. For all test cases, the DFE matched more closely with the 3-DOF FEM formulation than with Ahmed's 4-DOF FEM results when a similar number of elements are used. The reason for this is that the DFE is derived from the 3-DOF FEM and such a trend is expected. Ahmed [1] goes on to explain that the addition of an extra degree of freedom for each node has a tendency to lower the overall stiffness of a sandwich beam element causing an overall reduction in values of the natural frequencies. The mode shapes determined by the DFE formulation match the expectations based on previous knowledge on the behaviour of straight sandwich beams. The results of the applied DFE theory and methodology in the analysis of a curved symmetric sandwich beam demonstrate that DFE can be successfully extended from a straight beam case to produce a more general formulation.

Chapter 5: Conclusion

5.1 Overview of Research

Dynamic finite element theory was applied to analyze and study the free vibration behaviour of sandwich beams. Several different formulations were completed and the results were compared to FEM and DSM formulations in order to gauge the effectiveness of DFE as a vibrational analysis tool. The first and simplest model is a symmetrical sandwich beam where the face layers behave according to Euler-Bernoulli beam theory and the core is allowed to deform in shear only. The formulation was used to calculate the natural frequencies and mode shapes for a beam model used by Banerjee [8] and the results were compared to those presented by Banerjee as well as the results for a 20- and 40-element model constructed using classical FEM theory. All sets of results agreed very well with maximum error between the DFE model and DSM being less than 0.5%. The mode shapes found using the 20-element DFE also agree very well with the results found by Banerjee.

A more general formulation for a straight sandwich beam was needed that could be used to examine a wider range of systems than the first case. The second formulation consisted of face plates that behaved according to Rayleigh beam theory a core governed by Timoshenko beam theory. The newly developed model can be used to analyze asymmetrical sandwich beams where all three layers have different geometric and material properties. To test the accuracy of the DFE formulation several examples were taken from the literature and used as a benchmark, namely those studied by Banerjee and Sobey [10]. Alongside the DFE formulation, a model based on FEM theory was also produced. The first test case studied was an asymmetric sandwich beam with steel faces and a rubber core. The results of the DFE were impressive; for the first four natural frequencies a 2-element DFE model matched the exact DSM precisely (a 1-element DFE model had a maximum error of 0.044%), compared to a 15-element FEM mesh required to attain the same accuracy. The mode shapes determined from the DFE also matched those produced by DSM.

The second model considered was another asymmetric beam with steel face layers; however, the core was constructed from lead. The results from the numerical tests show that 20-element DFE and FEM models agree very well with each other with the maximum error between the DSM and DFE being 1.1%. The mode shapes produced by the DFE formulation also agree very well with those found by Banerjee and Sobey [10], with axial displacement having a more prominent role than in the case with a rubber core.

Lastly, the DFE formulation was tested for the degenerate symmetric straight beam case and the effect of including the core's bending and axial stiffness was explored. It was observed that, for all three DFE, FEM and DSM formulations, the inclusion of the bending and axial stiffness caused a slight increase in the natural frequencies of the beam and had little effect on its mode shapes. There was good agreement between the 30-element DFE and FEM models; however, both models calculated natural frequencies that were considerably lower than those found by Banerjee using DSM.

The third and final formulation completed was for a symmetric three-layered curved sandwich beam with the face layers governed by Euler-Bernoulli beam theory and the core assumed to deform in shear only. The formulation was tested against the FEM results presented by Ahmed [1, 2]. Two additional FEM models, with 3-DOFs and 4-DOFs per node, were also developed to be used in the testing. The same curved beam was used for each model with the boundary conditions being changed for each test case. Cantilever, doubly clamped and doubly simply supported boundary conditions were examined. For all three test cases, the results of 40-element DFE and 3-DOF FEM models agreed very well with each other. As also reported by Ahmed [1], the 4-DOF FEM formulation produced results that were consistently smaller than the 3-DOF FEM and the DFE. Ahmed explains that the additional degree of freedom per node has the overall effect of softening the element matrices thus driving down the natural frequencies of the beam. The results presented by Ahmed were produced using a 10-element 4-DOF FEM model and are higher than those calculated by all the models. Due to the low number of elements used in the model, it can be concluded that Ahmed's FEM model may not fully converge and that a model with additional elements will more closely match the DFE values. The mode shapes found for the test cases are also what are expected based on previous knowledge of the behaviour of straight beams. The final test case applies the curved beam formulation to a symmetric straight beam case. The results for the 40-element DFE and the Ahmed's FEM models agree well with each other with the maximum error between the two being 0.48% for the third natural frequency. The mode shapes calculated also match expectations for the vibration behaviour of a simply supported straight beam.

5.2 Future Work

Based on the successful implementation of DFE theory to the analysis of sandwich beams, the next logical step would be to extend the theory to other applications that impact the aerospace industry. The newly developed curved symmetric sandwich beam formulation can be extended to be more general and be applicable to asymmetric curved beams. The convergence rate of the curved beam formulation can also be increased by allowing the face layers to behave like Rayleigh beams and having the core to behave

according to Timoshenko beam theory. Banerjee [9] showed that the accuracy and convergence rate of sandwich beam models can be further increased with all three layers of the beam are assumed to follow Timoshenko beam theory. A final sandwich beam formulation where all layers behave as Timoshenko beams and is capable of analyzing an asymmetric beam that is either curved or straight is a goal that can be realized in the short term.

There are many numerical models that have been used to test and benchmark the analysis of sandwich beams. However, there is not an adequate amount of actual test data from experiments. Banerjee *et al.* [10] performed laboratory experiments on sandwich beams that they first analyzed using a numerical model in order to provide validation of DSM. The results of the testing showed that there was some disparity between the numerical model and actual model. Banerjee explains that the complex interaction at the interfaces of the layers is not taken into account and will lead into non-linearity if taken into account for the numerical model. Also, the assumptions made for the models with respect to the bonding of the different layers are almost impossible to achieve in actual sandwich beam construction. It is important that more experimental data on the vibrational behaviour of sandwich beams be collected in order to increase the accuracy and reliability of numerical models as well as provide additional results that can be used to benchmark future developments.

The DFE theory presented has also been used by Hashemi *et al.* [23-31] for the analysis of beams made of laminated composites which exhibit different couplings such as bending-torsion and extensional-torsion. This is of particular interest as the use of composites is increasing in the aerospace industry not only because of their high strength to weight ratio, but also because of the ability to have their mechanical properties customized to suit given design criteria such as additional stiffness in a certain direction or a desired coupling. York [44] has published several papers which provide definitive lists of the stacking sequences of layered composites that will yield desired behaviours. These lists can be used as a starting point to develop the DFE theory and methodology required to successfully derive a formulation that can be used to analyze layered composites. It serves as a good starting point because the vibration behaviour of the composites has been investigated using FEM tools such as MSC NASTRAN/PATRAN® and the results can be used to benchmark the results of future DFE formulations. Banerjee [4-7], Banerjee and Su [13, 14] and Banerjee and Williams [11, 12] have also applied DSM theory to the study of layered composites that have bending-torsion coupling under a number of different conditions including spinning and being centrifugally stiffened which adds to the number of applications of future DFE formulations. One could therefore envision the development of the governing differential equation, DSM, FEM and

DFE models for vibration analysis of composite sandwich configurations, where the face layers are composite laminates and the core is made of a light-weight material.

Another interesting field that can be another application of DFE theory is ultrasonic cutting which has been studied by Cardoni and Lucas [16], Lucas *et al.* [36] and MacBeath *et al.* [37]. Currently, ultrasonic cutting is being used mainly in the medical and food industry. Ultrasonic cutting has an operating frequency range of 20-40 kHz and uses a tuned blade resonant in the longitudinal mode in order to cut material via the method of crack propagation. DFE can aid in the design of ultrasonic blades by helping to identify the natural frequencies corresponding to longitudinal mode shapes. Due to the high frequencies used in ultrasonic cutting, the performance of DFE in accurately calculating higher order natural frequencies can be an asset by providing accurate results for the operating frequencies of the cutting blades while using less computing power and smaller DFE models.

5.3 Concluding Remarks

The derivation and application of the DFE formulation to sandwich beam construction has shown to be successful and provides results that match well with other free vibration analysis methods such as FEM and DSM. DFE performs reasonably against classical FEM theory and in certain cases, such as sandwich beams with very soft cores, outperforms FEM in terms of the convergence rate. Based on the findings of this research, extending the DFE methodology into other areas including composite sandwich configurations, ultrasonic cutting, etc., can provide some additional tools in the field of structural analysis. Dynamic finite element theory offers another tool for structural analysis meant to compliment the finite element method (FEM) and the dynamic stiffness (DSM) methods.

Appendix A1: FEM Program Flowchart

Figure A1-1 below shows the basic process flowchart for the FEM program used to calculate mode shapes.

Figure A1-1: Process flowchart for FEM program.

Appendix A2: DFE Program Flowcharts

The DFE program was separated into two separate parts because of the computing power required to perform the integrations of the DFE weak form differential equations of motion. The integrations are performed separately and the element matrices are calculated (the program process flowchart can be found in Figure A2-1) and then imported into the program that performs the frequency sweep and is actually used to find the natural frequencies (Figure A2-2).

Figure A2-1: Process flowchart for program used to calculate the element matrix for the DFE formulation.

Figure A2-2: Process flowchart for program used to perform the frequency sweep and find the natural frequencies for the DFE formulation.

Appendix A3: Development of Equations of Motion for Euler-Bernoulli/Shear Straight Symmetric Sandwich Beam Formulation

For the symmetric sandwich beam described in Section 2.2, where the faces deform according to Euler-Bernoulli beam theory and the core is allowed to deform in shear only, Banerjee [8] develops the equations of motion, (2 – 1) and (2 – 2) from the strain and kinetic energy of the system using Hamilton's principle. Hamilton's principle requires that:

$$\delta \int_{t_1}^{t_2} (T - V) dt = 0 \quad (A3 - 1)$$

Where δ is the variational operator, T and V are the kinetic and strain energies of the sandwich beam, respectively and t_1 and t_2 are an arbitrary time interval. The kinetic and strain energy of the system are defined in equations (A3 – 2) and (A3 – 3), below.

$$T = \int_0^L m \dot{u}^2 dx + \int_0^L m \dot{w}^2 dx \quad (A3 - 2)$$

$$V = \underbrace{E_1 A_1 \int_0^L (u_1')^2 dx + E_1 I_1 \int_0^L (w_1'')^2 dx}_{\text{Shear strain of the face layers}} + \underbrace{\frac{1}{2} k_2 A_2 G_2 \times \int_0^L \{2u/h_c + (1 + h_f/h_c)w'\}^2 dx}_{\text{Shear strain of the core}} \quad (A3 - 3)$$

In the equations above, a prime denotes differentiation with respect to x and an overhead dot denotes differentiation with respect to time, t . Substituting equations (A3 – 2) and (A3 – 3) into (A3 – 1) yields:

$$\delta \int_{t_1}^{t_2} \int_0^L \left[(m \dot{u}^2 + m \dot{w}^2) - E_1 A_1 (u')^2 - E_1 I (w'')^2 - (k_2 A_2 G_2 / 2) \{2u/h_c + (1 + h_f/h_c)w'\}^2 \right] dx dt = 0 \quad (A3 - 4)$$

Performing integration by parts and collecting terms in δu and δw gives the differential equations of motion:

$$E_1 A_1 u'' - (2 k_2 A_2 G_2 / h_c^2) u - (k_2 A_2 G_2 / h_c) (1 + h_f / h_c) w' - m \ddot{u} = 0 \quad (A3 - 5)$$

$$E_1 I_1 w'''' - (k_2 A_2 G_2 / 2) (1 + h_f / h_c)^2 w'' - (k_2 A_2 G_2 / h_c) (1 + h_f / h_c) u' + m \ddot{w} = 0 \quad (A3 - 6)$$

Assuming harmonic motion:

$$u(x, t) = U(x) e^{i\omega t} \quad \text{and} \quad w(x, t) = W(x) e^{i\omega t} \quad (A3 - 7)$$

Substituting (A3 - 7) into equations (A3 - 5) and (A3 - 6), performing the necessary differentiations and removing all common terms yield the differential equations of motion (2 - 1) and (2 - 2) seen in Section 2.2.

$$(E_1 A_1) u'' - (2 k_2 A_2 G_2 / h_c^2) u - (k_2 A_2 G_2 / h_c) (1 + h_f / h_c) w' + m \omega^2 u = 0 \quad (2 - 1)$$

$$(E_1 I_1) w'''' - (k_2 A_2 G_2 / 2) (1 + h_f / h_c)^2 w'' - (k_2 A_2 G_2 / h_c) (1 + h_f / h_c) u' - m \omega^2 w = 0 \quad (2 - 2)$$

Banerjee [8] provides a more in-depth explanation of development of the differential equations of motion if any clarification is needed.

Appendix A4: Development of Equations of Motion for Rayleigh/Timoshenko Straight Sandwich Beam Formulation

For the straight sandwich beam described in Section 3.2, where the faces deform according to Rayleigh beam theory and the core behaves according to Timoshenko beam theory, Banerjee [10] develops the equations of motion, (3 – 1), (3 – 2) and (3 – 3) from the strain and kinetic energy of the system using Hamilton's principle. Hamilton's principle requires that:

$$\delta \int_{t_1}^{t_2} (T - V) dt = 0 \quad (A4 - 1)$$

Where δ is the variational operator, T and V are the kinetic and strain energies of the sandwich beam, respectively and t_1 and t_2 are an arbitrary time interval. The kinetic and strain energy of the system are defined in equations (A4 – 2) and (A4 – 3), below.

$$\begin{aligned} T &= \frac{1}{2} \int_0^L \left[\underbrace{m_1(\dot{u}_1)^2 + \rho_1 I_1 (\dot{w}')^2}_{\text{Kinetic energy due to axial motion in layer 1}} \right. \\ &+ \underbrace{\frac{m_2}{3} \left\{ (\dot{u}_1)^2 + \dot{u}_1 \dot{u}_3 + (\dot{u}_3)^2 + \left(h_1 - \frac{h_3}{2} \right) \dot{u}_1 \dot{w}' + \left(\frac{h_1}{2} - h_3 \right) \dot{u}_3 \dot{w}' + \frac{1}{4} (h_1^2 - h_1 h_3 + h_3^2) (\dot{w}')^2 \right\}}_{\text{Kinetic energy due to axial motion of core layer}} \\ &+ \underbrace{m_3(\dot{u}_3)^2 + \rho_3 I_3 (\dot{w}')^2}_{\text{Kinetic energy due to axial motion of layer 3}} \\ &+ \underbrace{M \dot{w}^2}_{\text{Kinetic energy due to transverse motion of entire beam}} \left. \right] dx \quad (A4 - 2) \end{aligned}$$

$V =$

$$\frac{1}{2} \int_0^L \left[\underbrace{E_1 A_1 (u_1')^2 + E_1 I_1 (w'')^2}_{\text{Strain energy in layer 1 due to normal strain}} + \right. \\ \left. E_2 A_2 u_1'^2 + u_1' u_3' + u_3'^2 + h_1 - h_3 u_1' w'' + h_1 u_3' w'' + 14 h_1 u_3' w'' + h_1 h_3 + h_3 u_3' w'' \right] \text{Strain} \\ \text{energy in layer 2 due to normal strain} + E_3 A_3 u_3'^2 + E_3 I_3 w''^2 \text{Strain energy in layer 2} \\ \text{due to normal strain} + k_2 A_2 G_2 h_2 u_1 - u_3 + a w'^2 \text{Strain energy in layer 2 due to shear} \\ \text{strain} dx \quad (A4 - 3)$$

In the equations above, a prime denotes differentiation with respect to x and an overhead dot denotes differentiation with respect to time, t . Substituting equations (A4 - 2) and (A4 - 3) into equation (A4 - 1) yields:

$$\delta \frac{1}{2} \int_{t_1}^{t_2} \int_0^L \left\{ B_1 \dot{u}_1^2 + 2 B_6 \dot{u}_1 \dot{u}_3 + B_2 \dot{u}_3^2 + 2 B_5 \dot{u}_1 \dot{w}' + 2 B_4 \dot{u}_3 \dot{w}' + B_3 (\dot{w}')^2 + M \dot{w}^2 - C_1 (u_1')^2 - \right. \\ \left. 2 C_6 u_1' u_3' - 2 C_2 u_3'^2 - 2 C_5 u_1' w'' - 2 C_4 u_3' w'' - C_3 w''^2 - C_7 u_1^2 - 2 C_{12} u_1 u_3 - C_8 u_3^2 - 2 C_{11} u_1 \right. \\ \left. w' - 2 C_{10} u_3 w' - C_9 w'^2 \right\} dx \quad (A4 - 4)$$

Performing integration by parts and collecting terms in δu_1 , δu_3 and δw gives the differential equations of motion:

$$\left(-B_1 \frac{\partial^2}{\partial t^2} + C_1 \frac{\partial^2}{\partial x^2} - C_7 \right) u_1 + \left(-B_6 \frac{\partial^2}{\partial t^2} + C_6 \frac{\partial^2}{\partial x^2} - C_{12} \right) u_3 \\ + \left(-B_5 \frac{\partial^2}{\partial t^2} + C_5 \frac{\partial^2}{\partial x^2} - C_{11} \right) \frac{\partial w}{\partial x} = 0 \quad (A4 - 5)$$

$$\left(-B_6 \frac{\partial^2}{\partial t^2} + C_6 \frac{\partial^2}{\partial x^2} - C_{12} \right) u_1 + \left(-B_2 \frac{\partial^2}{\partial t^2} + C_2 \frac{\partial^2}{\partial x^2} - C_8 \right) u_3 \\ + \left(-B_4 \frac{\partial^2}{\partial t^2} + C_4 \frac{\partial^2}{\partial x^2} - C_{10} \right) \frac{\partial w}{\partial x} = 0 \quad (A4 - 6)$$

$$\left(-B_5 \frac{\partial^2}{\partial t^2} + C_5 \frac{\partial^2}{\partial x^2} - C_7 \right) \frac{\partial u_1}{\partial x} + \left(-B_4 \frac{\partial^2}{\partial t^2} + C_4 \frac{\partial^2}{\partial x^2} - C_{10} \right) \frac{\partial u_3}{\partial x} \\ + \left(-B_3 \frac{\partial^4}{\partial x^2 \partial t^2} + C_3 \frac{\partial^4}{\partial x^4} - C_9 \frac{\partial^2}{\partial x^2} + M \frac{\partial^2}{\partial t^2} \right) w = 0 \quad (A4 - 7)$$

Assuming harmonic motion:

$$u_1(x, t) = U_1(x)e^{i\omega t}; \quad u_3(x, t) = U_3(x)e^{i\omega t} \quad \text{and} \quad w(x, t) = W(x)e^{i\omega t} \quad (\text{A4} - 8)$$

Substituting (A4 - 8) into equations (A4 - 5), (A4 - 6) and (A4 - 7), performing the necessary differentiations and removing all common terms yields the differential equations of motion (3 - 1), (3 - 2) and (3 - 3) seen in Section 3.2.

$$\begin{aligned} & \left(-B_1 \frac{\partial^2}{\partial t^2} + C_1 \frac{\partial^2}{\partial x^2} - C_7 \right) u_1 + \left(-B_6 \frac{\partial^2}{\partial t^2} + C_6 \frac{\partial^2}{\partial x^2} - C_{12} \right) u_3 \\ & + \left(-B_5 \frac{\partial^2}{\partial t^2} + C_5 \frac{\partial^2}{\partial x^2} - C_{11} \right) \frac{\partial w}{\partial x} = 0 \end{aligned} \quad (3 - 1)$$

$$\begin{aligned} & \left(-B_6 \frac{\partial^2}{\partial t^2} + C_6 \frac{\partial^2}{\partial x^2} - C_{12} \right) u_1 + \left(-B_2 \frac{\partial^2}{\partial t^2} + C_2 \frac{\partial^2}{\partial x^2} - C_8 \right) u_3 \\ & + \left(-B_4 \frac{\partial^2}{\partial t^2} + C_4 \frac{\partial^2}{\partial x^2} - C_{10} \right) \frac{\partial w}{\partial x} = 0 \end{aligned} \quad (3 - 2)$$

$$\begin{aligned} & \left(-B_5 \frac{\partial^2}{\partial t^2} + C_5 \frac{\partial^2}{\partial x^2} - C_{11} \right) \frac{\partial u_1}{\partial x} + \left(-B_4 \frac{\partial^2}{\partial t^2} + C_4 \frac{\partial^2}{\partial x^2} - C_{10} \right) \frac{\partial u_3}{\partial x} \\ & + \left(-B_3 \frac{\partial^2}{\partial x^3 \partial t^2} + C_3 \frac{\partial^4}{\partial x^4} + C_9 \frac{\partial^2}{\partial x^2} - M \frac{\partial^2}{\partial t^2} \right) = 0 \end{aligned} \quad (3 - 3)$$

Banerjee [10] provides a more in-depth explanation of development of the differential equations of motion if any clarification is needed.

Appendix A5: Development of Equations of Motion for Euler-Bernoulli/Shear Curved Symmetric Sandwich Beam Formulation

For the symmetric curved sandwich beam described in Section 4.2, where the faces deform according to Euler-Bernoulli beam theory and the core deforms in shear only, Ahmed [1] develops the equations of motion, (4 – 1) and (4 – 2) from the strain and kinetic energy of the system using the principle of minimum total potential energy which requires that:

$$\delta(U - T) = 0 \quad (A5 - 1)$$

Where δ is the variational operator, U and T are the strain and kinetic energies of the sandwich beam, respectively and t_1 and t_2 are an arbitrary time interval. The kinetic and strain energy of the system are defined in equations (A5 – 2) and (A5 – 3), below.

$$U = \frac{1}{2} \int_0^S \left\{ \left[\alpha^2 \left(\frac{\partial v}{\partial y} \right)^2 + \left(\frac{w}{R} \right)^2 \right] + \gamma^2 \left(\frac{\partial^2 w}{\partial y^2} \right)^2 + \beta^2 \left(2v + h \frac{\partial w}{\partial y} \right)^2 \right\} dy \quad (A5 - 2)$$

$$T = \frac{1}{2} \omega^2 \int_0^S (Q_1 v^2 + Q_2 w^2) dy \quad (A5 - 3)$$

Substituting equations (A5 – 2) and (A5 – 3) into equation (A5 – 1) yields:

$$\delta \left(\frac{1}{2} \int_0^S \left\{ \left[\alpha^2 \left(\frac{\partial v}{\partial y} \right)^2 + \left(\frac{w}{R} \right)^2 \right] + \gamma^2 \left(\frac{\partial^2 w}{\partial y^2} \right)^2 + \beta^2 \left(2v + h \frac{\partial w}{\partial y} \right)^2 - (Q_1 v^2 + Q_2 w^2) \right\} dy \right) = 0 \quad (A5 - 4)$$

Performing integration by parts and collecting terms in δv , and δw gives the differential equations of motion (4 – 1) and (4 – 2):

$$\frac{\partial^2 v}{\partial y^2} + \frac{1}{\alpha^2} (\omega^2 Q_1 - 4\beta^2) v - \frac{2h\beta^2}{\alpha^2} \frac{\partial w}{\partial y} = 0 \quad (4 - 1)$$

$$\frac{\partial^4 w}{\partial y^4} - \frac{h^2 \beta^2}{\gamma^2} \frac{\partial^2 w}{\partial y^2} + \frac{1}{\gamma^2} \left(\frac{\alpha^2}{R^2} - \omega^2 Q_2 \right) w - \frac{2h\beta^2}{\alpha^2} \frac{\partial v}{\partial y} = 0 \quad (4 - 2)$$

Ahmed [1] provides a more in-depth explanation of development of the differential equations of motion if any clarification is needed.

Appendix A6: Development of Dynamic Finite Element Stiffness Matrix for Euler-Bernoulli/Shear Straight Symmetric Sandwich Beam Formulation

The Dynamic Finite Element stiffness matrix for a symmetric straight sandwich beam is developed from equations (2 – 12) and (2 – 13) found in Section 2.4. Applying the approximations for the element variables and the test functions shown in equations (2 – 20) and (2 – 21) to equations (2 – 12) and (2 – 13) yield the DFE stiffness matrix defined in equation (2 – 22).

Looking first at equation (2 – 12):

$$\begin{aligned}
 W_u^k = & \underbrace{- \int_0^l ((\delta u' E_1 A_1)' + \delta u m \omega^2) u dx}_{(*)} + \underbrace{\int_0^l \delta u (2k_2 A_2 G_2 / h_c^2) u dx + [\delta u' (E_1 A_1) u]_0^l}_{[k]_u^k \text{ Uncoupled}} \\
 & + \underbrace{\int_0^l \delta u (k_2 A_2 G_2 / h_c) (1 + h_f / h_c) w' dx}_{[K_{uw}]_{2 \times 4} \text{ Coupling}}
 \end{aligned} \tag{2 – 12}$$

The first term in equation (2 – 12), (*), becomes zero due to the choice of the basis function for $u(x)$. The next two terms, produce a symmetric 2x2 matrix $[k]_u^k$ that contains all the uncoupled stiffness matrix elements associated with the displacement $u(x)$. The final term, produces a 2x4 matrix $[k_{uw}]$ that contain all the terms that couple the displacement $u(x)$ with $w(x)$.

$$[k]_u^k = \begin{bmatrix} k_u(1,1) & k_u(1,2) \\ k_u(2,1) & k_u(2,2) \end{bmatrix} \tag{A6 – 1}$$

$$[k_{uw}] = \begin{bmatrix} k_{uw}(1,1) & k_{uw}(1,2) & k_{uw}(1,3) & k_{uw}(1,4) \\ k_{uw}(2,1) & k_{uw}(2,2) & k_{uw}(2,3) & k_{uw}(2,4) \end{bmatrix} \tag{A6 – 2}$$

Looking next at equations (2 – 13):

$$\begin{aligned}
W_w^k = & \underbrace{\int_0^l \left(\delta w''''(E_1 I_1) - \delta w''(k_2 A_2 G_2 / 2) (1 + h_f / h_c)^2 - \delta w(m \omega^2) \right) w dx}_{(**)} \\
& + \underbrace{[\delta w''(E_1 I_1) w']_0^l - [\delta w'''(E_1 I_1) w]_0^l + [\delta w'(k_2 A_2 G_2 / 2) (1 + h_f / h_c)^2 w]_0^l}_{[k]_w^k \text{ Uncoupled}} \\
& + \underbrace{\int_0^l \delta w'(k_2 A_2 G_2 / h_c) (1 + h_f / h_c) u dx}_{[K_{wu}]_{4 \times 2} \text{ Coupling}} \quad (2-13)
\end{aligned}$$

The first integral term in equation (13), (**), becomes zero due to the choice of the basis function for $w(x)$. The next three terms, produce a symmetric 4x4 matrix $[k]_w^k$ that contain all the uncoupled stiffness matrix elements associated with the displacement $w(x)$. The final term, produces a 4x2 matrix $[k_{wu}]$ that contain all the terms that couple the displacement $w(x)$ with $u(x)$. It is important to note that $[k_{wu}] = [k_{uw}]^T$.

$$[k]_w^k = \begin{bmatrix} k_w(1,1) & k_w(1,2) & k_w(1,3) & k_w(1,4) \\ & k_w(2,2) & k_w(2,3) & k_w(2,4) \\ & & k_w(3,3) & k_w(3,4) \\ \text{sym.} & & & k_w(4,4) \end{bmatrix} \quad (A6-3)$$

$$[k_{wu}] = \begin{bmatrix} k_{wu}(1,1) & k_{wu}(1,2) \\ k_{wu}(2,1) & k_{wu}(2,2) \\ k_{wu}(3,1) & k_{wu}(3,2) \\ k_{wu}(4,1) & k_{wu}(4,2) \end{bmatrix} \quad (A6-4)$$

Matrices (A6-1), (A6-2), (A6-3) and (A6-4) are added according to equation (2-22) in order to obtain the 6x6 element stiffness matrix for a symmetric straight sandwich beam.

$$[K] = \begin{bmatrix} k_u(1,1) & k_{uw}(1,1) & k_{uw}(1,2) & k_u(1,2) & k_{uw}(1,3) & k_{uw}(1,4) \\ & k_w(1,1) & k_w(1,2) & k_{wu}(1,2) & k_w(1,3) & k_w(1,4) \\ & & k_w(2,2) & k_{wu}(2,2) & k_w(2,3) & k_w(2,4) \\ & & & k_u(2,2) & k_{uw}(2,3) & k_{uw}(2,4) \\ & & & & k_w(3,3) & k_w(3,4) \\ \text{sym.} & & & & & k_w(4,4) \end{bmatrix} \quad (A6-5)$$

Appendix A7: Development of Dynamic Finite Element Stiffness Matrix for Rayleigh/Timoshenko Straight Symmetric Sandwich Beam Formulation

The Dynamic Finite Element stiffness matrix for a straight sandwich beam is developed from equations (3 – 19), (3 – 20) and (3 – 21) found in Section 3.4. Applying the approximations for the element variables and the test functions shown in equations (3 – 30), (3 – 31) and (3 – 32) to equations (3 – 19), (3 – 20) and (3 – 21) yield the DFE stiffness matrix defined in equation (3 – 33).

Looking first at equation (3 – 19):

$$\begin{aligned}
 W_{u1}^i = & - \underbrace{\int_0^l u_1 (C_1 \delta u_1'' + \omega^2 B_1 \delta u_1) dx}_{(***)} + \underbrace{\int_0^l \delta u_1 C_7 u_1 dx}_{[K]_{u1}^k \text{ Uncoupled Shear Term}} + \underbrace{[\delta u_1' C_1 u_1]_0^l}_{[K]_{u1}^k \text{ Uncoupled}} \\
 & + \underbrace{\int_0^l \delta u_1' C_6 u_3' dx - \int_0^l \delta u_1 (\omega^2 B_6 - C_{12}) u_3 dx}_{[K_{u1u3}]_{2 \times 2} \text{ Coupling}} \\
 & + \underbrace{\int_0^l \delta u_1' C_5 w'' dx - \int_0^l \delta u_1 (\omega^2 B_5 - C_{11}) w' dx}_{[K_{u1w}]_{2 \times 4} \text{ Coupling}}
 \end{aligned} \tag{3 – 19}$$

For frequencies below the changeover frequency of the top face layer, the first term (***) become zero as a result of the choice of basis function and the second and third terms yield a symmetric 2x2 matrix $[k]_{u1}^k$ that contains all the uncoupled stiffness matrix elements associated with the displacement $u_1(x)$. For frequencies above the changeover frequency of the top layer, the first two integral terms of equation (3 – 19) are zero, and the third term of equation (3 – 19) yield a symmetric 2x2 matrix $[k]_{u1}^k$ that contains all the uncoupled stiffness matrix elements associated with the displacement $u_1(x)$.

$$[k]_{u1}^k = \begin{bmatrix} k_{u1}(1,1) & k_{u1}(1,2) \\ k_{u1}(2,1) & k_{u1}(2,2) \end{bmatrix} \tag{A7 – 1}$$

The fourth term of equation (3 – 19) produces a 2x2 coupling matrix $[k_{u1u3}]$ that contain all the terms that couple the displacements $u_1(x)$ with $u_3(x)$. The final term in equation (3 – 19) produces a 2x4 coupling matrix $[k_{u1w}]$ that contain all the terms that couple the displacements $u_1(x)$ and $w(x)$.

$$[k]_{u1u3}^k = \begin{bmatrix} k_{u1u3}(1,1) & k_{u1u3}(1,2) \\ k_{u1u3}(2,1) & k_{u1u3}(2,2) \end{bmatrix} \quad (A7 - 2)$$

$$[k_{u1w}] = \begin{bmatrix} k_{u1w}(1,1) & k_{u1w}(1,2) & k_{u1w}(1,3) & k_{u1w}(1,4) \\ k_{u1w}(2,1) & k_{u1w}(2,2) & k_{u1w}(2,3) & k_{u1w}(2,4) \end{bmatrix} \quad (A7 - 3)$$

Studying equation (3 – 20) next:

$$\begin{aligned} W_{u3}^i = & - \underbrace{\int_0^l u_3 (C_2 \delta u_3'' + \omega^2 B_2 \delta u_3) dx}_{(****)} + \underbrace{\int_0^l \delta u_3 C_8 u_3 dx}_{[K]_{u3}^k \text{ Uncoupled Shear Term}} + \underbrace{[\delta u_3' C_2 u_3]_0^l}_{[K]_{u3}^k \text{ Uncoupled}} \\ & + \underbrace{\int_0^l \delta u_3' C_6 u_1' dx - \int_0^l \delta u_3 (\omega^2 B_6 - C_{12}) u_1 dx}_{[K_{u3u1}]_{2 \times 2} \text{ Coupling}} \\ & + \underbrace{\int_0^l \delta u_3' C_4 w'' dx - \int_0^l \delta u_3 (\omega^2 B_4 - C_{10}) w' dx}_{[K_{u3w}]_{2 \times 4} \text{ Coupling}} \end{aligned} \quad (3 - 18)$$

For frequencies below the changeover frequency of the bottom face layer, the first term (****) become zero as a result of the choice of basis function and the second and third terms yield a symmetric 2x2 matrix $[k]_{u3}^k$ that contains all the uncoupled stiffness matrix elements associated with the displacement $u_3(x)$. For frequencies above the changeover frequency of the top layer, the first two integral terms of equation (3 – 20) are zero, and the third term of equation (3 – 20) yield a symmetric 2x2 matrix $[k]_{u3}^k$ that contains all the uncoupled stiffness matrix elements associated with the displacement $u_l(x)$.

$$[k]_{u3}^k = \begin{bmatrix} k_{u3}(1,1) & k_{u3}(1,2) \\ k_{u3}(2,1) & k_{u3}(2,2) \end{bmatrix} \quad (A7 - 4)$$

The fourth term of equation (3 – 20) produces a 2x2 coupling matrix $[k_{u3u1}]$ that contain all the terms that couple the displacements $u_3(x)$ with $u_l(x)$. It should be stated that $[k_{u3u1}] = [k_{u1u3}]^T$. The final term in equation (3 – 20) produces a 2x4 coupling matrix $[k_{u3w}]$ that contain all the terms that couple the displacements $u_3(x)$ and $w(x)$.

$$[k]_{u3u1}^k = \begin{bmatrix} k_{u3u1}(1,1) & k_{u3u1}(1,2) \\ k_{u3u1}(2,1) & k_{u3u1}(2,2) \end{bmatrix} \quad (A7 - 5)$$

$$[k_{u3w}] = \begin{bmatrix} k_{u3w}(1,1) & k_{u3w}(1,2) & k_{u3w}(1,3) & k_{u3w}(1,4) \\ k_{u3w}(2,1) & k_{u3w}(2,2) & k_{u3w}(2,3) & k_{u3w}(2,4) \end{bmatrix} \quad (A7 - 6)$$

Looking lastly at equation (3 – 21):

$$\begin{aligned}
W_w^i = & \underbrace{\int_0^l w(C_3 \delta w'''' - (C_9 - \omega^2 B_3) \delta w'' - \omega^2 M \delta w) dx}_{(*)} \\
& - \underbrace{[\delta w''' C_3 w]_0^l + [\partial w'' C_3 w']_0^l - [\delta w'(\omega^2 B_3 - C_9) w]_0^l}_{[K]_w^k \text{ Uncoupled}} \\
& + \underbrace{\int_0^l \delta w'' C_5 u_1' dx - \int_0^l \delta w'(\omega^2 B_5 - C_{11}) u_1 dx}_{[K_{wu1}]_{4 \times 2} \text{ Coupling}} \\
& + \underbrace{\int_0^l \delta w'' C_4 u_3' dx - \int_0^l \delta w'(\omega^2 B_4 - C_{10}) u_3 dx}_{[K_{wu3}]_{4 \times 2} \text{ Coupling}}
\end{aligned} \tag{3 – 21}$$

The first integral term in equation (3 – 21), (*), becomes zero due to the choice of the basis function for $w(x)$. The next three terms, produce a symmetric 4x4 matrix $[k]_w^k$ that contain all the uncoupled stiffness matrix elements associated with the displacement $w(x)$. The fifth and sixth terms, produce a 4x2 matrix $[k_{wu1}]$ that contain all the terms that couple the displacement $w(x)$ with $u_1(x)$. It is important to note that $[k_{wu1}] = [k_{u1w}]^T$. The seventh and eighth terms a 4x2 matrix $[k_{wu3}]$ that contain all the terms that couple the displacement $w(x)$ with $u_3(x)$. It is important to note that $[k_{wu3}] = [k_{u3w}]^T$.

$$[k]_w^k = \begin{bmatrix} k_w(1,1) & k_w(1,2) & k_w(1,3) & k_w(1,4) \\ & k_w(2,2) & k_w(2,3) & k_w(2,4) \\ & & k_w(3,3) & k_w(3,4) \\ \text{sym.} & & & k_w(4,4) \end{bmatrix} \tag{A7 – 7}$$

$$[k_{wu1}] = \begin{bmatrix} k_{wu1}(1,1) & k_{wu1}(1,2) \\ k_{wu1}(2,1) & k_{wu1}(2,2) \\ k_{wu1}(3,1) & k_{wu1}(3,2) \\ k_{wu1}(4,1) & k_{wu1}(4,2) \end{bmatrix} \tag{A7 – 8}$$

$$[k_{wu3}] = \begin{bmatrix} k_{wu3}(1,1) & k_{wu3}(1,2) \\ k_{wu3}(2,1) & k_{wu3}(2,2) \\ k_{wu3}(3,1) & k_{wu3}(3,2) \\ k_{wu3}(4,1) & k_{wu3}(4,2) \end{bmatrix} \tag{A7 – 9}$$

Matrices (A7 – 1), (A7 – 2), (A7 – 3), (A7 – 4), (A7 – 5), (A7 – 6), (A7 – 7), (A7 – 8) and (A9 – 9) are added according to equation (3 – 33) in order to obtain the 8x8 element stiffness matrix for a straight sandwich beam.

$[K] =$

$$\begin{bmatrix}
 k_{u1}(1,1) & k_{u1u3}(1,1) & k_{u1w}(1,1) & k_{u1w}(1,2) & k_{u1}(1,2) & k_{u1u3}(1,2) & k_{u1w}(1,3) & k_{u1w}(1,4) \\
 & k_{u3}(1,1) & k_{u3w}(1,1) & k_{u3w}(1,2) & k_{u3u1}(1,2) & k_{u3}(1,2) & k_{u3w}(1,3) & k_{u3w}(1,4) \\
 & & k_w(1,1) & k_w(1,2) & k_{wu1}(1,2) & k_{wu3}(1,2) & k_w(1,3) & k_w(1,4) \\
 & & & k_w(2,2) & k_{wu1}(2,2) & k_{wu3}(2,2) & k_w(2,3) & k_w(2,4) \\
 & & & & k_{u1}(2,2) & k_{u1u3}(2,2) & k_{u1w}(2,3) & k_{u1w}(2,4) \\
 & & & & & k_{u3}(2,2) & k_{u3w}(2,3) & k_{u3w}(2,4) \\
 & & & & & & k_w(3,3) & k_w(3,4) \\
 \text{sym.} & & & & & & & k_w(4,4)
 \end{bmatrix}$$

(A7 - 10)

Appendix A8: Development of Dynamic Finite Element Stiffness Matrix for Euler-Bernoulli/Shear Curved Symmetric Sandwich Beam Formulation

The Dynamic Finite Element stiffness matrix for a symmetric curved sandwich beam is developed from equations (4 – 12) and (4 – 13) found in Section 4.4. Applying the approximations for the element variables and the test functions shown in equations (4 – 20) and (4 – 21) to equations (4 – 12) and (4 – 13) yield the DFE stiffness matrix defined in equation (4 – 22).

Looking first at equation (4 – 12):

$$W_v^k = - \underbrace{\int_0^l (\delta v'' \alpha^2 + \delta v \omega^2 Q_1) v dy}_{\text{*****}} + \underbrace{\int_0^l \delta v 4 \beta^2 v dy + [\delta v' \alpha^2 v]_0^l}_{[k]_v^k \text{ Uncoupled}} + \underbrace{\int_0^l \delta v 2 h \beta^2 w' dy}_{[k_{vw}]_{2 \times 4} \text{ Coupling}} \quad (4 - 12)$$

The first term in equation (4 – 12), (*****), becomes zero due to the choice of the basis function for $v(y)$. The next two terms, produce a symmetric 2x2 matrix $[k]_v^k$ that contains all the uncoupled stiffness matrix elements associated with the displacement $v(y)$. For the test cases, the changeover frequency for the faces is well above the range of frequencies being studied meaning the shear term is left out of the integral term (*****). The final term, produces a 2x4 matrix $[k_{uw}]$ that contain all the terms that couple the displacement $v(y)$ with $w(y)$.

$$[k]_v^k = \begin{bmatrix} k_v(1,1) & k_v(1,2) \\ k_v(2,1) & k_v(2,2) \end{bmatrix} \quad (A8 - 1)$$

$$[k_{vw}] = \begin{bmatrix} k_{vw}(1,1) & k_{vw}(1,2) & k_{vw}(1,3) & k_{vw}(1,4) \\ k_{vw}(2,1) & k_{vw}(2,2) & k_{vw}(2,3) & k_{vw}(2,4) \end{bmatrix} \quad (A8 - 2)$$

Looking next at equations (4 – 13):

$$W_w^k = \underbrace{\int_0^l \left(\delta w'''' \gamma^2 - \delta w'' h^2 \beta^2 + \delta w \left(\frac{\alpha^2}{R^2} - \omega^2 Q_2 \right) \right) w dy}_{\text{*****}} + \underbrace{[\delta w'' \gamma^2 w']_0^l - [\delta w'''' \gamma^2 w]_0^l + [\delta w' h^2 \beta^2 w]_0^l}_{[k]_w^k \text{ Uncoupled}} + \underbrace{\int_0^l \delta w' 2 \beta^2 h v dy}_{[k_{vw}]_{4 \times 2} \text{ Coupling}} \quad (4 - 13)$$

The first integral term in equation (4 – 13), (*****), becomes zero due to the choice of the basis function for $w(y)$. The next three terms, produce a symmetric 4x4 matrix $[k]_w^k$ that contain all the uncoupled stiffness matrix elements associated with the displacement $w(y)$. The final term, produces a 4x2 matrix $[k_{wv}]$ that contain all the terms that couple the displacement $w(y)$ with $v(y)$. It is important to note that $[k_{wv}] = [k_{vw}]^T$.

$$[k]_w^k = \begin{bmatrix} k_w(1,1) & k_w(1,2) & k_w(1,3) & k_w(1,4) \\ & k_w(2,2) & k_w(2,3) & k_w(2,4) \\ & & k_w(3,3) & k_w(3,4) \\ \text{sym.} & & & k_w(4,4) \end{bmatrix} \quad (A8 - 3)$$

$$[k_{wv}] = \begin{bmatrix} k_{wv}(1,1) & k_{wv}(1,2) \\ k_{wv}(2,1) & k_{wv}(2,2) \\ k_{wv}(3,1) & k_{wv}(3,2) \\ k_{wv}(4,1) & k_{wv}(4,2) \end{bmatrix} \quad (A8 - 4)$$

Matrices (A8 – 1), (A8 – 2), (A8 – 3) and (A8 – 4) are added according to equation (4 – 22) in order to obtain the 6x6 element stiffness matrix for a symmetric straight sandwich beam.

$$[K] = \begin{bmatrix} k_v(1,1) & k_{vw}(1,1) & k_{vw}(1,2) & k_v(1,2) & k_{vw}(1,3) & k_{vw}(1,4) \\ & k_w(1,1) & k_w(1,2) & k_{wv}(1,2) & k_w(1,3) & k_w(1,4) \\ & & k_w(2,2) & k_{wv}(2,2) & k_w(2,3) & k_w(2,4) \\ & & & k_v(2,2) & k_{vw}(2,3) & k_{vw}(2,4) \\ & & & & k_w(3,3) & k_w(3,4) \\ \text{sym.} & & & & & k_w(4,4) \end{bmatrix} \quad (A8 - 5)$$

Appendix A9: Additional Test Cases for Rayleigh/Timoshenko Formulation

The Rayleigh/Timoshenko DFE formulation for straight sandwich beams was applied to additional numerical test cases taken from Howson and Zare [33] and compared to the DSM results. However, the agreement between the values calculated using DFE theory and additional FEM analysis was unsatisfactory compared to the results presented by Howson and Zare. The main reason for the discrepancies can be attributed to the differences in the assumptions made in the displacement fields and general behaviour of the sandwich beam which were then used to derive the equations of motions and eventually the values of the natural frequencies. For posterity, the results of have been included for three different test cases.

The first test case is a symmetric sandwich beam with cantilever end conditions. The beam is of length, $L = 0.7112$ m and width, $b = 25.4$ mm with the top and bottom faces of thickness, $t_1 = t_2 = 0.4572$ mm and a core thickness, $t_c = 12.7$ mm. The material properties of the face layers are: Young's modulus $E_1 = E_2 = 68.9$ GPa and density, $\rho_1 = \rho_2 = 2680$ kg/m³. The core layer has material properties: shear modulus $G = 82.68$ MPa and density $\rho_c = 32.8$ kg/m³. The results of the DFE analysis of the natural frequencies are presented in Table A9-1 below along with the DSM results of Howson and Zare [33] and a FEM model developed and used for additional testing.

Table A9-1: Natural frequencies of a symmetric sandwich beam with cantilever end conditions taken from Howson and Zare [33].

ω_n (rad/s)	Howson DSM [33]	DFE 20 Elements	DFE 30 Elements	DFE 40 Elements	FEM 20 Elements	FEM 30 Elements	FEM 40 Elements
ω_1	212.067	213.50	211.86	211.46	211.01	210.95	210.92
ω_2	1250.30	1224.00	1215.30	1212.37	1214.09	1211.59	1210.64
ω_3	3218.92	3059.64	3037.32	3029.77	3042.48	3032.87	3028.22
ω_4	5700.73	5272.36	5233.15	5219.80	5262.51	5232.79	5221.33
ω_5	8480.10	7674.08	7616.33	7596.50	7675.12	7622.87	7602.77
ω_6	11409.13	10149.00	10072.00	10046.00	10163.73	10088.51	10057.26
ω_7	14403.89	10353.00	10353.00	10353.00	10356.00	10354.53	10354.00
ω_8	17418.43	12643.00	12547.00	12514.00	12671.64	12569.56	12531.36

The next beam example is a symmetric sandwich beam with the same material and geometric properties as the beam described above with the exception of length. The length used is $L = 0.9144$ m and both ends are simply supported. The results of the natural frequency calculations are summarized in Table A9-2 below along with the results found by Howson and Zare [33] and an FEM model developed for the purpose of gaining extra results for additional comparisons. It should be noted that for the simply

supported end conditions, only the flexural displacements are fixed and the axial displacements are not, which lead to a rigid body mode at a frequency of zero.

Table A9-2: Natural frequencies of a symmetric sandwich beam with simply supported end conditions taken from Howson and Zare [33].

ω_n (rad/s)	Howson DSM [33]	DFE 20 Elements	DFE 30 Elements	DFE 40 Elements	FEM 20 Elements	FEM 30 Elements	FEM 40 Elements
ω_1	358.99	856.10	846.87	843.75	356.80	356.61	356.55
ω_2	1379.69	2150.30	2125.45	2117.05	1349.48	1346.85	1345.93
ω_3	2922.76	3830.65	3785.04	3769.55	2799.65	2789.28	2785.65
ω_4	4826.60	5715.00	5645.66	5621.91	4525.92	4501.48	4492.94
ω_5	6953.48	7691.77	7597.30	7564.79	6396.54	6352.87	6337.57
ω_6	9205.49	9698.86	9579.15	9537.70	8332.14	8265.99	8242.70
ω_7	11520.53	11706.00	11562.00	11512.00	10289.97	10199.80	10167.86
ω_8	13861.90	13701.00	13533.00	13474.00	12248.77	12134.29	12093.43
ω_9	16209.30	15677.00	15487.00	15420.00	14198.87	14060.56	14010.77
ω_{10}	18552.05	16105.00	16105.00	16105.00	16121.76	15975.30	15916.73

The final beam case studied is a symmetric sandwich with length, $L = 1.21872$ m, width, $b = 25.4$ mm, face layer thicknesses, $t_1 = t_2 = 0.40624$ mm and core thickness, $t_c = 6.3475$ mm with both ends simply supported. The material Young's modulus, $E_1 = E_2 = 68.9$ GPa and density, $\rho_1 = \rho_2 = 2687.3$ kg/m³. The core material has a shear modulus, $G = 68.9$ MPa and density $\rho_c = 119.69$ kg/m³. The results of the DFE and FEM analysis are presented in Table A9-3 below. Table A9-3 also includes DSM results from Howson and Zare [33] for reference.

Table A9-3: Natural frequencies of a symmetric sandwich beam with fixed end conditions taken from Howson and Zare [33].

ω_n (rad/s)	Howson DSM [33]	DFE 20 Elements	DFE 30 Elements	DFE 40 Elements	FEM 20 Elements	FEM 30 Elements	FEM 40 Elements
ω_1	217.38	229.60	221.24	218.35	216.34	215.71	215.48
ω_2	584.96	611.59	589.14	581.40	579.30	575.74	574.47
ω_3	1113.10	1150.32	1107.70	1092.96	1096.21	1085.43	1081.56
ω_4	1776.78	1813.62	1745.77	1722.26	1739.42	1715.60	1707.04
ω_5	2553.02	2574.00	2476.78	2443.03	2484.33	2440.89	2425.32
ω_6	3420.13	3407.72	3277.95	3232.82	3308.36	3239.02	3214.19
ω_7	4359.19	4295.55	4130.99	4073.65	4192.34	4091.67	4055.61
ω_8	5354.24	5222.34	5021.65	4951.50	5120.46	4984.21	4935.32
ω_9	6392.20	6176.30	5938.95	5855.63	6080.05	5905.23	5842.32
ω_{10}	7462.54	7148.35	6874.58	6777.94	7061.16	6845.97	6768.23

List of Contributions

1. Adique, E. and Hashemi, S. M. (2007). Free vibration of sandwich beams using the dynamic finite element method. *CD Proceedings of the 11th International Conference on Civil, Structural and Environmental Engineering Computing (CC07)*, September 18-21, 2007 St. Julians, Malta, 1-9.
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