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DEPARTMENT OF AEROSPACE ENGINEERING

AER870 - AEROSPACE ENGINEERING THESIS

Simulations of LQR Control on Linear Spacecraft Dynamics about Asteroid Equilibrium Points

FINAL REPORT

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ABSTRACT

Future exploratory missions to asteroids may require a spacecraft to perform attitude and position change maneuvers within small perturbations of the equilibrium point to conduct measurements and make observations based on mission requirements. The non-linear dynamics of the spacecraft can be approximated to be linear given that the system operates about an equilibrium point and the signals are small. Based on this, the linearized system is equivalent to the non-linear system within a limited operating range. This project follows this precedent and applies a closed loop LQR controller to perturbations of 1%, 2% and 5% from asteroid 101955 Bennu's equilibrium points. The LQR controller methodology requires that weighting matrices **Q** and **R** which penalize the states and the controls respectively to be iterated for – depending on the application requirements. The iteration procedure is the primary objective of this project and is conducted for six different spacecraft orientations about eight different equilibrium points. The procedure examines the settling times and response plots to critique the performance of the controller. This paper presents the underlying control theory, the modelling scenario, the simulation procedure, results and some patterns discovered in the results. This project was conducted in conjunction with [3] and presents the results of simulations based on linearized spacecraft dynamics. The results of the non-linear spacecraft dynamics simulations can be found in [3].

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1. INTRODUCTION

A growing field of interest in planetary science is the exploration of asteroids for a variety of reasons such as – abundance of metals scarce on Earth, cache of water for future deep space missions, and as they are remnants of the early solar system and can advance the understanding of the formation of Earth along with the origins of life on the planet. However, due to the irregular shape and mass distribution of the asteroid, the weak gravitational field, and the deviations due solar radiation pressure, the orbital and attitude dynamics of the spacecraft close to the asteroid is complicated [1] [2]. Furthermore, exploratory missions to asteroids would require a spacecraft to perform attitude and position change maneuvers within small perturbations of the equilibrium point to conduct measurements and make observations based on mission requirements. In order to satisfy such requirements, this project applies the *Linear Quadratic Regulator (LQR)* control method and simulates the performance of the controller based on linearized dynamics of the spacecraft. This paper discusses the underlying control theory, the scenario in which the spacecraft is modelled, the simulation procedure, and the results of the project.

2. PROJECT OBJECTIVE

The objective of this thesis project is to tune an LQR controller for stabilization at different equilibrium points about the asteroid 101955 Bennu at varying orientations based on linearized dynamics of a 7x7x50 m spacecraft. The attitude dynamics of the spacecraft along with the details of the controller are already established and this project specifically focuses on the controller design using linearized dynamics. The full details of controller, attitude dynamics and the non-linearized implementation can be found in [3].

3. LITERATURE REVIEW

3.1 LQR Controllers

The *Linear Quadratic Regulator (LQR)* controller belongs to the family of *Quadratic Optimal Regulator Systems* and provides a systematic method of computing the state feedback gain matrix. For a system with the state equation [4]:

$$\dot{\boldsymbol{x}} = \boldsymbol{Ax} + \boldsymbol{Bu} \tag{1}$$

where **A** is the state matrix, **x** is the state vector, **B** is the input matrix and **u** is the control input. The performance of a controller on such a system can be judged using the performance index, **J**, given by [4]:

$$J = \int_{\boldsymbol{0}}^{\infty} (\boldsymbol{x^T Q x} + \boldsymbol{u^T R u})dt \tag{2}$$

where **Q** is the weighting matrix that penalizes the states and **R** is the weighting matrix that penalizes the control. As there is no analytical method to find the values of **Q** and **R**, they must be iterated upon based on the system performance requirements as dictated by the designer.

If the system given by Equation 1 is controllable, an optimal control input vector **u(t)** can be found where the performance index J is minimized. The optimal control input vector is proportional to the optimal gain K and the relationship is the **Optimal Control Law** given by [4]:

$$\boldsymbol{u(t)} = -\boldsymbol{Kx(t)} \tag{3}$$

If the elements of the matrix **K** is found which minimizes the performance index, J, then the control input matrix given by Equation 3 is optimal for any initial state **x(t).** Note that although the system that minimizes the performance index is called optimal, in practice the system may not necessarily provide an "optimal" configuration. However, a system designed based on the quadratic performance index does ensure that the design yields a stable control system [4]. The optimal configuration of the system is shown below in Figure 1.

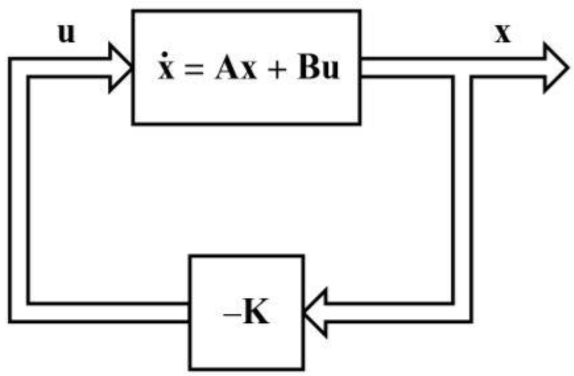


Figure 1: Block Diagram of the Optimal Regulator System [4]

To optimize the performance of the controller, the optimal gain matrix **K** must be found relative to the system state, $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$. Following a series of derivations beyond the scope of this project, the optimal gain matrix, **K**, can be found by [4]:

$$\boldsymbol{K} = \boldsymbol{R}^{-1}\boldsymbol{B}^{\boldsymbol{T}}\boldsymbol{P} \tag{4}$$

where **P** is a positive-definite matrix. To find the optimal gain in Equation 4, the elements of **P** must be first found using the *Reduced Matrix Ricatti Equation* given by [4]:

$$\boldsymbol{A}^{\boldsymbol{T}}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} - \boldsymbol{P}\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\boldsymbol{T}}\boldsymbol{P} + \boldsymbol{Q} = \boldsymbol{0} \tag{5}$$

In summary, the design procedure is as follows:

1. Determine the state matrix **A**, and the input matrix **B** based on the system dynamics.
2. Iterate to find the values of the state weights, **Q** and the input weights **R** based on the desired system performance.
3. Solve the Reduced-Matrix Ricatti Equation (Equation 5) for the matrix **P**.
4. Substitute the matrix **P** into the optimal gain matrix **K** expression in Equation 4. The result is the optimal matrix **K** used to find the optimal control input vector $\boldsymbol{u}(\boldsymbol{t}) = -\boldsymbol{K}\boldsymbol{x}(\boldsymbol{t})$**.**

Steps 3 and 4 can be simulated in MATLAB by using the command: `[K,P,E] = lqr(A,B,Q,R)` - which returns the gain value **K**, the solution to the Riccatti's equation **P**, and the eigenvalue vector **E**.

3.2 Attitude Dynamics

3.2.1 Problem Scenario

This project applies the LQR controller method to the attitude stabilization of a spacecraft about the asteroid 101955 Bennu around its eight equilibrium points (E1, E2, E3, E4, E5, E6, E7, E8) as shown in Figure 2.

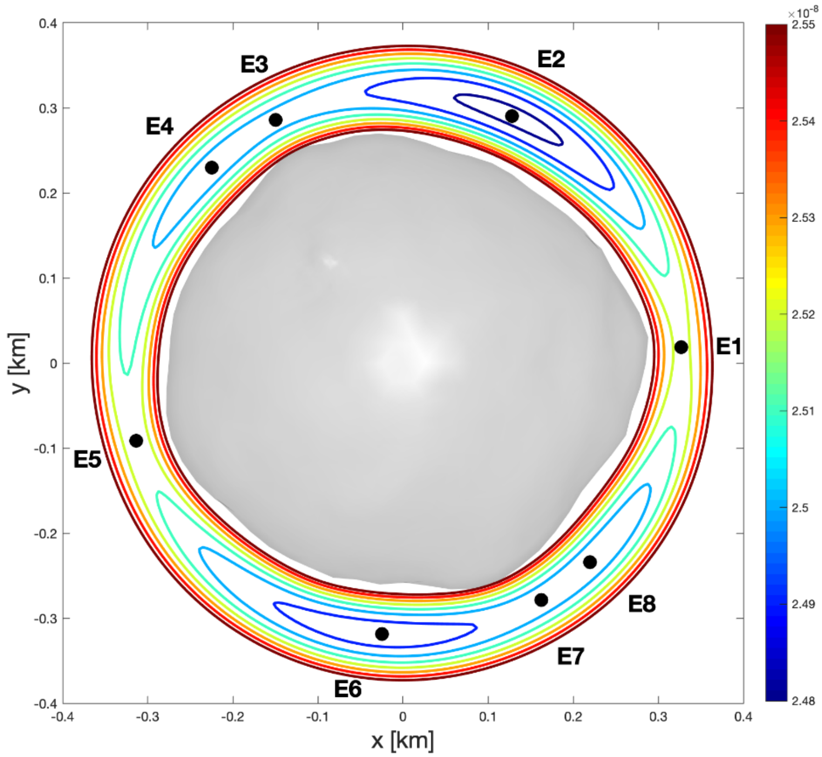


Figure 2: Equilibrium Points about Bennu [3]

About the eight equilibrium points, the spacecraft can also be placed in six different orientations, Radial Max, Radial Min, In Plane Max, In Plane Min , Out of Plane Max and Out of Plane Min as shown in Figure 3, Figure 4 and Figure 5 respectively.

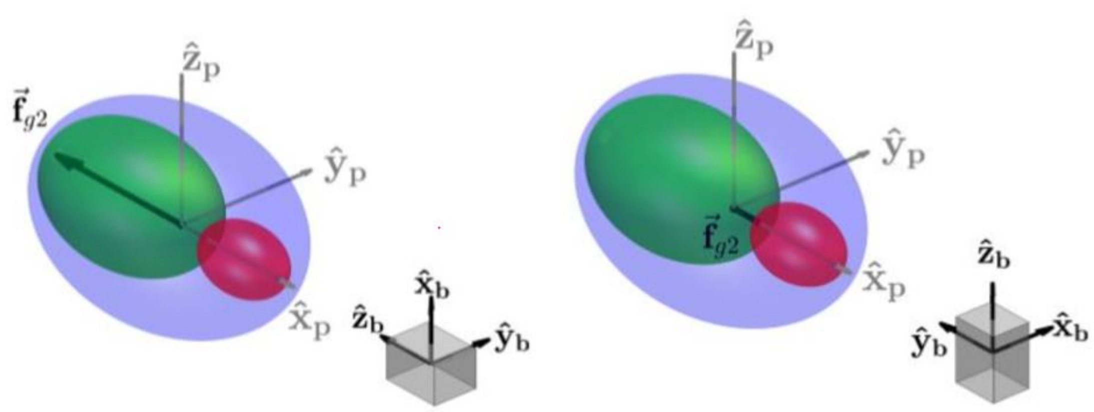


Figure 3: Radial Max (L) and Radial Min (R) Orientations [5]

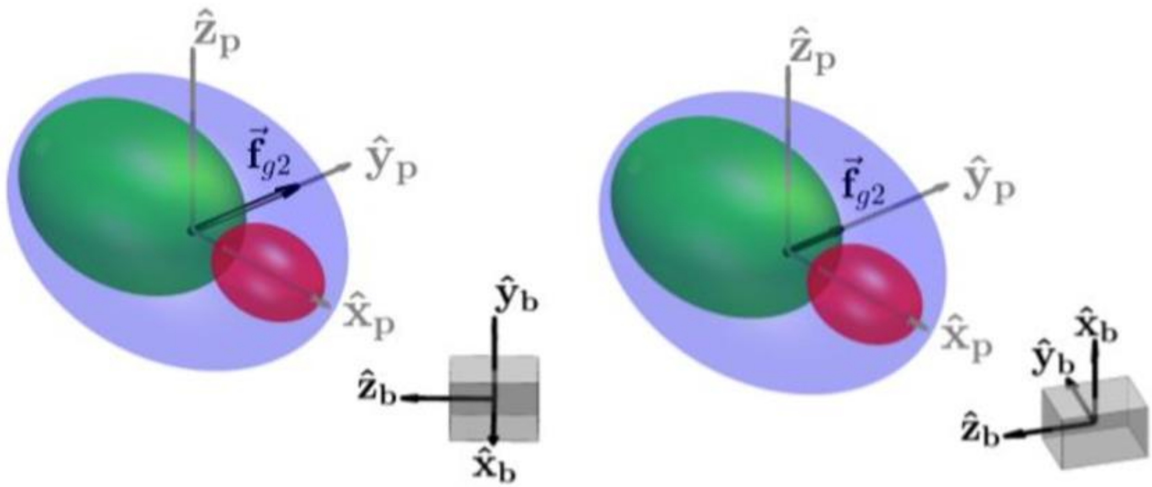


Figure 4: In Plane Max (L) and In Plane Min (R) Orientations [5]

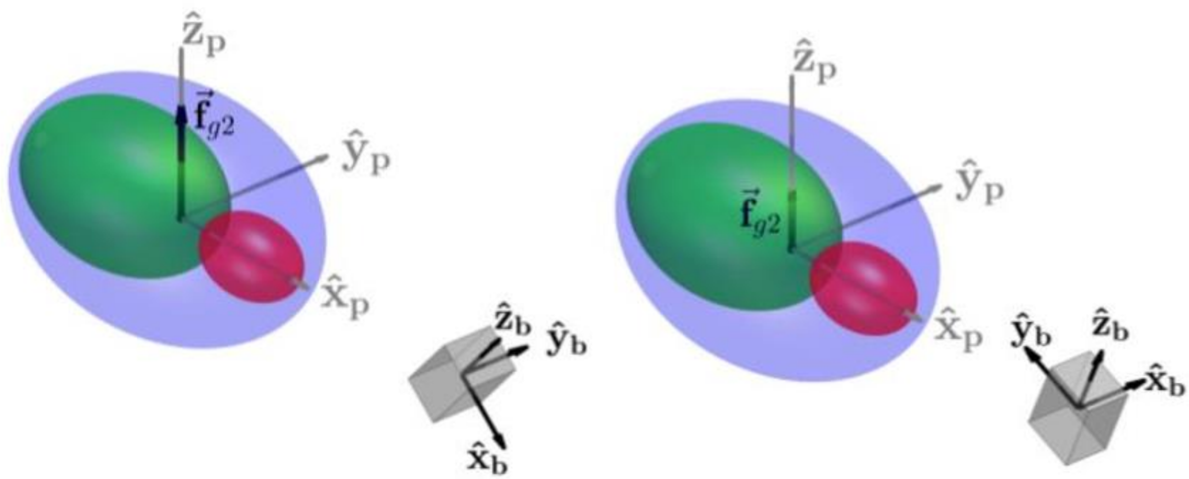


Figure 5: Out of Plane Max (L) and Out of Plane Min (R) Orientations [5]

The combination of eight equilibrium points about the asteroid along with the six different orientations of the spacecraft means that there are 48 possible configurations.

3.2.2 Application to the Controller

Now with the problem scenario defined, the state vector **x** given as [5]:

$$\boldsymbol{x} = \begin{bmatrix} \phi \\ \theta \\ \psi \\ \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \tag{6}$$

Where (ϕ, θ, ψ) and $(\dot{\phi}, \dot{\theta}, \dot{\psi})$ are the Euler rotation angles (roll, pitch and yaw) and the angular rates of the spacecraft body frame $\mathcal{F}_b$ relative to the asteroid body frame $\mathcal{F}_o$. The control input vector **u** is defined as [5]:

$$\boldsymbol{u} = \begin{bmatrix} T_{x_c} \\ T_{y_c} \\ T_{z_c} \end{bmatrix} \tag{7}$$

where $\left(T_{x_c}, T_{y_c}, T_{z_c}\right)$ are the control torques applied to the spacecraft. Then, the attitude dynamics of the spacecraft can be defined using the state matrix **A** and the input matrix **B** as [5]:

$$\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{4\omega_o^2}{I_x}\left(I_z - I_y\right) & 0 & 0 & 0 & 0 & \frac{\omega_o}{I_x}\left(I_x - I_y + I_z\right) \\ 0 & \frac{3\omega_o^2}{T_y}\left(I_z - I_y\right) & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\omega_o^2}{I_z}\left(I_x - I_y\right) & \frac{\omega_o}{I_z}\left(I_y - I_x - I_z\right) & 0 & 0 \end{bmatrix} \tag{8}$$

$$\boldsymbol{B} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{I_x} & 0 & 0 \\ 0 & \frac{1}{I_y} & 0 \\ 0 & 0 & \frac{1}{I_z} \end{bmatrix} \tag{9}$$

where (I_x, I_y, I_z) are the spacecraft's *principal moments of inertia*, and ω_o is the *orbital angular rate*. From the matrices of **A** and **B,** the state-space model of the spacecraft can be defined using Equation 1.

The non-linear dynamics of the spacecraft can then be approximated to be linear given that the system operates about an equilibrium point and the signals are small. From this, the linearized system is equivalent to the non-linear system within a limited operating range. This project follows this precedent and applies the controller to perturbations of 1%, 2% and 5%, while the full linearization procedure and the results of the non-linear dynamics simulations can be found in [3].

4. SIMULATION PROCEDURE

The primary objective of the simulation procedure is to adjust the **Q** and **R** matrices. As discussed in Section 3.1, the **Q** and **R** pairs are applied to the performance index, J, from which an optimal control input **u(t)** can be found. Thus, it must be considered that the control input needs to be small enough to avoid spinning the spacecraft uncontrollably, but also high enough to return to the equilibrium point in a reasonable amount of time. As discussed in Section 3.1, there is no analytical procedure for finding **Q** and **R** and thus these values must be iterated upon for each of the 48 different configurations. The full MATLAB script for the simulation, `LQR_controller_initialization.m`, `LQR_Q_and_R_initialization.m`, `figures.m` scripts and the Simulink model `system_block_diagram_closed_loop_LQR.slx` ,can be found in [3] and the procedure depicted in this section demonstrates that used for equilibrium point E1 in the In Plane Max orientation at 1% perturbation.

4.1 Controller Initialization

First the controller must be initialized about the asteroid equilibrium point and the procedure is as follows:

1. The asteroid and spacecraft data is loaded onto the MATLAB workspace.
2. In the `LQR_controller_initialization.m` script, the spacecraft orientation and the equilibrium points can be specified in Lines 21 and 29 respectively as shown in Figure 6.

```
12      % Code description for s/c attitude selection. Possible options:
13      % - 'fixed'
14      % - 'radial-max'
15      % - 'radial-min'
16      % - 'in-plane-max'
17      % - 'in-plane-min'
18      % - 'out-of-plane-max'
19      % - 'out-of-plane-min'
20
21      sc_orientation = 'in-plane-max';
22
23      disp('~~~~~~~~~~~~~~~~~~~~~~');
24      disp('S/C orientation');
25      disp(sc_orientation);
26      disp('~~~~~~~~~~~~~~~~~~~~~~');
27
28      % Code for equilibrium point: E1, E2, etc.
29      eq_point = 'E1';
```

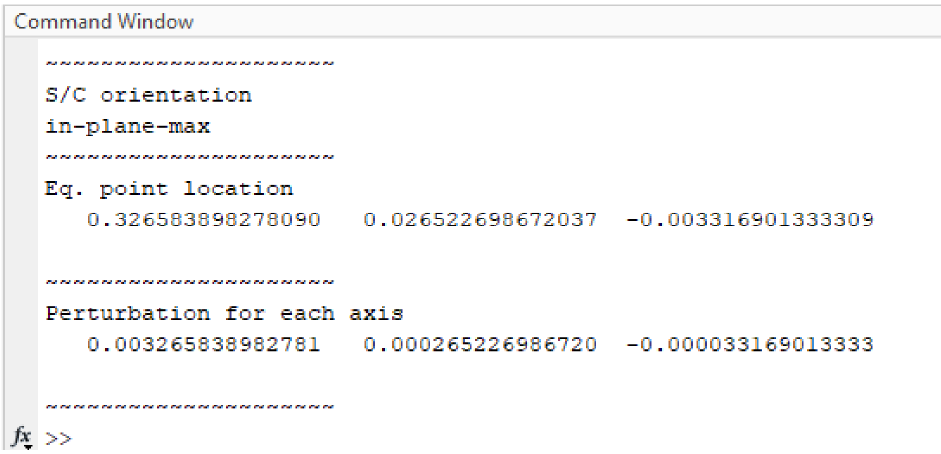
Figure 6: Spacecraft Orientation and Equilibrium Point Specification

3. The perturbation value can also be specified in Line 53 as shown in Figure 7.

```
52      %% Automatically sets ups perturbation to 1% in each component for X0
53      X0 = 0.01 .* eq_point_data;
```

Figure 7: Perturbation Value Specification

4. The initial **Q** and **R** pairs are also specified in the script as a 6x6 and 3x3 identity matrices, respectively. The identity matrices are then multiplied by $1E10$ and $1E6$ for **Q** and **R,** respectively. This value pair was chosen arbitrarily and is used the initialization of all 48 configurations.
5. Then the script is run, and the equilibrium point along with the associated perturbation can be seen in the workspace as shown in Figure 8.



```
Command Window
~~~~~~~~~~~~~~~~~~~~~~
S/C orientation
in-plane-max
~~~~~~~~~~~~~~~~~~~~~~
Eq. point location
   0.326583898278090   0.026522698672037  -0.003316901333309

~~~~~~~~~~~~~~~~~~~~~~
Perturbation for each axis
   0.003265838982781   0.000265226986720  -0.000033169013333

~~~~~~~~~~~~~~~~~~~~~~
fx >>
```

Figure 8: Script Output in the MATLAB workspace

6. Now the Simulink model as shown in Figure 9 is run with the integrator set as ode4 and with a 1 second time step as depicted in Figure 10.

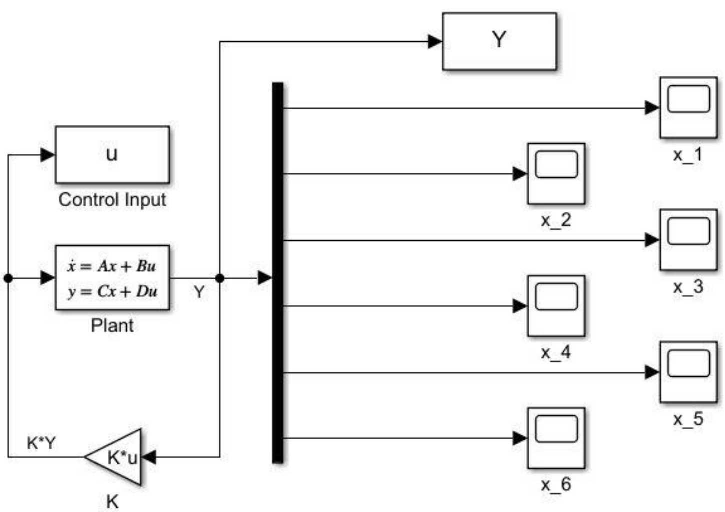


Figure 9: Simulink Model of the System

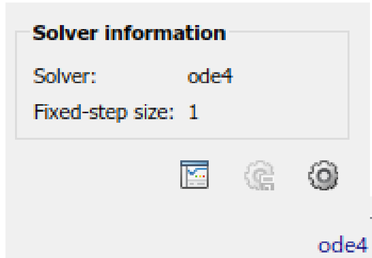


Figure 10: Integrator Settings

7. Now executing the `figures.m` script outputs three response plots – *Output States, Control Input 'u'* and *Orientation Response* as shown in Figure 11, 12 and 13, respectively.

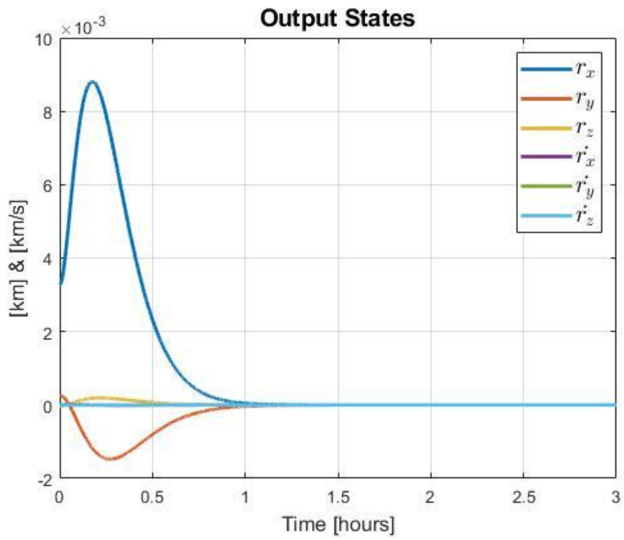


Figure 11: Output States of E1 - In Plane Max

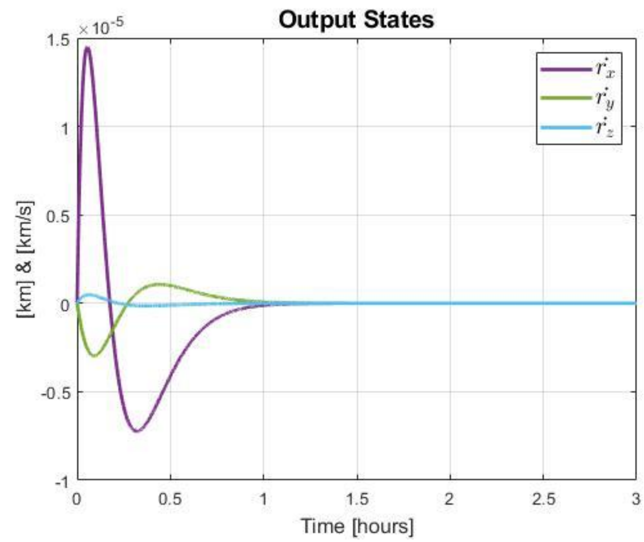


Figure 12: Rates of Spacecraft Motion in E1 - In Plane Max

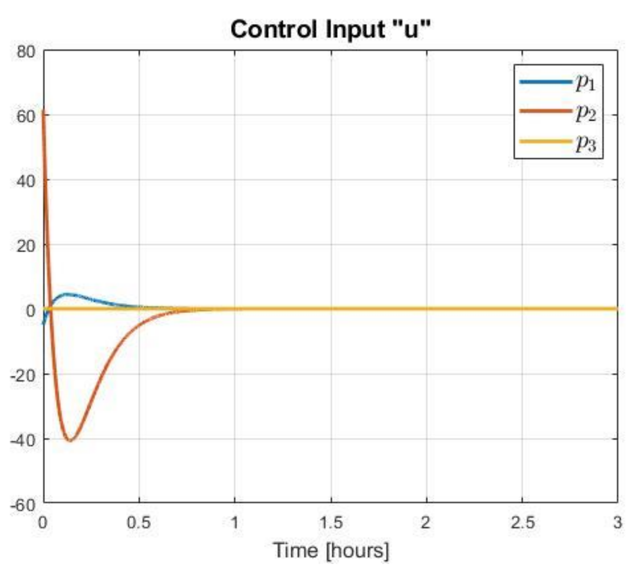


Figure 13: Control Input Response for E1 - In Plane Max

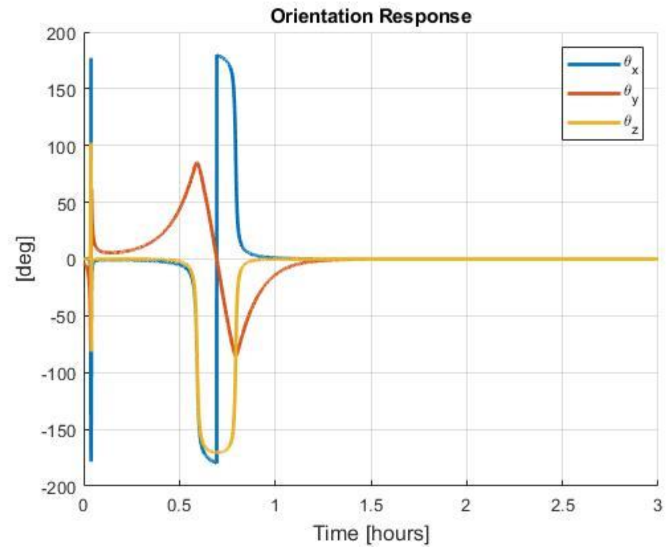


Figure 14: Orientation Response for E1 - In Plane Max

8. The *Output States* response shows the translational motion of the spacecraft following a perturbation with its position with respect to time shown in Figure 11 and the rate of motion shown in Figure 12. The *Control Input "u"* response in Figure 13 shows the spacecraft orientations in Modified Rodrigues Parameters (MRP). Transforming the MRP control input into Euler angles gives the *Orientation Response* plot in Figure 14. Since the MRP is beyond the scope of the project, the *Output States* and *Orientation Response* plots will be primarily examined throughout the project. These plots must all converge to the y = 0, representing that the spacecraft has returned to the equilibrium point and is no longer in any motion – translational or rotational.
9. Using the values of $\boldsymbol{Y}$ and $(\boldsymbol{r_x},\ \boldsymbol{r_y},\ \boldsymbol{r_z})$ used to plot the *Output States* plot and the *Orientation Response* plot respectively, the settling time of the responses can also be calculated using the MATLAB commands `stepinfo()` as shown in Figure 15.

```
Y1 = stepinfo(Y.Data(:,1),Y.Time);
Y2 = stepinfo(Y.Data(:,2),Y.Time);
Y3 = stepinfo(Y.Data(:,3),Y.Time);
Y4 = stepinfo(Y.Data(:,4),Y.Time);
Y5 = stepinfo(Y.Data(:,5),Y.Time);
Y6 = stepinfo(Y.Data(:,6),Y.Time);

Y1_TS = Y1.SettlingTime;
Y2_TS = Y2.SettlingTime;
Y3_TS = Y3.SettlingTime;
Y4_TS = Y4.SettlingTime;
Y5_TS = Y5.SettlingTime;
Y6_TS = Y6.SettlingTime;
```

```
r1 = stepinfo(r_x,u.Time);
r2 = stepinfo(r_y,u.Time);
r3 = stepinfo(r_z,u.Time);

r1_TS = r1.SettlingTime;
r2_TS = r2.SettlingTime;
r3_TS = r3.SettlingTime;
```

Figure 15: Finding the Settling Times

The settling time is the primary parameter used to judge the performance of the controller and the settling times found from the initialized system will then be compared to that found from the iterated **Q** and **R** as shown below in Section 4.2.

4.2 Q and R Tuning

With the settling time responses found in Section 4.1, the **Q** and **R** pairs can be iterated upon using the `LQR_Q_and_R_initialization.m` script.

1. With the asteroid and spacecraft data already loaded onto the workspace, the values of **Q** and **R** can be iterated from Line 6 and 7 respectively as shown in Figure 16.

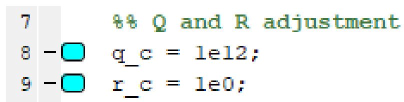


Figure 16: Q and R adjustments

2. As the controller is applied to spacecraft mission where the amount of available fuel and thus control torque is limited, the **R** is kept to a value of 1 and **Q** is iterated upon. The first iteration value of **Q** is randomly selected and iterated upon with the objective of minimizing the settling times.
3. Starting with a value of **Q** = 1E12 and running the Simulink model and the `figures.m` script the following response plots are returned:

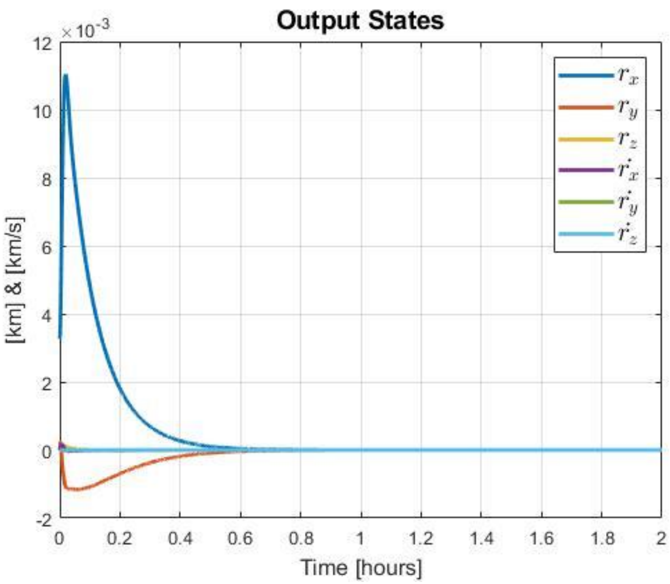


Figure 17: Translation Motion - Iteration 1

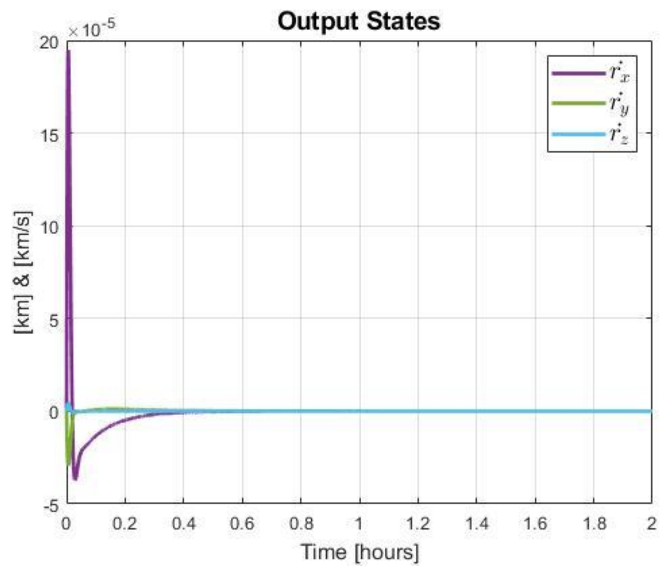


Figure 18: Rate of Translation - Iteration 1

4. The responses shown in Figure 17, 18 and 19, although, reduced the time needed for the system to reach a steady state response, the dynamics of the spacecraft would not be reasonable. Thus, sharp changes in any of the response plots is to be avoided as to not apply a large control torque and cause uncontrollable spinning of the spacecraft.

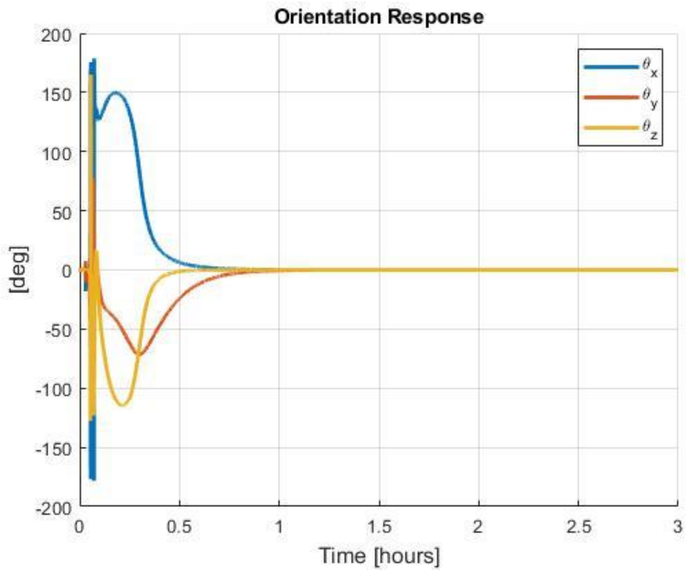


Figure 19: Orientation Response for Iteration 1

5. Following Steps 1-3 for a few more iterations, a value of **Q** = 3E07 returns the following response plots:

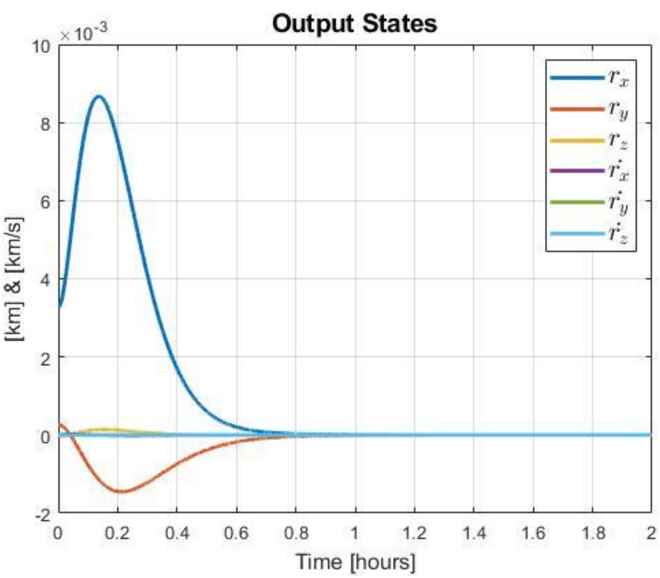


Figure 20: Translation Motion - Final Iteration

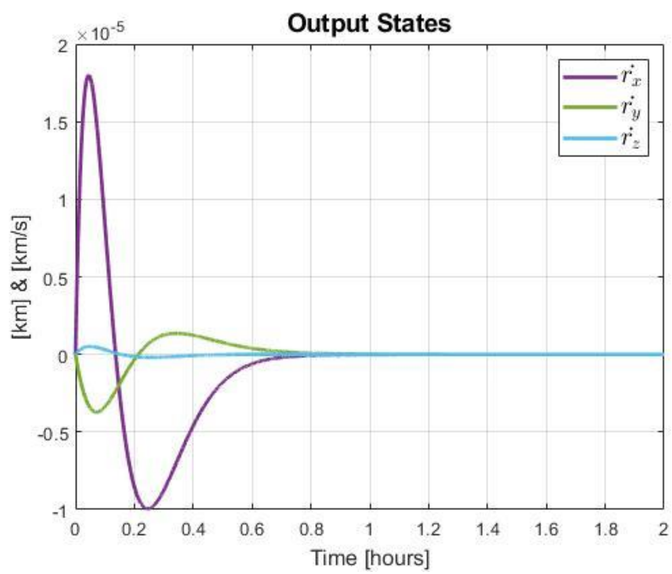


Figure 21: Rate of Translation – Final Iteration

6. The response plots obtained with the iterated value of **Q** shown in Figure 20, 21 and 22 shows a much more reasonable response while also reducing the time needed to reach a steady state response. The settling times of the initialized simulation, the first simulation and the final simulation are presented below in Table 1.

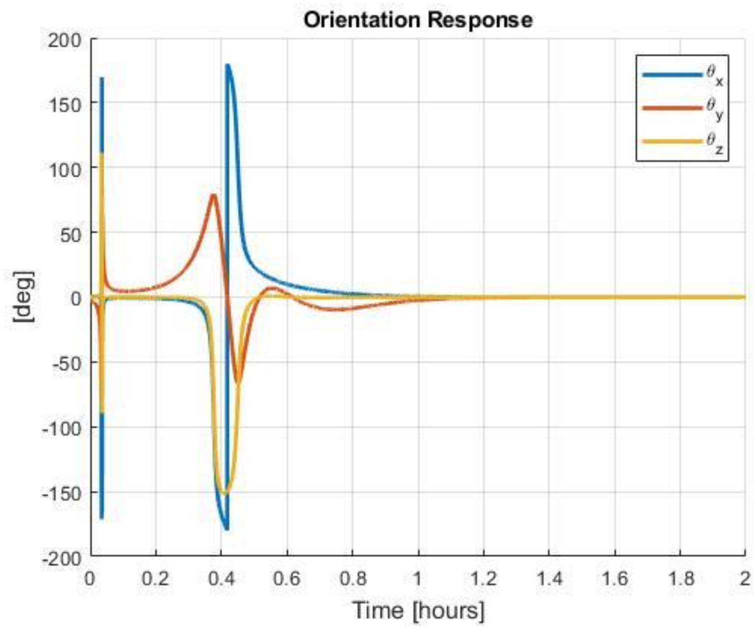


Figure 22: Orientation Response - Final Iteration

Table 1: Settling Times for Varying Q and R pairs

		Settling Times (s)			
Iteration	**Position**	**Velocity**	**Euler Angles**	**Q**	**R**
Initial	3367	3392	3547	1E+10	1E+06
1	1986	806	2225	1E+12	1E+00
Optimal	2599	2378	2748	3E+07	1E+00

From Table 1, it can be seen that although higher values of **Q** reduces the settling times, the response and dynamics of the spacecraft may not necessarily be reasonable and thus the iteration procedure must consider both the settling times and the dynamics of the response plots through the iteration of **Q** and **R.**

4.3 Simulation Procedure – Summary

In summary the iteration procedure is as follows:

1. Load the asteroid and spacecraft data onto the workspace, change the equilibrium point and orientation of the spacecraft as desired and run the scripts `LQR_controller_initialization.m`, the Simulink model `system_block_diagram_closed_loop_LQR.slx` and the script `figures.m` respectively.
2. Record the settling times of the initialized system.
3. Iterate the values of **Q** and **R** in `LQR_Q_and_R_initialization.m`, and run the script, the Simulink model and `figures.m` examining the response plots and iterating until a adequate combination of settling times and feasible response in obtained.

5. RESULTS

Following the iteration procedure discussed in Section 4, each unique configuration of the spacecraft from the combination of equilibrium points and orientations at 1%, 2% and 5% perturbations were simulated. The following section presents and discusses the results obtained from the In-Plane Max and Radial Min orientations at a 1% perturbation. Due to the large volume of data, the simulations conducted for all the remaining simulations are presented in Appendix A.

5.1 1% Perturbation

5.1.1 In Plane Max

Table 2: Settling Times and Respective Q & R for each Equilibrium Point at In Plane Max

		Position (s)	Velocity (s)	Euler Angles (s)	Q	R
E1	**Initial**	3367	3392	3547	1E+10	1E+06
	Optimal	2599	2378	2748	3E+07	1E+00
E2	**Initial**	4333	4136	4188	1E+10	1E+06
	Optimal	2380	2436	2829	1E+08	1E+00
E3	**Initial**	2618	2729	3242	1E+10	1E+06
	Optimal	1367	1657	1833	3E+11	1E+00
E4	**Initial**	3115	3322	3524	1E+10	1E+06
	Optimal	1951	2022	2479	6E+07	1E+00
E5	**Initial**	2242	2630	2481	1E+10	1E+06
	Optimal	1719	1930	1491	1E+08	1E+00
E6	**Initial**	4673	4648	4775	1E+10	1E+06
	Optimal	4313	3873	3586	2E+08	1E+00
E7	**Initial**	2389	2715	2707	1E+10	1E+06
	Optimal	1937	2051	1767	1E+08	1E+00
E8	**Initial**	2388	2709	2696	1E+10	1E+06
	Optimal	1936	2050	1762	1E+08	1E+00

Examining the initial and optimal settling times in Table 2 shows that the iteration procedure for **Q** and **R** was effective in reducing the settling times of the response.

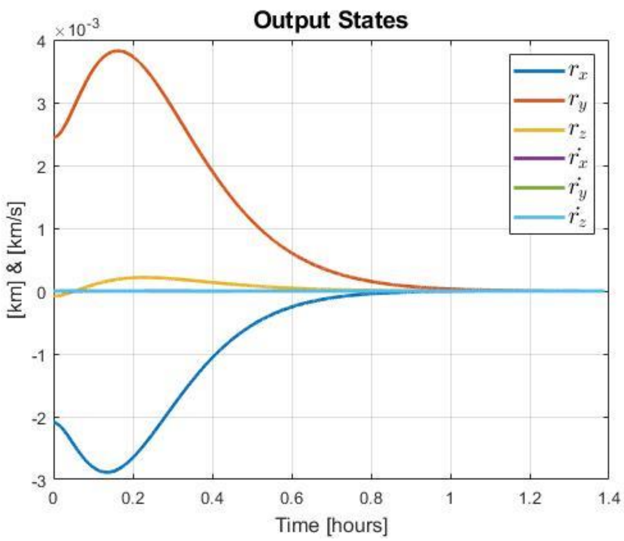


Figure 23: E4 Output States – Initial

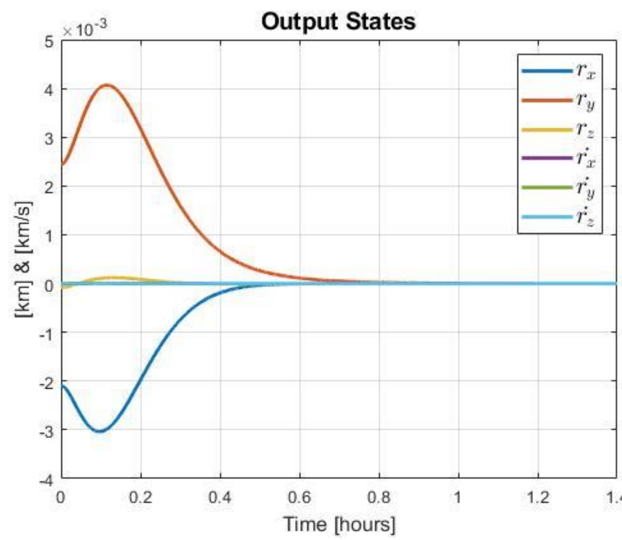


Figure 24: E4 Output States – Optimal

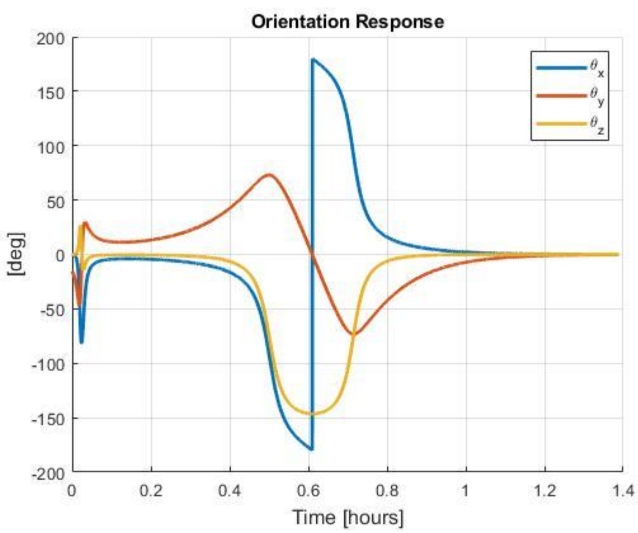


Figure 25: E4 Orientation Response – Initial

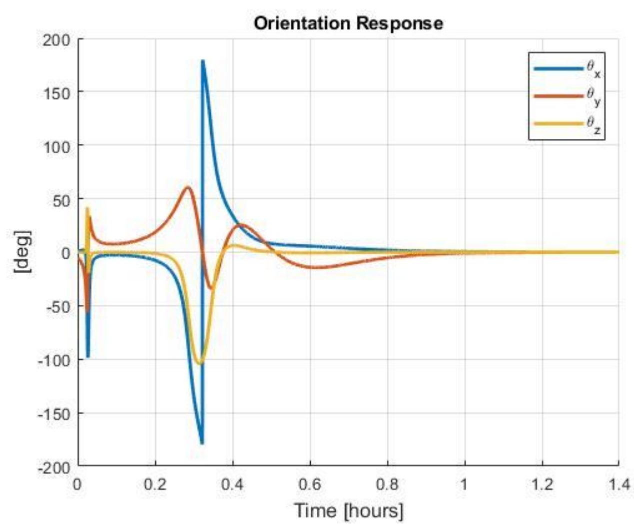


Figure 26: E4 Orientation Response – Optimal

Figures 23 to 26 demonstrate the response plots for the iterated **Q** and **R** values at equilibrium point E4 shown in Table 2.

5.1.2 Radial Min

Table 3: Settling Times and Respective Q & R for each Equilibrium Point at Radial Min

		Position (s)	Velocity (s)	Euler Angles (s)	Q	R
E1	**Initial**	3052	3015	5174	1E+11	1E+06
	Optimal	2092	1951	4090	5E+09	1E+00
E2	**Initial**	4814	4822	7004	1E+10	1E+06
	Optimal	2905	2707	5371	2E+09	1E+00
E3	**Initial**	2860	2865	4683	1E+10	1E+06
	Optimal	2062	1950	3302	6E+08	1E+00
E4	**Initial**	3189	2650	5744	1E+10	1E+06
	Optimal	2575	2142	4246	1E+09	1E+00
E5	**Initial**	2424	2351	3850	1E+10	1E+06
	Optimal	1588	1571	2866	2E+10	1E+00
E6	**Initial**	5684	5784	7862	1E+10	1E+06
	Optimal	3670	3819	5965	1E+09	1E+00
E7	**Initial**	2326	2322	5047	1E+10	1E+06
	Optimal	1721	1724	3760	1E+11	1E+00
E8	**Initial**	3078	3017	4926	1E+10	1E+06
	Optimal	2013	1992	3252	1E+10	1E+00

Table 3 presents the initial and optimal settling times in the Radial Min orientation with the respective **Q** and **R**. The settling time for the position, velocity and orientations (Euler angles) are improved.

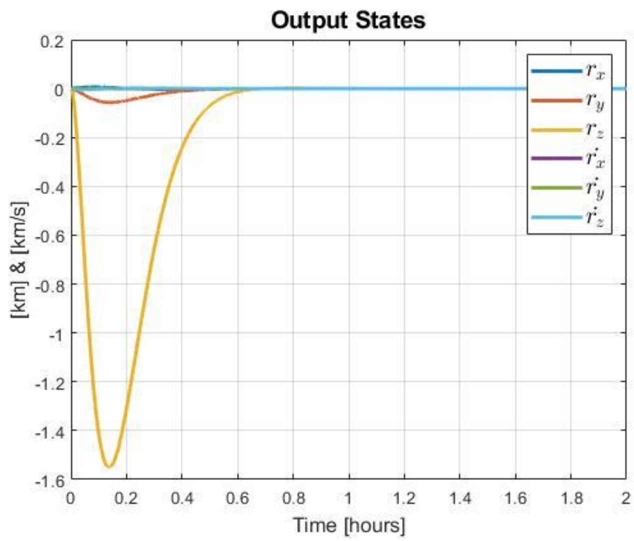


Figure 27: E7 Output States – Initial

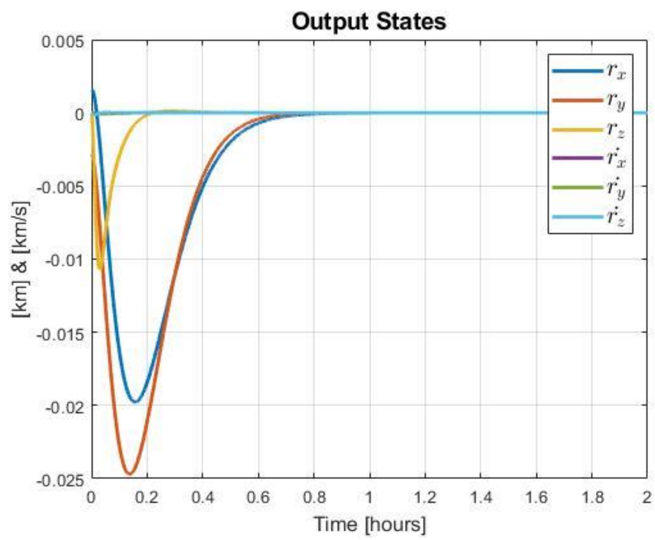


Figure 28: E7 Output States – Optimal

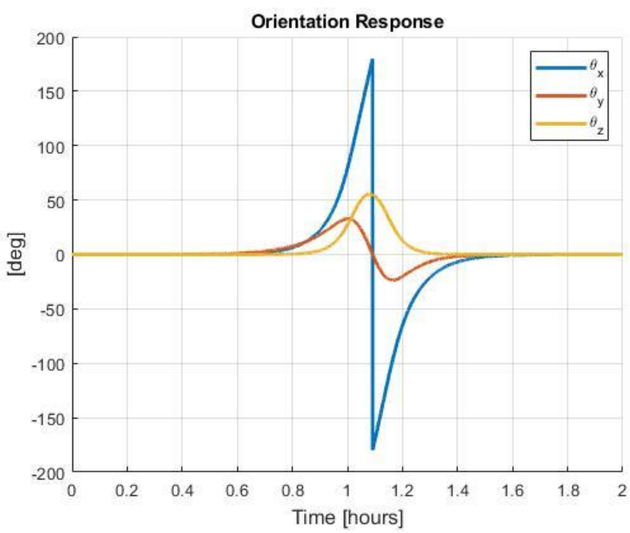


Figure 29: E7 Orientation Response – Initial

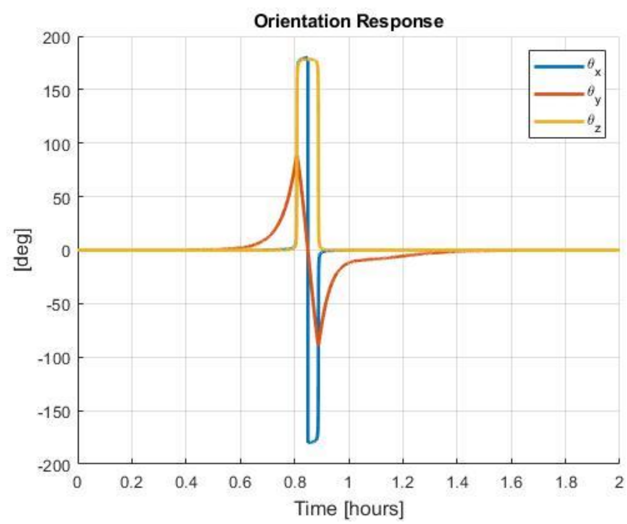


Figure 30: E7 Orientation Response – Optimal

Figures 23 to 26 demonstrate the response plots for the iterated **Q** and **R** values at equilibrium point E7 shown in Table 2.

6. ANALYSIS OF RESULTS

The iteration procedure conducted for 144 possible scenarios' (48 configurations, 3 different perturbation) showed the following patterns:

- Proportionality of Q and R
- Common settling times for states regardless of perturbations

6.1 Proportionality of Q and R

The relationship between **Q** and **R** is shown in Equation 2, where **Q** is the weighting matrix that penalized the states, while **R** is the weighting matrix that penalizes the control. Iterations throughout the project showed that settling times obtained from a **Q** and **R** pair were not unique and the same settling times could be found for a different **Q** and **R** pair. This relationship can be better examined from the results shown in Table 4.

Table 4: Settling Times for Varying Q and R Pairs

Iteration	Position(s)	Velocity (s)	Euler Angles (s)	Q	R
1	1734	1694	3543	1E+10	1E+00
2	1734	1694	3543	1E+12	1E+02
3	1734	1694	3543	1E+13	1E+03
4	1734	1694	3543	1E+07	1E-03

Table 4 depicts the simulations conducted for the Radial Min orientation at equilibrium point E5 at a 5% perturbation. The results show that despite the varying **Q** and **R** pairs, the settling times of the simulation remains the same. Further examination of the **Q** and **R** pairs shows that the exponent of **R** subtracted from the exponent of **Q** always equals 10, or mathematically – letting x be the exponent of **Q** and y the exponent of **R**:

$$x - y = 10$$

Generalizing the above relationship as:

$$x - y = z \tag{9}$$

Where z is an integer – for any integer combination of x and y that results in z, the settling times of the system will be the same.

The response plots of the above iterations can also be verified to be the same:

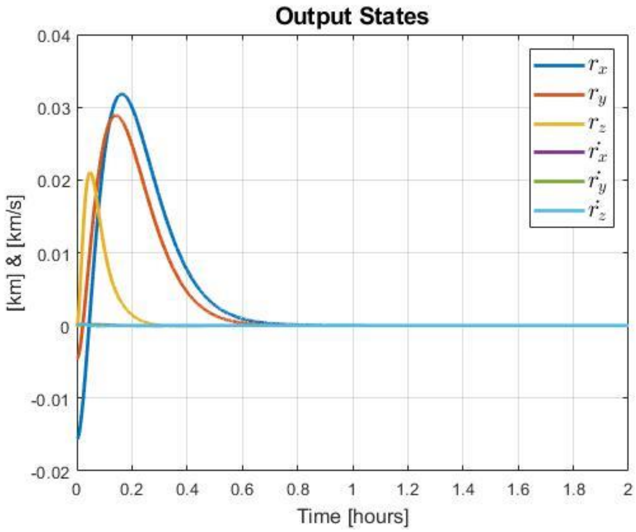


Figure 31: Output States - Iteration 1

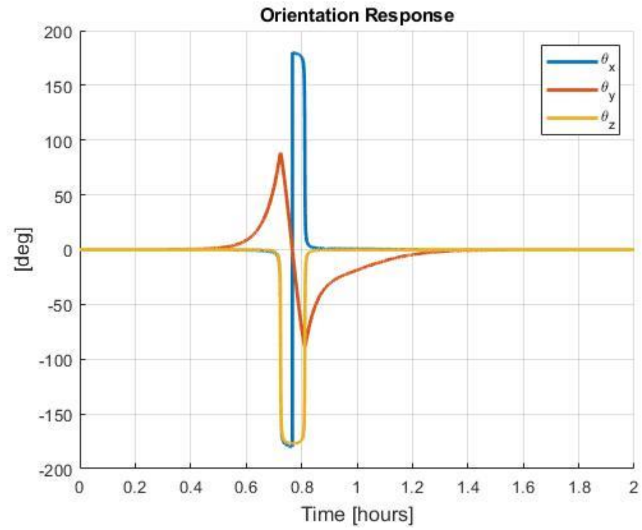


Figure 32: Orientation Response - Iteration 1

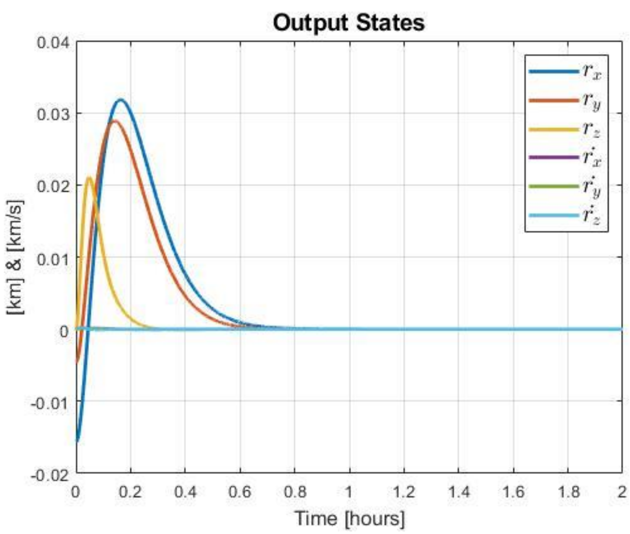


Figure 33: Output States - Iteration 4

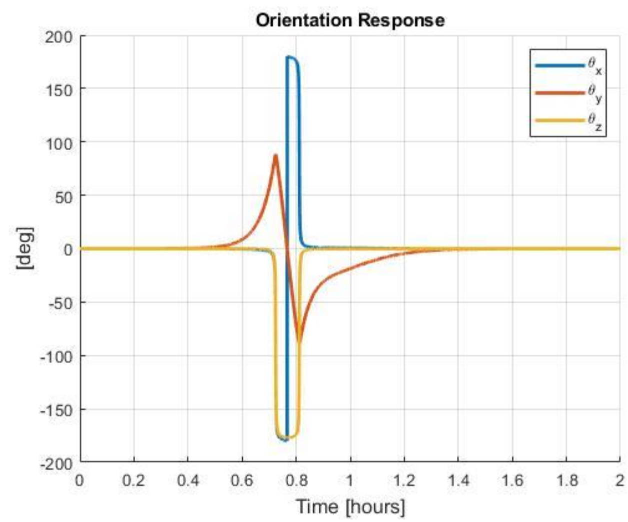


Figure 34: Orientation Response - Iteration 4

If a value of z is found that provides an ideal settling time and response plots combination, then Q and R values can be varied using the relationship in Equation 9 based on the mission requirements. The simulation procedures conducted here minimizes the penalty on control due to a limit on the control torque available, however for mission scenarios where further control torque may be available the weighing of Q and R can be adjusted resulting in the same system performance.

6.2 Common Settling Times for States

Another overlying pattern found through the iterations was that for a common configuration – the settling times for the Position and Velocity response was found to be the same for similar Q and R pairs but different perturbation values. This relationship is expanded upon in Table 5.

Table 5: Settling Times for Varying Perturbations

Perturbation	Position(s)	Velocity (s)	Euler Angles (s)	Q	R
1%	4532	4530	5823	1E+10	1E+06
2%	4532	4530	6104	1E+10	1E+06
5%	4533	4531	6497	1E+10	1E+06

Table 4 presents the settling times at differing perturbations for the same Q and R pair at Out of Plane Max at E4. It can be observed that the settling times for translational motion is the same regardless of the perturbation whereas the settling times for rotational motion increases with perturbation. An increased perturbation means that the spacecraft is at a greater distance from the equilibrium point and if the time required to return to equilibrium is the same then the spacecraft must be travelling at a greater speed. Examining the Output States velocity response plots as shown in Figure 27, 29 and 31 shows this where the velocity in the x and y direction increases along with the perturbation. Despite the spacecraft arriving at the equilibrium point in the same amount of time, the spacecraft takes longer to return to its nominal orientation. This is likely due to a larger control torque being applied meaning the spacecraft takes longer to decelerate and settle into its original orientation.

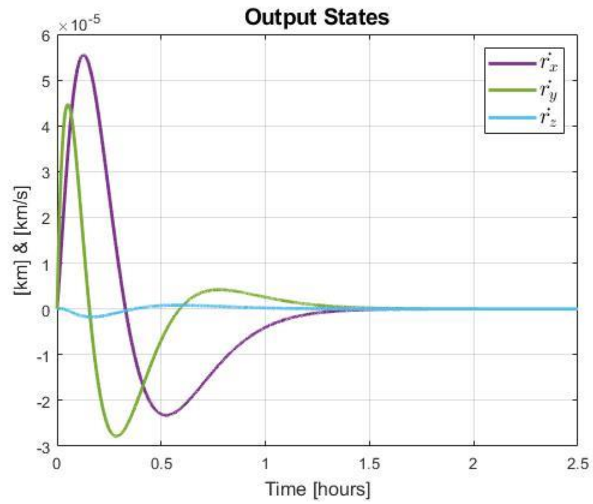


Figure 35: Output State Response - 1%

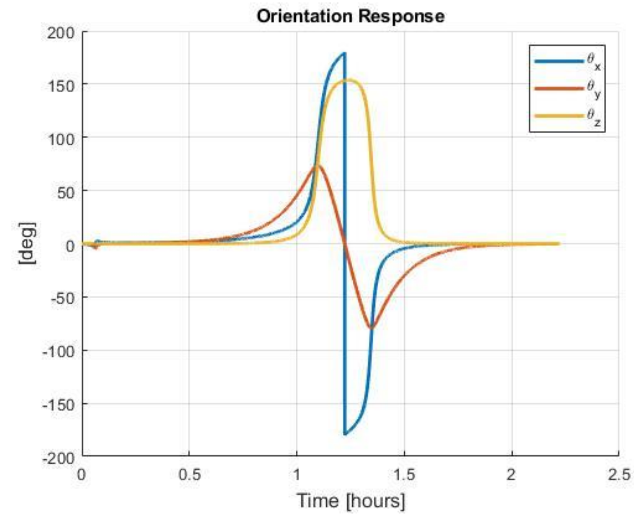


Figure 36: Orientation Response - 1%

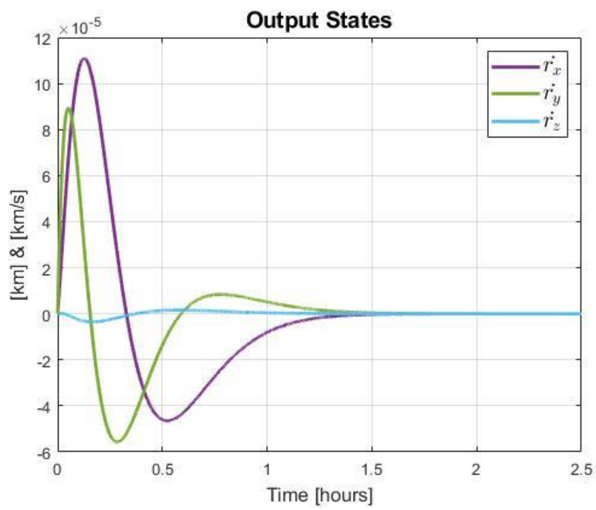


Figure 37: Output State Response - 2%

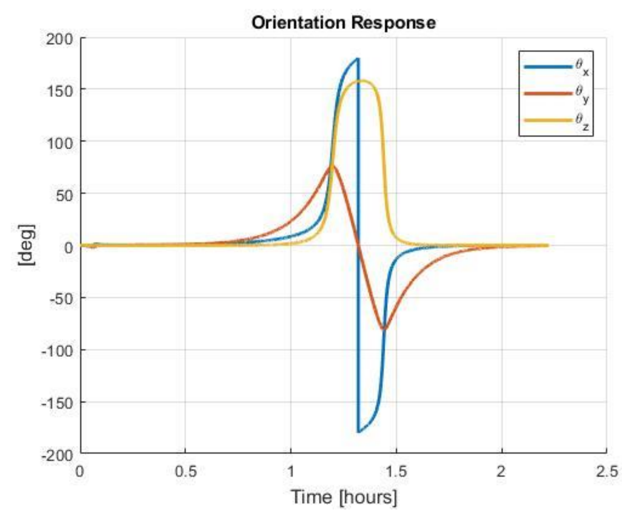


Figure 38: Orientation Response - 2%

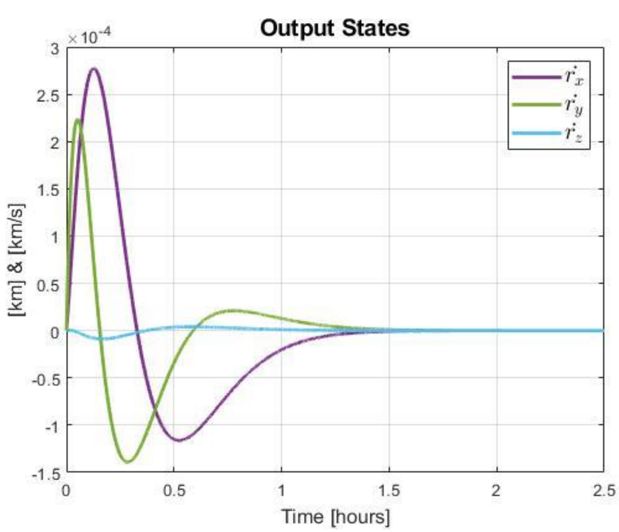


Figure 39: Output State Response - 5%

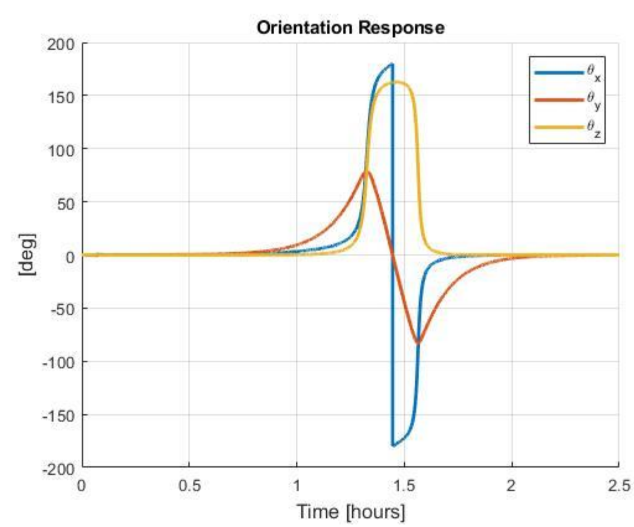


Figure 40: Orientation Response - 5%

7. CONCLUSION

This thesis project presents the simulations of a spacecraft about varying equilibrium points of the asteroid 101955 Bennu. Underlying control theory states that a typically non-linear system can be linearized given that the system operates about an equilibrium point and the signals are small. Furthermore, the linearized system can considered to be equivalent to the non-linear system within a limited operating range. Following this precedent, this project applied a closed loop LQR controller to spacecraft perturbations of 1%, 2% and 5% from each of the eight different asteroid equilibrium points. Subsequently, the LQR methodology requires that weighting matrices **Q** and **R** be iterated upon based on the system requirements which was the primary objective of this project. The iteration procedure is presented in Section 4 and uses the settling time and plots of the system response to assess the controller performance. The results obtained from the simulations are tabulated and presented in Section 5 and Appendix A. Some patterns observed from the simulations were:

- proportionality of **Q** and **R**: settling times obtained from a **Q** and **R** pair were not unique and the same settling times could be found for a different **Q** and **R** pair
- common settling times for different perturbations: for a common configuration – the settling times for the Position and Velocity response was found to be the same with similar Q and R pairs but different perturbation values.

This project presents the results based on linearized spacecraft dynamics and simulations for non-linear spacecraft dynamics can be found in [3].

8. BIBLIOGRAPHY

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APPENDIX A - SIMULATION RESULTS

A.1 1% Perturbation

A.1.2 In Plane Min

Table 6: Settling Times and Respective Q & R for each Equilibrium Point at In Plane Min – 1%

		Position (s)	Velocity (s)	Euler Angles (s)	Q	R
E1	**Initial**	3367	3392	3596	1E+10	1E+06
	Optimal	2562	2274	2838	4E+07	1E+00
E2	**Initial**	4116	4304	4460	1E+10	1E+06
	Optimal	2528	2611	3496	5E+07	1E+00
E3	**Initial**	2618	2729	3372	1E+10	1E+06
	Optimal	1746	1903	2345	3E+07	1E+00
E4	**Initial**	3115	3323	3586	1E+10	1E+06
	Optimal	1952	2023	2435	1E+08	1E+00
E5	**Initial**	2242	2630	2705	1E+10	1E+06
	Optimal	1719	1930	2040	1E+08	1E+00
E6	**Initial**	4739	5226	4981	1E+10	1E+06
	Optimal	4552	4090	3872	1E+08	1E+00
E7	**Initial**	2389	2716	1187	1E+10	1E+06
	Optimal	1937	2051	1935	1E+08	1E+00
E8	**Initial**	2389	2715	1187	1E+10	1E+06
	Optimal	1937	2051	1935	1E+08	1E+00

A.1.3 Radial Max

Table 7: Settling Times and Respective Q & R for each Equilibrium Point at Radial Max – 1%

		Position (s)	Velocity (s)	Euler Angles (s)	Q	R
E1	**Initial**	3544	3545	4578	1E+10	1E+06
	Optimal	2555	2671	3494	3E+07	1E+00
E2	**Initial**	4517	4492	5487	1E+10	1E+06
	Optimal	3217	3080	4148	2E+07	1E+00
E3	**Initial**	2928	3057	4386	1E+10	1E+06
	Optimal	2244	2217	3081	3E+07	1E+00
E4	**Initial**	4013	4032	5524	1E+10	1E+00
	Optimal	2471	2133	3902	2E+08	1E+00
E5	**Initial**	2543	2562	3702	1E+10	1E+06
	Optimal	1870	1991	2749	3E+07	1E+00
E6	**Initial**	5720	5749	6304	1E+10	1E+06
	Optimal	4137	4116	4740	2E+07	1E+00
E7	**Initial**	2597	2602	3708	1E+10	1E+06
	Optimal	1924	1872	2501	4E+07	1E+00
E8	**Initial**	3938	3917	4698	1E+10	1E+06
	Optimal	2882	2856	3730	4E+07	1E+00

A.1.5 Out of Plane Max

Table 8: Settling Times and Respective Q & R for each Equilibrium Point at Out of Plane Max – 1%

		Position (s)	Velocity (s)	Euler Angles (s)	Q	R
E1	**Initial**	2699	2888	4968	1E+10	1E+06
	Optimal	1783	1288	2744	1E+10	1E+00
E2	**Initial**	4756	4765	5597	1E+10	1E+06
	Optimal	2446	2390	3829	3E+08	1E+00
E3	**Initial**	2858	2849	4675	1E+10	1E+06
	Optimal	1837	1347	2804	2E+09	1E+00
E4	**Initial**	4532	4530	5823	1E+10	1E+06
	Optimal	2836	2952	4424	5E+07	1E+00
E5	**Initial**	2865	2907	4103	1E+10	1E+06
	Optimal	1980	2376	3091	1E+08	1E+00
E6	**Initial**	6587	6568	7531	1E+10	1E+06
	Optimal	3346	3215	5174	2E+08	1E+00
E7	**Initial**	2776	2785	4024	1E+10	1E+06
	Optimal	1965	1938	2981	1E+08	1E+00
E8	**Initial**	3546	3489	4286	1E+10	1E+06
	Optimal	4289	1325	3003	6E+08	1E+00

A.1.6 Out of Plane Min

Table 9: Settling Times and Respective Q & R for each Equilibrium Point at Out of Plane Min – 1%

		Position (s)	Velocity (s)	Euler Angles (s)	Q	R
E1	**Initial**	2699	2888	4829	1E+10	1E+06
	Optimal	1783	1288	2821	1E+10	1E+00
E2	**Initial**	4742	5081	5901	1E+10	1E+06
	Optimal	2345	2220	3865	5E+08	1E+00
E3	**Initial**	2858	2849	4202	1E+10	1E+06
	Optimal	1921	1454	2814	1E+09	1E+00
E4	**Initial**	4532	4530	5078	1E+10	1E+00
	Optimal	2866	2655	4234	1E+08	1E+00
E5	**Initial**	2865	2907	4216	1E+10	1E+06
	Optimal	1980	2043	3104	1E+08	1E+00
E6	**Initial**	6587	6568	7844	1E+10	1E+06
	Optimal	3150	2893	5296	4E+08	1E+00
E7	**Initial**	2776	2785	4142	1E+10	1E+06
	Optimal	2055	2110	3298	3E+07	1E+00
E8	**Initial**	3546	3489	4826	1E+10	1E+06
	Optimal	1884	1869	2874	1E+08	1E+00

A.2 2% Perturbation

A.2.1 In Plane Max

Table 10: Settling Times and Respective Q & R for each Equilibrium Point at In Plane Max – 2%

		Position (s)	Velocity (s)	Euler Angles (s)	Q	R
E1	**Initial**	3367	3392	3803	1E+10	1E+06
	Optimal	2417	2534	3126	2E+07	1E+00
E2	**Initial**	4115	4303	4741	1E+10	1E+06
	Optimal	2654	2804	4044	2E+07	1E+00
E3	**Initial**	2618	2729	3476	1E+10	1E+06
	Optimal	1886	2042	2663	2E+07	1E+00
E4	**Initial**	3115	3322	3780	1E+10	1E+06
	Optimal	1951	2022	2970	1E+08	1E+00
E5	**Initial**	2242	2630	2712	1E+10	1E+06
	Optimal	1897	2215	2079	1E+07	1E+00
E6	**Initial**	4613	4970	5046	1E+10	1E+06
	Optimal	4584	4775	4467	1E+07	1E+00
E7	**Initial**	2389	2716	2960	1E+10	1E+06
	Optimal	1922	2192	2242	1E+07	1E+00
E8	**Initial**	2389	2715	2956	1E+10	1E+06
	Optimal	2046	4195	2011	4E+07	1E+00

A.2.2 In Plane Min

Table 11: Settling Times and Respective Q & R for each Equilibrium Point at In Plane Min – 2%

		Position (s)	Velocity (s)	Euler Angles (s)	Q	R
E1	**Initial**	3367	3392	3765	1E+10	1E+06
	Optimal	2599	2378	3079	3E+07	1E+00
E2	**Initial**	4115	4303	4782	1E+10	1E+06
	Optimal	2497	2623	3939	3E+07	1E+00
E3	**Initial**	2618	2729	3676	1E+10	1E+06
	Optimal	1886	2042	2611	2E+07	1E+00
E4	**Initial**	3121	3331	3968	1E+10	1E+06
	Optimal	2062	2114	2780	6E+07	1E+00
E5	**Initial**	2242	2630	2893	1E+10	1E+06
	Optimal	1785	2021	2409	5E+07	1E+00
E6	**Initial**	4736	5212	5408	1E+10	1E+06
	Optimal	4639	4281	4410	5E+07	1E+00
E7	**Initial**	2389	2716	1258	1E+10	1E+06
	Optimal	2060	1749	1780	3E+07	1E+00
E8	**Initial**	2389	2716	1258	1E+10	1E+06
	Optimal	1863	1941	1992	2E+08	1E+00

A.2.3 Radial Max

Table 12: Settling Times and Respective Q & R for each Equilibrium Point at Radial Max – 2%

		Position (s)	Velocity (s)	Euler Angles (s)	Q	R
E1	**Initial**	3544	3545	4934	1E+10	1E+06
	Optimal	2713	2823	3650	2E+07	1E+00
E2	**Initial**	4524	4501	5895	1E+10	1E+06
	Optimal	3335	3442	4574	1E+07	1E+00
E3	**Initial**	2928	3057	4689	1E+10	1E+06
	Optimal	2602	2603	3984	1E+07	1E+00
E4	**Initial**	4015	4034	5844	1E+10	1E+06
	Optimal	2926	3035	4295	2E+07	1E+00
E5	**Initial**	2543	2562	3937	1E+10	1E+06
	Optimal	2005	2111	3083	2E+07	1E+00
E6	**Initial**	5720	5749	6473	1E+10	1E+06
	Optimal	4460	4250	5496	1E+07	1E+00
E7	**Initial**	2597	2602	3839	1E+10	1E+06
	Optimal	1996	2098	2792	2E+07	1E+00
E8	**Initial**	3939	3919	5012	1E+10	1E+06
	Optimal	3086	2939	4200	2E+07	1E+00

A.2.4 Radial Min

Table 13: Settling Times and Respective Q & R for each Equilibrium Point at Radial Min – 2%

		Position (s)	Velocity (s)	Euler Angles (s)	Q	R
E1	**Initial**	3052	3014	5385	1E+10	1E+06
	Optimal	2200	2082	4387	3E+09	1E+00
E2	**Initial**	4814	4822	7327	1E+10	1E+06
	Optimal	2905	2707	5800	2E+09	1E+00
E3	**Initial**	2860	2865	4929	1E+10	1E+06
	Optimal	2101	1984	3578	5E+08	1E+00
E4	**Initial**	3189	2651	5996	1E+10	1E+06
	Optimal	2575	2142	4532	1E+09	1E+00
E5	**Initial**	3121	3331	3968	1E+10	1E+06
	Optimal	4066	4028	3019	2E+10	1E+00
E6	**Initial**	5684	5785	8150	1E+10	1E+06
	Optimal	3834	3981	7521	7E+08	1E+00
E7	**Initial**	2326	2322	5267	1E+10	1E+06
	Optimal	3668	3820	3689	1E+11	1E+00
E8	**Initial**	3078	3017	5187	1E+10	1E+06
	Optimal	2130	2088	3725	7E+09	1E+00

A.2.5 Out of Plane Max

Table 14: Settling Times and Respective Q & R for each Equilibrium Point at Out of Plane Max – 2%

		Position (s)	Velocity (s)	Euler Angles (s)	Q	R
E1	**Initial**	2699	2888	5222	1E+10	1E+06
	Optimal	2296	2336	4152	4E+07	1E+00
E2	**Initial**	4756	4765	5918	1E+10	1E+06
	Optimal	2446	2390	4174	3E+08	1E+00
E3	**Initial**	2858	2849	5015	1E+10	1E+06
	Optimal	1837	1347	3032	2E+09	1E+00
E4	**Initial**	4532	4530	6104	1E+10	1E+06
	Optimal	2866	2655	4172	1E+08	1E+00
E5	**Initial**	2865	2907	4308	1E+10	1E+06
	Optimal	1997	2089	3361	5E+07	1E+00
E6	**Initial**	6595	6577	7974	1E+10	1E+06
	Optimal	3646	3596	5956	1E+08	1E+00
E7	**Initial**	2776	2785	4230	1E+10	1E+06
	Optimal	2032	1822	3271	6E+07	1E+00
E8	**Initial**	3546	3489	4608	1E+10	1E+06
	Optimal	1397	1371	3315	5E+08	1E+00

A.2.6 Out of Plane Min

Table 15: Settling Times and Respective Q & R for each Equilibrium Point at Out of Plane Min – 2%

		Position (s)	Velocity (s)	Euler Angles (s)	Q	R
E1	**Initial**	2699	2888	5148	1E+10	1E+06
	Optimal	1908	1438	3190	3E+09	1E+00
E2	**Initial**	4756	4765	6282	1E+10	1E+06
	Optimal	4568213	2390	4316	3E+08	1E+00
E3	**Initial**	2858	2849	4520	1E+10	1E+06
	Optimal	1575	1025	2869	4E+10	1E+00
E4	**Initial**	4532	4530	6377	1E+10	1E+06
	Optimal	2250	1771	3789	2E+09	1E+00
E5	**Initial**	2865	2907	4437	1E+10	1E+06
	Optimal	1218	1181	2485	1E+10	1E+00
E6	**Initial**	6587	6568	8351	1E+10	1E+06
	Optimal	3346	3215	5982	2E+08	1E+00
E7	**Initial**	2776	2785	4361	1E+10	1E+06
	Optimal	1965	1938	3163	1E+08	1E+00
E8	**Initial**	3546	3489	5218	1E+10	1E+06
	Optimal	1278	1207	2994	1E+09	1E+00

A.3 5% Perturbation

A.3.1 In Plane Max

Table 16: Settling Times and Respective Q & R for each Equilibrium Point at In Plane Max – 5%

		Position (s)	Velocity (s)	Euler Angles (s)	Q	R
E1	**Initial**	3367	3392	4134	1E+10	1E+06
	Optimal	2417	2534	3688	2E+07	1E+00
E2	**Initial**	4115	4303	5174	1E+10	1E+06
	Optimal	2976	3155	4816	1E+07	1E+00
E3	**Initial**	2618	2729	3778	1E+10	1E+06
	Optimal	1886	2042	3036	2E+07	1E+00
E4	**Initial**	3114	3322	4093	1E+10	1E+06
	Optimal	1952	2023	3394	1E+08	1E+00
E5	**Initial**	2242	2630	3008	1E+10	1E+06
	Optimal	1897	2215	2333	1E+07	1E+00
E6	**Initial**	4710	5153	5736	1E+10	1E+06
	Optimal	4829	4708	5559	1E+07	1E+00
E7	**Initial**	2389	2716	3286	1E+10	1E+06
	Optimal	1950	2227	2607	9E+06	1E+00
E8	**Initial**	2389	2715	3277	1E+10	1E+06
	Optimal	1949	2227	2598	9E+06	1E+00

A.3.2 In Plane Min

Table 17: Settling Times and Respective Q & R for each Equilibrium Point at In Plane Min – 5%

		Position (s)	Velocity (s)	Euler Angles (s)	Q	R
E1	**Initial**	3367	3392	4325	1E+10	1E+06
	Optimal	2417	2534	3818	2E+07	1E+00
E2	**Initial**	4115	4303	5366	1E+10	1E+06
	Optimal	2976	3155	4875	1E+07	1E+00
E3	**Initial**	2618	2729	4017	1E+10	1E+06
	Optimal	2118	2266	3024	1E+07	1E+00
E4	**Initial**	3115	3322	4330	1E+10	1E+06
	Optimal	2090	2347	3580	2E+07	1E+00
E5	**Initial**	2242	2630	3168	1E+10	1E+06
	Optimal	1798	2040	2622	2E+07	1E+00
E6	**Initial**	4736	5212	5910	1E+10	1E+06
	Optimal	4701	4537	5100	2E+07	1E+00
E7	**Initial**	2389	2716	1352	1E+10	1E+06
	Optimal	1922	2192	1056	1E+07	1E+00
E8	**Initial**	2389	2715	1348	1E+10	1E+06
	Optimal	2060	1748	2039	3E+07	1E+00

A.3.3 Radial Max

Table 18: Settling Times and Respective Q & R for each Equilibrium Point at Radial Max – 5%

		Position (s)	Velocity (s)	Euler Angles (s)	Q	R
E1	**Initial**	3544	3545	5392	1E+10	1E+06
	Optimal	2974	3060	4247	1E+07	1E+00
E2	**Initial**	4524	4501	6378	1E+10	1E+06
	Optimal	3332	3441	5084	1E+07	1E+00
E3	**Initial**	2928	3057	4996	1E+10	1E+06
	Optimal	2602	2603	4218	1E+07	1E+00
E4	**Initial**	4015	4034	6215	1E+10	1E+06
	Optimal	2926	3035	4807	2E+07	1E+00
E5	**Initial**	2543	2562	4359	1E+10	1E+06
	Optimal	2209	2285	3652	1E+07	1E+00
E6	**Initial**	4757	5797	7098	1E+10	1E+06
	Optimal	4460	4250	6230	1E+07	1E+00
E7	**Initial**	2597	2602	4307	1E+10	1E+06
	Optimal	2223	2290	3182	1E+07	1E+00
E8	**Initial**	3939	3919	5578	1E+10	1E+06
	Optimal	3146	3236	4948	1E+07	1E+00

A.3.4 Radial Min

Table 19: Settling Times and Respective Q & R for each Equilibrium Point at Radial Min – 5%

		Position (s)	Velocity (s)	Euler Angles (s)	Q	R
E1	**Initial**	3052	3015	5858	1E+10	1E+06
	Optimal	2200	2082	4887	3E+09	1E+00
E2	**Initial**	4813	4821	7706	1E+10	1E+06
	Optimal	2906	2707	6307	2E+09	1E+00
E3	**Initial**	2860	2865	5251	1E+10	1E+06
	Optimal	2146	2023	3946	4E+08	1E+00
E4	**Initial**	3189	2651	6648	1E+10	1E+06
	Optimal	2658	2204	5036	6E+08	1E+00
E5	**Initial**	2424	2351	4196	1E+10	1E+06
	Optimal	4425	4299	3543	1E+10	1E+00
E6	**Initial**	5684	5784	8860	1E+10	1E+06
	Optimal	3719	3868	8072	9E+08	1E+00
E7	**Initial**	2326	2323	5575	1E+10	1E+06
	Optimal	3806	3722	4155	1E+11	1E+00
E8	**Initial**	3078	3017	5525	1E+10	1E+06
	Optimal	2289	2219	4439	4E+09	1E+00

A.3.5 Out of Plane Max

Table 20: Settling Times and Respective Q & R for each Equilibrium Point at Out of Plane Max – 5%

		Position (s)	Velocity (s)	Euler Angles (s)	Q	R
E1	**Initial**	2699	2888	5567	1E+10	1E+06
	Optimal	2362	2404	4528	3E+07	1E+00
E2	**Initial**	4759	4769	6369	1E+10	1E+06
	Optimal	2833	2856	4920	1E+08	1E+00
E3	**Initial**	2858	2849	5254	1E+10	1E+06
	Optimal	2207	2007	3838	1E+08	1E+00
E4	**Initial**	4533	4531	6497	1E+10	1E+06
	Optimal	2059	1532	4022	1E+10	1E+00
E5	**Initial**	2865	2907	4578	1E+10	1E+06
	Optimal	2238	2291	3974	3E+07	1E+00
E6	**Initial**	6595	6577	8495	1E+10	1E+06
	Optimal	3646	3596	6602	1E+08	1E+00
E7	**Initial**	2776	2785	4501	1E+10	1E+06
	Optimal	1920	1996	3626	4E+07	1E+00
E8	**Initial**	3546	3489	5020	1E+10	1E+06
	Optimal	1447	1431	3762	4E+08	1E+00

A.3.6 Out of Plane Min

Table 21: Settling Times and Respective Q & R for each Equilibrium Point at Out of Plane Min – 5%

		Position (s)	Velocity (s)	Euler Angles (s)	Q	R
E1	**Initial**	2699	2888	5557	1E+10	1E+06
	Optimal	1908	1772	3533	3E+09	1E+00
E2	**Initial**	4758	4768	6700	1E+10	1E+06
	Optimal	2003	1470	4203	2E+10	1E+00
E3	**Initial**	2858	2849	4898	1E+10	1E+06
	Optimal	1921	1454	3420	1E+09	1E+00
E4	**Initial**	4533	4531	6763	1E+10	1E+06
	Optimal	2250	1771	4253	2E+09	1E+00
E5	**Initial**	2865	2907	4725	1E+10	1E+06
	Optimal	1153	1023	2719	4E+10	1E+00
E6	**Initial**	6595	6577	9012	1E+10	1E+06
	Optimal	3646	3596	6753	1E+08	1E+00
E7	**Initial**	2776	2785	4647	1E+10	1E+06
	Optimal	1965	1938	3335	1E+08	1E+00
E8	**Initial**	3546	3489	5600	1E+10	1E+06
	Optimal	2251	2213	3936	4E+07	1E+00