SPANNING WITH BINARY OPTIONS: A VECTOR LATTICE APPROACH

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Abstract

The issue of portfolio insurance is one of the prime concerns of the investors who want to insure their asset at minimum or appropriate cost. Static hedging with binary options is a popular strategy that has been explored in various option models (see e.g. (2; 3; 4; 7)). In this thesis, we propose a static hedging algorithm for discrete time models. Our algorithm is based on a vector lattice technique. In chapter 1, we give the necessary background on the theory of vector lattices and the theory of options. In chapter 2, we reveal the connection of lattice-subspaces with the minimum-cost portfolio insurance strategy. In chapter3, we outline our algorithm and give applications to binomial and trinomial option models. In chapter 4, we perform simulations and analyze the hedging errors of our algorithm for European, Barrier, Geometric Asian, Arithmetic Asian, and Lookback options. The study has revealed that static hedging could be suitable strategy for the European, Barrier, and Geometric Asian options as these options have shown less inclination to the rollover effect.

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Chapter 1

Introduction

The first chapter will review the theoretical aspects of the study: Vector lattice theory followed by option theory and stock price models.

1.1 Vector Lattices

Vector lattice theory is a branch of pure mathematics that has produced enormous applications in Mathematical Finance and Economics. In this section, we will give the necessary background of the theory needed for the exposition of this thesis.

Ordered Vector Spaces A vector space $\mathcal{X}$ equipped with a partial order [1] is an ordered vector space if it satisfies $X + Z \geq Y + Z$, and $\alpha X \geq \alpha Y$ for each Z, $X \geq Y$ and $\alpha \geq 0$.

Upper Bound and Lower Bound Let $(\mathcal{X}, \geq)$ be an ordered vector space. A point $X \in \mathcal{X}$ is an upper bound (resp. lower bound) of a subset $\mathcal{M} \subseteq \mathcal{X} \iff Y \leq X (resp. X \leq Y)$, for all $Y \in \mathcal{M}$.

Vector Lattices An ordered vector space $(\mathcal{X}, \geq)$ is said to be a vector lattice if and only if the least upper bound of $\{X, Y\}$ exists for each $X, Y \in \mathcal{X}$. The least upper bound of $\{X, Y\}$ is denoted by $X \vee Y$ and the greatest lower bound of $\{X, Y\}$ is denoted by $X \wedge Y$.

The space $\mathbb{R}^m$ eqquiped with the pointwise order $\geq$ is a vector lattice. In the following we will give some basic terminology and facts of $(\mathbb{R}^m, \geq)$.

Pointwise Order Relation in $\mathbb{R}^m$ It is defined as $X \leq Y \iff X(i) \leq Y(i)$ for all $X, Y \in \mathbb{R}^m$, $i = 1...m$.

[1] Partially Ordered Set: A set $(\mathcal{X}, \geq)$ is partially ordered if $\geq$ is reflexive, antisymmetric, and transitive.

Lattice Operations and Identities in $\mathbb{R}^m$ For any two vectors $X, Y \in \mathbb{R}^m$, the following identities hold:

$$X \vee Y = (\max(X(1), Y(1)),\ \max((X(2), Y(2)),\ \ldots,\ \max(X(m), Y(m))) \tag{1.1}$$

$$X \wedge Y = (\min(X(1), Y(1)),\ \min(X(2), Y(2)),\ \ldots,\ \min(X(m), Y(m))) \tag{1.2}$$

$$X^+ = X \vee 0 \tag{1.3}$$

$$X^- = (-X) \vee 0 \tag{1.4}$$

$$|X| = X \vee (-X) \tag{1.5}$$

$$X = X^+ - X^- \tag{1.6}$$

$$|X| = X^+ + X^- \tag{1.7}$$

The set of all positive elements of $\mathbb{R}^m$ is denoted by $\mathbb{R}^m_+ = \{X \in \mathbb{R}^m \mid X \geq 0\}$.

Subspaces of $\mathbb{R}^m$ A subset $\mathcal{M} \subseteq \mathbb{R}^m$ is a subspace of $\mathbb{R}^m$ if $\mathcal{M}$ satisfies the following conditions

1. if X and Y are two vectors in $\mathcal{M}$, then $X + Y$ is also in $\mathcal{M}$

2. if t is a real number and X is in $\mathcal{M}$, then tX is also in $\mathcal{M}$

A base for a subspace $\mathcal{M}$ is a collection of vectors $\{D_1,\ \ldots,\ D_k\}$ in M that are linearly independent and have the following property

$$X \in \mathcal{M} \text{ if and only if } X = \sum_{i=1}^{k} \lambda_i D_i,\ \text{for some } \lambda_i \in \mathbb{R} \tag{1.8}$$

The minimal subspace that contains the vectors $\{Z_1,\ Z_2,\ \ldots,\ Z_r\}$ is denoted by $span\{Z_1,\ Z_2,\ \ldots,\ Z_r\}$ and is given by the following formula

$$span\{Z_1,\ Z_2,\ \ldots,\ Z_r\} = \{\sum_{i=1}^{r} \lambda_i Z_i \mid \lambda_i \in \mathbb{R}\} \tag{1.9}$$

Lattice Subspaces of $\mathbb{R}^m$ A subspace $\mathcal{M} \subset \mathbb{R}^m$ is said to be a lattice subspace if $(\mathcal{M}, \geq)$ is a vector lattice on its own.

Sublattices of $\mathbb{R}^m$ A subspace $\mathcal{S}$ is said to be a sublattice of $\mathbb{R}^m$ whenever $X^+ = X \vee 0 \in \mathcal{S}, \forall\ X \in \mathcal{S}$.

A base $\{B_i \mid i = 1, \ldots, k\}$ for a sublattice $\mathcal{S}$ is said to be a positive base whenever

$$X \vee Y = \sum_{i=1}^{k} (\lambda_i \vee \mu_i) B_i,\ X = \sum_{i=1}^{k} \lambda_i B_i,\ Y = \sum_{i=1}^{k} \mu_i B_i \tag{1.10}$$

The minimal sublattice that contains the subspace $\mathcal{Z}$ is denoted by $\mathcal{S}(\mathcal{Z})$. In the following we will give an algorithm to calculate $\mathcal{S}(\mathcal{Z})$.

For the following fix some linear independent vectors $Z_1, ..., Z_r$ in $\mathbb{R}^m_+$ and $\mathcal{Z} = span\{Z_1,\ Z_2,\ \ldots,\ Z_r\}$. We first define $Z = \sum_{j=1}^{r} Z_j$. That is

$$Z(i) = \sum_{j=1}^{r} Z_j(i),\ \forall i \in \{1,\ ...,\ m\} \tag{1.11}$$

For $Z(i) > 0$ we define the basic function $\beta : \{1, ..., m\} \to \mathbb{R}^r_+$ as follows

$$\beta(i) = (\frac{Z_1(i)}{Z(i)},\ \frac{Z_2(i)}{Z(i)},\ \ldots,\ \frac{Z_r(i)}{Z(i)}) \tag{1.12}$$

The range of β is denoted as $R(\beta)$ and the cardinality of $R(\beta)$ is the number of elements of $R(\beta)$ denoted as card$R(\beta)$. $R(\beta)$ is shown as

$$R(\beta) = \{\beta(i) \mid i = 1,\ 2,\ \ldots,\ m,\ Z(i) > 0\} \tag{1.13}$$

The subspace $\mathcal{Z}$ is a sublattice of $\mathbb{R}^m$ if and only if card $R(\beta) = r$. In this case, a positive basis $\{B_1,\ B_2,\ \ldots,\ B_r\}$ of $\mathcal{Z}$ is given by

$$\begin{bmatrix} B_1 \\ \vdots \\ B_r \end{bmatrix} = A^{-1} \begin{bmatrix} Z_1 \\ \vdots \\ Z_r \end{bmatrix} \tag{1.14}$$

$$A = [\beta(1),\ \beta(1),\ \ldots,\ \beta(r)] \tag{1.15}$$

Sublattice Algorithm

In the case where the input matrix $\mathcal{Z}$ is not a sublattice of $\mathbb{R}^m$ we have the following method to generate $\mathcal{S}(\mathcal{Z})$.

1. Calculate the function β along with $R(\beta) = \{P_1,\ P_2,\ \cdots,\ P_\mu\}$, where the first r vectors are linearly independent.

2. Then $\mathcal{S}(\mathcal{Z}) = span\{Z_1,\ Z_2,\ \ldots,\ Z_r,\ Z_{r+1},\ \ldots,\ Z_\mu\}$, where the vectors $Z_{r+k}, k = 1,\ 2,\ \ldots,\ \mu - r$ are defined as follows

$$Z_{r+k}(i) = Z(i) \text{ if } i \in I_{r+k} \text{ and } Z_{r+k}(i) = 0 \text{ if } i \notin I_{r+k},$$

where $I_{r+k} = \{i \in \{1,\ 2,\ \ldots,\ m\} \mid \beta(i) = P_{r+k}\}$ for each $k = 1,\ 2,\ \ldots,\ \mu - r$.

3. The positive basis of $\mathcal{S}(\mathcal{Z})$ is given by applying the formula 1.14.

Working Example : Consider the following three vectors Z_1, Z_2, Z_3 in $\mathbb{R}^5$ and $\mathcal{Z} = span\{Z_1, Z_2, Z_3\}$.

$$\begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 4 & 1 \\ 2 & 3 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

The sum vector is $Z = Z_1 + Z_2 + Z_3 = (4, 4, 1, 5, 3)$ and the function β is defined as follows:

$$\beta(1) = (\frac{1}{4}, \frac{2}{4}, \frac{1}{4}),$$

$$\beta(2) = (\frac{1}{4}, \frac{3}{4}, 0),$$

$$\beta(3) = (0, 0, 1),$$

$$\beta(4) = (\frac{4}{5}, 0, \frac{1}{5}),$$

$$\beta(5) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}),$$

We see that card $R(\beta) = 5$ and thus $\mathcal{S}(\mathcal{Z}) = \mathbb{R}^5$.

1.2 Options and Stock Price Models

This section presents the concept of derivatives, options, and hedging . Besides this, the section includes Black-Scholes, and discrete times models. Let us begin with the concept of derivatives.

A derivative is a contract whose value is derived from the behavior of an underlying asset such as stocks, currency or bond. It is assumed that they have existed for hundred of years in their primitive form; however, rapid industrialization has resulted highly sophisticated form, and explosive increase in its volume, use, and types. Therefore, a concept of security may have aroused in order to reduce the risk associated with the derivatives. Hedging is one of strategies adopted to the financial market to minimize the risk. A best example of hedging could be buying or selling of options.

1.2.1 Options

An option is a contract that one party sells to another. The holder of the option has a choice or right to buy or sell the underlying asset at a given price(strike price) before or on the expiration date depending on the nature of the options. A traditional or vanilla European call and put option can be exercised at maturity date . On the other hand, derivative like American call and put options can be exercised at anytime during its expiration time period.

An option can be classified into path dependent and path independent. A path dependent option is an exotic option whose value depends not only the price of the underlying asset but the entire path that asset takes during its life. Asian, lookback, and barrier are the example of path dependent options. The path dependent options can further be classified as soft path dependent, and hard path dependent. A soft path dependent assumes the lowest or highest value in trading history or it could be a triggering event such as the underlying touching a specific price. Barrier, and lookback options fall into this category. A hard path dependent option takes into account the entire trading history of the underlying asset. Options type in this group include Asian options. On the other hand, path independent options do not depend on the path travelled during its period, rather its payoff solely depends on the price at the expiration time. A prime example of this category is a European option where the price for the buyer depends exclusively on the terminal price. This study has focused on both path dependent and independent options. A path independent European call option and path dependents call options such as knockout, lookback, Asian (both geometric and Arithmetic Average) have been considered in this study.

European Options

A European call option gives the owner the right to buy a stock at a given price at some time in future (the maturity). It is strictly a right, but not an obligation. If the market price is below the strike, the owner will not execute the transaction. On the other hand, if the market price is above the strike, the owner can buy the stock at the strike and immediately sell it in the market. Thus, the payoff of a European call option is

$$\max(S - K, 0)$$

where S is stock price at maturity and K is the strike price. Similarly, a European put option gives the owner the right to sell a stock at a given price at some time in the future.

The payoff of a European put option is

$$\max(K - S, 0)$$

The figure 1.1 presents the payoff European put and call options. Beyond calls and puts, a wide variety of other option exists. These are contracts that differ from traditional or vanilla put or call options in their payment, structures, expiration dates, and strikes prices. In other words, exotic options are variation of American and European option styles. In this section, we will briefly discuss the exotic options: binary, Asian, barrier, and lookback.

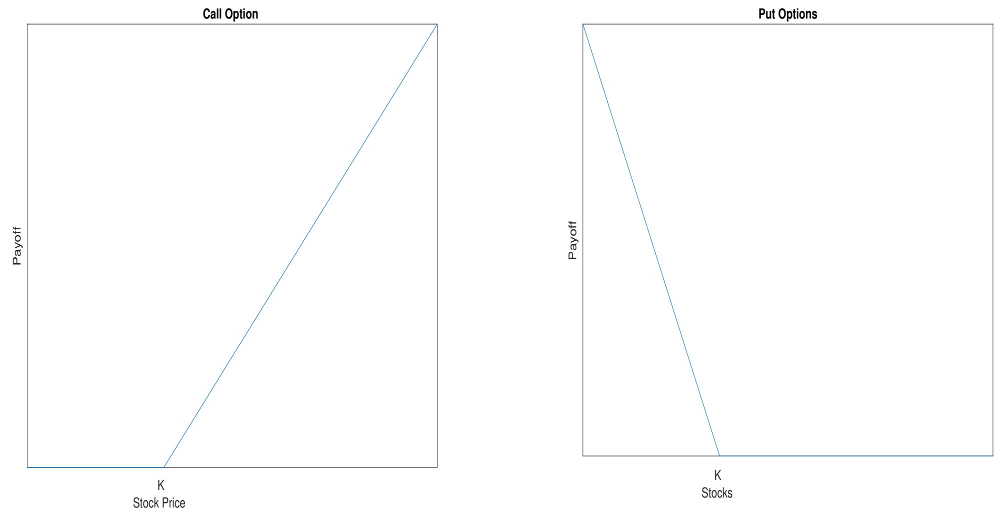


Figure 1.1: Payoffs for European Calls and Puts .

Binary Options

The simplest exotic option is the binary or digital options. Unlike traditional call option, in which final payout increase incrementally with each rise in the underlying asset price above the strikes, binaries pay a finite lump-sum if the asset is above strike. In a similar manner, a buyer of binary option is paid a finite lump-sum if the asset closes below the stated exercised price. In other words, this pays off nothing if the asset price ends up below or above based on the nature of options.

It pays the fixed amount Q if the asset price is below the strike price, and nothing if it is above the strike price. The payoffs for a binary call and put with initial price S, strike price K, and time to maturity T are

$$C(S,\ K,\ T) = \begin{cases} Q & S_T > K \\ 0 & S_T \leq K \end{cases} \tag{1.16}$$

$$P(S,\ K,\ T) = \begin{cases} Q & S_T < K \\ 0 & S_T \geq K \end{cases} \tag{1.17}$$

Asian Options

The option that are fully path dependent and their payoffs depend on the history of random walk of asset price via some sort of average which provides different type of cash-flow despite to the more standard American and European options are known as Asian options. The options for both calls and puts have two variations, fixed strike and floating strike options. The payoff of a fixed strike Asian option is the positive difference between the average underlying price over the time period and the predetermined fixed strike at maturity where as the payoff is the positive difference between the average underlying price over the time period and the underlying price at maturity. In this study, we will focus on fixed strike geometric average options and arithmetic average options . The payoffs for a fixed strike call and put with initial price S, strike price K, and time to maturity T are

$$C(S,\ K,\ T) = max(0,\ S_{avg} - K) \tag{1.18}$$

$$P(S,\ K,\ T) = max(0,\ K - S_{avg}) \tag{1.19}$$

where avg means geometric or arithmetic average based on their context.

Barrier Options

Barrier options are similar to plain vanilla calls and puts; however, the payoff depends whether the underlying asset price reaches a certain level over the life time options. In other words, it only becomes activated when underlying asset hits a present price level. In this sense, the value of the option jumps up or down in leaps instead of changing the price in small increments. Examples of this options are foreign exchange and equity market. These options can be classified as either knock-out options or knock-in options. A knock-out option ceases to exist when the underlying asset price reaches a certain barrier; a knock-in option comes into existence only when the underlying asset price reaches a barrier.In principle, barrier option can be applied to any options. this paper analyzes knock-out barrier options that have vanilla payoffs if the underlying asset does not hit the predetermined barrier level H. The payoffs for a knockout call and put with initial price S, strike price K, and time to maturity T are

$$C(S,\ K,\ T) = \begin{cases} max(0,\ S_T - K) & S < H \\ 0 & S \geq H \end{cases} \tag{1.20}$$

$$P(S,\ K,\ T) = \begin{cases} max(0,\ K - S_T) & S > H \\ 0 & S \leq H \end{cases} \tag{1.21}$$

Lookback Options

Lookback options are type of exotic options whose payoff depends on the optimal underlying asset's price occurring over the life of the option. The option allows the holder the advantage of knowing history(lookback) to determine the payoff. There exists two kind of lookback options: with floating strike and with fixed strike. Like standard European options, the option's strike price is fixed; however, the difference is that the price is not exercised at the maturity price. In this case, the payoff is maximum difference between the optimal underlying asset price and the strike.

The payoff functions for the lookback call (LC) and the lookback put(LP), respectively, are given by:

$$C(S,\ K,\ T) = max(0,\ S_{max} - K) \tag{1.22}$$

$$P(S,\ K,\ T) = max(0,\ K - S_{min}) \tag{1.23}$$

where as $S_{\max}$ is the asset's maximum price during the life of the option, $S_{\min}$ is the asset's minimum price during the life of the option, and K is the strike price.

On the other hand, the payoff functions for the lookback call (LC) and the lookback put(LP) with floating strike prices , respectively, are given by:

$$C(S,\ K,\ T)_{float} = max(0,\ S_T - S_{min}) \tag{1.24}$$

$$P(S,\ K,\ T)_{float} = max(0,\ S_{max} - S_T) \tag{1.25}$$

where as $S_{\max}$ is the asset's maximum price during the life of the option, $S_{\min}$ is the asset's minimum price during the life of the option, S_T is the underlying asset's price at maturity T . In this study, we will discuss about the lookback call option with fixed strike price.

To sum-up, many exotic options are path- dependent while Plain-vanilla options are path independent; their payoff only depends upon the price at maturity. In generals, America options are also path dependent.

1.2.2 Hedging

The main purpose of options is hedging. A hedging is an investment to reduce the risk of adverse price movement in an asset. A hedge consists of taking an offsetting position in a related security. In other words, hedging is the strategy of reducing the sensitivity of a portfolio to the movement of an underlying asset by taking opposite positions in different financial instruments. It is analogous to taking out an insurance policy. In a classical hedging, put options are used for downside protection. Let's illustrate with an example. Sushma is an investor; her money is an index fund. She prefers the stocks market over bonds because she knows that historical return is much greater though she has knowledge that stock market is riskier than bond market. She is worried, but is willing to take some risk. At any chances of losses, she wants(plans) to limit her losses to ten percent. To achieve the plan, she has two strategies. The first one is to stop a loss order. Suppose price of Sushma's fund is 100. If price drops below 90, she will immediately sell it. This immediate selling is possible only under normal market condition. However, in a crash situation, the prices drops so fast she will not able to sell her portfolio at 90; she could lose much more. She wants insurance against it. So her another strategy will be buying a put option with a strike price of 90 that will yield her desired protection. This simple examples illustrates how option can be used as insurance. Insurance is just one applications of hedging with options (Chou A. 1997) as the author explains the hedging concept in the unpublished Doctoral dissertation 'Static Replication of Exotic Options'.

Most hedging strategies are classified as either dynamic or static hedging. Dynamic hedging requires the hedge position to constantly be updated in response to market movements. Delta hedging is a popular dynamic hedging strategy for options that matches the option delta by constantly going long and short on the underlying asset according to market movements. The primary drawbacks of delta hedging include the constant need to monitor the markets and the transaction costs involved. Depending on the movement of the stock the trader has to frequently buy and sell securities in order to avoid being under or overhedged and will incur transaction costs causing lower returns.

Static hedging is the practice of taking a hedge position that has a similar maturity to investment portfolio. In the context of this paper we are looking to hedge exotic with binary options that have identical maturity. In this case, there is no need to constantly monitor the market and constantly buy and sell assets as investors would do in a dynamic hedging strategy. The tradeoff in dynamic versus static hedging is between accuracy and cost, as to perfectly delta hedge this would require infinitesimal time periods between rebalancing. In this paper we outline an algorithm that determines a static hedge for any exotic option, using binary options under binomial and trinomial tree assumptions. This leads to situations where exotic options can have different payoffs at identical maturity prices and the corresponding vanilla option hedge returns identical payoffs. In this case, one cannot create a perfect hedge; however, by minimizing the error we can receive the benefits of static hedging over dynamic hedging.

The next section will discuss about the most celebrated model in mathematical finance: the Black-Scholes options pricing model. It has tremendous theoretical and practical implications. However, it has some limitations.

1.2.3 Stock Price Models

Black-Scholes Model

The model is one of the most important concepts in modern financial theory. It was developed in 1973 by Fischer Black, Robert Merton, and Myron Scholes, and still has wide application and regarded as one of the best ways of determining fair prices of options.

Basically, the Black-Scholes model requires five input variables: the strike price of an option, current stock price S, the time to expiration T, the risk free rate r, and volatilityσ. With certain assumptions, they derived the following European call (C) and put (P) options :

$$C = SN(d_1) - KN(d_2) \tag{1.26}$$

$$P = KN(-d_2) - SN(-d_1) \tag{1.27}$$

where,

$$d_1 = \frac{\ln(\frac{S}{K}) + (r - \rho + \frac{\sigma^2}{2}(T - t)}{\sigma\sqrt{T - t}}$$

and

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

If the underlying asset pays continuously fixed dividend ρ, then the call and put options will be:

$$C = Se^{-\rho(T-t)}N(d_1) - Ke^{-\rho(T-t)}N(d_2) \tag{1.28}$$

$$P = Ke^{-\rho(T-t)}N(-d_2) - Se^{-\rho(T-t)}N(-d_1) \tag{1.29}$$

where $N(.)$ is cumulative normal distribution with mean zero and standard deviation one.

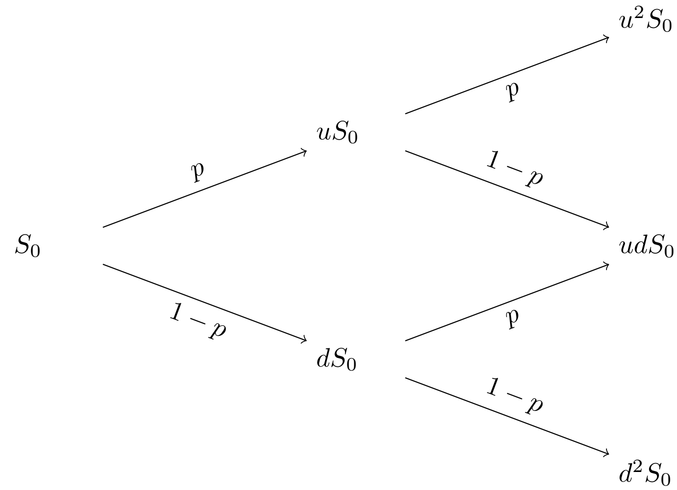
There are multiple derivations of this formula. However, the most three important are differential equation method, the binomial model , and risk neutral probability measure. The original derivations in 1973 used differential equation. In 1979, Cox, Ross, Rubinstein proposed binomial model, which has important practical application because it yields numerical solutions. Those with unfamiliar with stochastic process or differential equations, binomial method provides a nice combinatorial interpretations of the Black-Scholes model. Later, Phelim Boyle(1986) extended binomial model to trinomial model. The next section will explain the binomial, and trinomial option pricing model.

Financial Mathematics or quantitative finance models the security price in terms of discrete or continuous time frame. Discrete time model assumes trades take place instantaneously, as a result its prices depends upon finite number of states. This section focus the security price movement on discrete time frame only.

Binomial Model

The binomial option pricing model provides a numerical methods for the valuations of options proposed by Cox, Ross, and Rubenstien in 1979. The model uses a discrete rime (lattice based) model of the price variation for an asset over time. The valuation of options is computed using a binomial tree, for a number of time steps between the valuation and expiration dates. Each node in the lattice represents a possible price of underlying assets at a given point in time. The working procedure involves iterative approach, starting at each of the final nodes, and working backwards through the tree towards the first node (valuation date). At each step, it is assumed that underlying asset will move up u or down d by a specific factor per step of the tree. If S_0 is the current price , then in the next period the price will either be $S_u = uS_0$ or $S_d = dS_0$, where $u \geq 1, 0 < d \leq 1$. The up and down factors which are calculated using the volatility σ, time duration of step t (measured in years) are $u = e^{\sigma\sqrt{t}}$, $d = e^{-\sigma\sqrt{t}} = \frac{1}{u}$. The diagram below shows the price variation under binomial tree model.

Price Variation under Binomial Tree Model

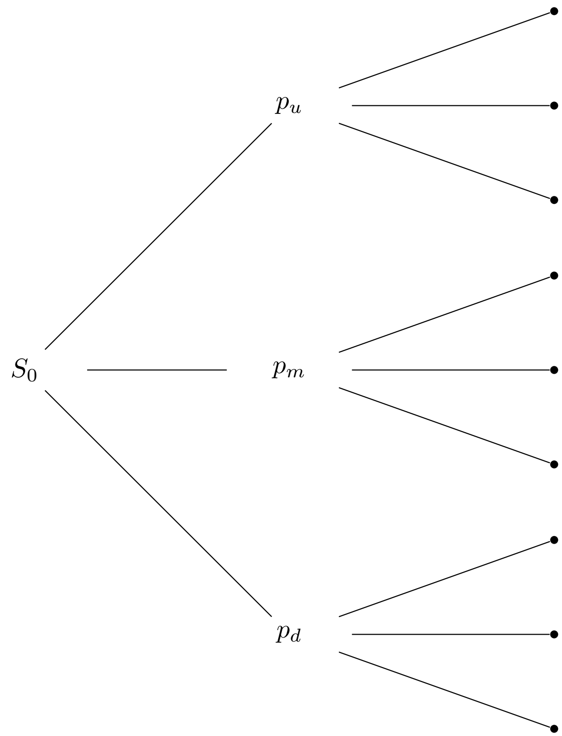


Various options that are path dependent such as American options are prime examples of effective applications of the model, which is still used today. It has similar assumptions to the Black-Scholes model, with the binomial model providing a discrete time approximation to the continuous time processes that the Black-Scholes equation models. It has been proved that as the number of time periods in the binomial increases towards infinity the binomial tree approximation will converge to the Black-Scholes result with the same parameters.

Trinomial Model

The trinomial model is an extension to the binomial model, and is considered as more advanced model. In this model, the price increases by a factor of u with probability p_u, decreases by a factor of d with the probability p_d or remains unchanged with probabilities $p_m = 1 - p_u - p_d$ where as $ud = 1$. The diagram below presents the price variation under trinomial tree model.

Two- Step Trinomial Pricing Tree



Above trinomial tree is formulated as

$$S(t+\Delta t) = \begin{cases} \text{S(t)u with probability p}_u \\ \text{S(t) with probability 1 - p}_u - p_d \\ \text{S(t)d with probability p}_d \end{cases} \tag{1.30}$$

with its parameter $u = e^{\sigma\sqrt{2\Delta t}}$, $d = e^{-\sigma\sqrt{2\Delta t}}$, and transition probabilities p_u, p_d, p_m

The trees provide an effective method of numerical calculation of option prices within Black-Scholes share pricing model. The overall calculation of option price is similar to the binomial model where the option value is calculated backwards from the final nodes according to price and probability of three potential nodes instead of two. However, trinomial incorporates a third value: a zero change in value over a time period. This assumption makes the model more relevant to real life situations as the underlying asset may not change over a time period , such as a month a year. For instance, pricing exotic options which are more complex than simple vanilla options, the trinomial model is considered more stable and accurate. Both models are utilized in this paper in order to preform a complete evaluation of the algorithm.

Chapter 2

Portfolio Insurance

In this chapter will discuss applications of the vector lattice theory to the minimum cost portfolio insurance problem.

2.1 The problem

Generally, investors do not considered portfolio insurance as a policy, rather they take it as an investment strategy. In other words, it is a hedging strategy developed to limit the losses an investor might face from declining stock index. In practices, Investors want to balance stocks and options on stocks to achieve a risk free portfolio. For instance, buying an index put option is one way to create a risk free or less risk associated portfolio. This approach allows an investor to preserve the upside gains but limit downside risk. Therefore, they might use Portfolio insurance when the market direction is uncertain or volatile.

The concept of portfolio insurance was originated by Hayne Leland when investors dropped out of the market as they foreseen their investment would be at risk due to highly volatile market. Hayne Leland and Mark Rubinstein in 1976 developed strategies to cope with this issue. Then, they began offering a portfolio insurance service. This service provided investors to hedge against risk. For example, with enough put options at the right price, the profit from selling the stocks can offset most or all of the losses from a bad market swing. However, it is not free from drawback. It costs money to buy put options. If a portfolio does not move downward, the option's price will reduce the profit. Therefore, the stockholders want to pay the minimum cost to buy the insurance in order to maximize the profit.

Minimum- cost portfolio insurance is an investment strategy that enables an investor to avoid losses and capture gains of a payoff of a portfolio at minimum cost. The next section will explains under which conditions there is an existence of the minimum-cost portfolio insurance. For more details on this problem we refer the reader to (1).

2.2 Existence results

We will denote with $\mathcal{M}$ the asset span and we assume that $\mathcal{M}$ is a subspace of $\mathbb{R}^s$.

Theorem 1 *The minimum-cost insured portfolio exists and is price independent for every portfolio and at every floor if and only if the asset span is lattice-subspace of $\mathbb{R}^s$. In this case, the minimum -cost insured portfolio θ^k satisfies*

$$X(\theta^k) = X(\theta) \vee_{\mathcal{M}} \mathbf{k}$$

Proof Assume that minimum cost insured portfolio exist and is price independent . For any portfolio η such that $X(\eta) \geq X(\theta) \vee \mathbf{k}$, we have that $p.\eta \geq p.(\theta)^k$ for every arbitrage -free price vector p. By Lemma 2.1, $X(\eta) \geq X(\theta)^k$. This implies payoff $X(\theta)^k$ is the supremum of $X(\theta)$ and $\mathbf{k}$ in $\mathcal{M}$.

For any two arbitrary payoffs y_1 and y_2 in , we need to show that there exists supremum $y_1 \vee_{\mathcal{M}} y_2$. The simple lattice identity $y_1 \vee_{\mathcal{M}} y_2 = [(y_1 - y_2 + k) \vee_{\mathcal{M}} \mathbf{k}] + \mathbf{y_2} - \mathbf{k}$ proves that supremum exists . Hence, $\mathcal{M}$ is a lattice subspace.

Conversely, if the asset span $\mathcal{M}$ is a lattice-subspace, then the supremum $X(\theta) \vee_{\mathcal{M}} k$ exists for every portfolio (θ) and every $\mathbf{k}$. Now, let the portfolio $(\theta)^k)$ be such that $X(\theta^k) = X(\theta) \vee_{\mathcal{M}} \mathbf{k}$. Then, for every portfolio η satisfying $X(\eta) \geq X(\theta) \vee \mathbf{k}$, we have that $X(\eta) \geq X(\theta)^k$. This inequalities implies $p.\eta \geq p.(\theta)^k$ for every arbitrage-free price vector p. Consequently, the portfolio (θ^k) is the minimum cost insured portfolio for every arbitrage free price. This ends the proof of the theorem.

The above theorem requires that the asset span should be lattice subspace for the existence of price-independent minimum-cost portfolio insurance. However, there is still a question in which condition the asset span will be a lattice subspace. The following theorem (by Abramovich-Aliprantis-Polyrakis, 1994) reveals the requirement for the asset span $\mathcal{M}$ to be a lattice subspace. We will present this theorem without proof.

Theorem 2 *The asset span $\mathcal{M}$ is a lattice subspace of $\mathbb{R}^s$ if and only if there is fundamental set of states.*

Before presenting a method of finding the minimum-cost insured portfolio, we will describe a relation of contingent claims $y_1, y_2 \in R^s$ in a fundamental set of states F.

For a fundamental set of states $\mathbf{F}$ and two contingent claims $y_1, y_2 \in R^s$, $y_1 \geq_F y_2$ if y_1 dominates y_2 in the fundamental states. That is , $y_1 \geq_F y_2$ if y_1 for every state $s \in F$.
Similarly , we write $y_1 =_F y_2$ if y_1, if y_1 and y_2 are the same in the fundamental states. Here, we will only state the following theorem.

Theorem 3 *If $F = \{s_1, s_2, ..., s_N$ is a fundamental set of states and y_1, y_2 are payoffs in the asset span $\mathcal{M}$, then $y_1 \geq y_2$ if and only if $y_1 \geq_F y_2$*

The next theorem emphasis to the fundamental set of states for the minimum cost portfolio insurance. It says that if there exists a fundamental set of states, then only the fundamental states are relevant for the minimum-cost portfolio insurance .

Theorem 4 *Suppose that there exists a fundamental set of states F for the asset span $\mathcal{M}$. Then for every arbitrage- free price system p and for every portfolio θ and floor k , the minimum- cost insured portfolio θ^k is the unique portfolio that replicates the insured payoff $X(\theta) \vee k$ in the fundamental states. That is ;*

$$X(\theta^k) =_F X(\theta) \vee k$$

The portfolio θ^k is the solution to the equation

$$X_F(\theta^k) =_F X(\theta) \vee k$$

that is ,

$$\theta^k = X_F^{-1}[X(\theta) \vee k]_F$$

Proof *If there exists a fundamental set of states , then theorem 2 implies that the asset span $\mathcal{M}$ is a lattice subspace. If $\mathcal{M}$ is a lattice subspace, then applying theorem 1, the minimum cost insured portfolio satisfies $X(\theta^k) = X(\theta) \vee_{\mathcal{M}} \mathbf{k}$. Now, wee need to show $_F X(\theta) \vee k = X(\theta) \vee_{\mathcal{M}} \mathbf{k}$.*
Let $z \in \mathcal{M} \implies z =_F X(\theta) \vee k$. We note here that X_F is non-singular which implies that payoff always exists and unique.
Let us verify z is the supremum of $X(\theta)$ and $\mathbf{k}$ relative to the asset span $\mathcal{M}$. First, let us claim that z is an upper bound of $X(\theta)$ and $\mathbf{k}$. Indeed , since $z =_F X(\theta) \vee k$, it follows that $z \geq_F X(\theta)$ and $z \geq_F k$, and so from theorem 3, we can infer that $z \geq X(\theta)$ and $z \geq k$ in $\mathcal{M}$. z is the least upper bound of $X(\theta)$ and $\mathbf{k}$ in $\mathcal{M}$. let $y \in \mathcal{M}$ satisfy $y \geq X(\theta)$ and $y \geq k$. Then, $y \geq X(\theta) \vee \mathbf{k}$, and hence $y \geq_F X(\theta) \vee k$. Consequently $y \geq_F z$. Now, theorem 3 implies $y \geq z$. Therefore, z is the supremum $X(\theta) \vee_{\mathcal{M}} \mathbf{k}$. This completes the theorem.

Working Example We consider two securities with payoffs in three states $x_1 = (1, 1, 1), x_2 = (1, 1, 1)$. We will compute minimum-cost insured portfolio at every arbitrage- free price.

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

The matrix whose columns are the components of the vectors X_1, X_2

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$$

Here, $\{y_1 = (1,0), y_3 = (1,2)\}$ is fundamental set because

$(1,1) = \frac{1}{2}(1,0) + \frac{1}{2}(1,2)$ i.e. there exist non negative scalars $\frac{1}{2}$, $\frac{1}{2}$, such that $X(s) = \sum_{j=1}^{N} \alpha_i^s x(s_i)$, $s \notin \mathcal{F}, \alpha_i \geq 0$. Therefore, $\mathcal{M}$ is lattice subspace. The contingent $x_2 \vee 1$ is not marketed since it does not belong to asset span. We can show that minimum cost insured portfolio $(1, \frac{1}{2})$ will create maximum payoff $x_2 \vee_M 1 = (1, \frac{3}{2}, 2)$.

Chapter 3

Hedging Algorithm

In this chapter, we will present a static hedging algorithm based. Our algorithm is based on the following elegant result of Ross.

Theorem 5 (Ross Theorem, (9)) *In the discrete model, the payoff of all path independent options coincide with the sublattice generated by **1** and* S_T

3.1 Static Hedging Algorithm

We will develop an algorithm that generates a portfolio of binary options written on a primary security S that can be used to hedge statically an arbitrary option Z. We assume here that the underlying asset price S at period t is modelled as a vector $S_t = (S_t(1), ..., S_t(m))' \in \mathbb{R}^m$, where m is the number of end states. we will get the price vector at period t using binomial or trinomial tree model.

We also denote with $\mathbb{1} = (\underbrace{1, ..., 1}_{m})'$ the returns of of a riskless bond with zero interest rate.

At the terminal period $t = T$, Matrix of returns of returns is modeled as below:

$$M = [\mathbb{1}\ S_T] = \begin{bmatrix} 1 & S_T(1) \\ 1 & S_T(2) \\ \vdots & \vdots \\ 1 & S_T(m) \end{bmatrix} \tag{3.1}$$

A positive basis $X_1, ..., X_n$ for $\mathcal{S}_M$ can be calculated by the sublattice algorithm explained in the second chapter.

$$PB_M = [X_1,\ \cdots,\ X_n] = \begin{bmatrix} X_1(1) & \dots & X_n(1) \\ \vdots & \ddots & \vdots \\ X_1(m) & \cdots & X_n(m) \end{bmatrix} \tag{3.2}$$

We denote here that $X_i(j) \in \{0,1\}$ for each $i = 1,..,n, j = 1,...,m$ and thus each X_i can be viewed as the terminal payoff of a suitable portfolio of binary options. After calculating the positive basis we find the best approximation of the target option with a portfolio $\vartheta = (\theta_1, ..., \theta_n)'$ of the binary options $X_1, ..., X_n$. The portfolio value V^ϑ at the terminal period $t = T$ is given by the following formula

$$V^\vartheta = PB_M \cdot \vartheta = \sum_{j=1}^{n} \theta_j X_j \tag{3.3}$$

For any $Z \in \mathbb{R}^m$, let $P_{\mathcal{S}_M}[Z]$ be the orthogonal projection of Z onto $\mathcal{S}_M$, that is

$$P_{\mathcal{S}_M}[Z] = argmin\{||Z - W||_2 \mid W \in \mathcal{S}_M\}, \tag{3.4}$$

Then $V^\vartheta := P_{\mathcal{S}_M}[Z]$ and the portfolio ϑ can be calculated by the following formula.

$$\vartheta = (PB'_M PB_M)^{-1} \cdot PB'_M \cdot Z \tag{3.5}$$

The corresponding hedging error is calculated as follows

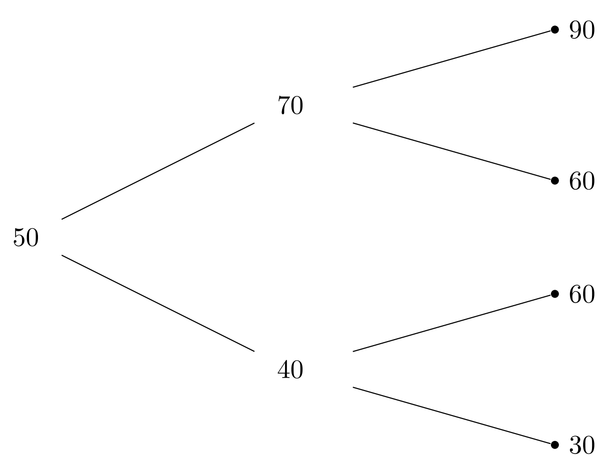
$$RMSE = ||V^\vartheta - Z||_2 \tag{3.6}$$

3.2 Working Examples

To illustrate the algorithm, we will use the example of a geometric average fixed strike Asian call with strike $K = 55$. The payoff of this option is calculated using the geometric mean of the underlying price over time with payoff

$$GAC(S,\ K,\ T) = max(0,\ S_{ave} - K) \tag{3.7}$$

Binomial Model



$$Z = \begin{bmatrix} max(0, \ \sqrt[3]{50 * 70 * 90} - 55) \\ max(0, \ \sqrt[3]{50 * 70 * 60} - 55) \\ max(0, \ \sqrt[3]{50 * 40 * 60} - 55) \\ max(0, \ \sqrt[3]{50 * 40 * 30} - 55) \end{bmatrix} = \begin{bmatrix} 13.04 \\ 4.44 \\ 0 \\ 0 \end{bmatrix} \tag{3.8}$$

The input matrix M for the algorithm containing all the asset prices at maturity and the returns of the risk free bonds is

$$M = \begin{bmatrix} 1 & 90 \\ 1 & 60 \\ 1 & 60 \\ 1 & 30 \end{bmatrix} \tag{3.9}$$

Using the *sublat.m* function we calculate the corresponding positive basis for M as

$$PB_M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.10}$$

$$\mathcal{S}_M = span\{(1, \ 0, \ 0, \ 0), \ (0, \ 1, \ 1, \ 0), \ (0, \ 0, \ 0, \ 1)\} \tag{3.11}$$

$$Z \notin \mathcal{S}_M \tag{3.12}$$

The approximation using PB_M is determined by

$$\vartheta = (PB_M' PB_M)^{-1} \cdot PB_M' \cdot Z \tag{3.13}$$

$$V^{\vartheta} = PB_M \cdot \vartheta \tag{3.14}$$

with the resulting values

$$\vartheta = \begin{bmatrix} 13.04 \\ 2.22 \\ 0.00 \end{bmatrix} \tag{3.15}$$

$$V^{\vartheta} = \begin{bmatrix} 13.04 \\ 2.22 \\ 2.22 \\ 0 \end{bmatrix} \tag{3.16}$$

$$RMSE = 1.5698 \tag{3.17}$$

Trinomial Model

The trinomial option pricing model, proposed by Phelim Boyle in 1986, is considered to be more accurate than the binomial model. Unlike binomial model as explained in the first chapter, in this model the price increases by a factor of u with probability p_u, decreases by a factor of d with the probability p_d or remains unchanged with probabilities $p_m = 1 - p_u - p_d$. The fig 3.2 and diagram below reveal how the trinomial tree generates the underlying asset price at different time periods.

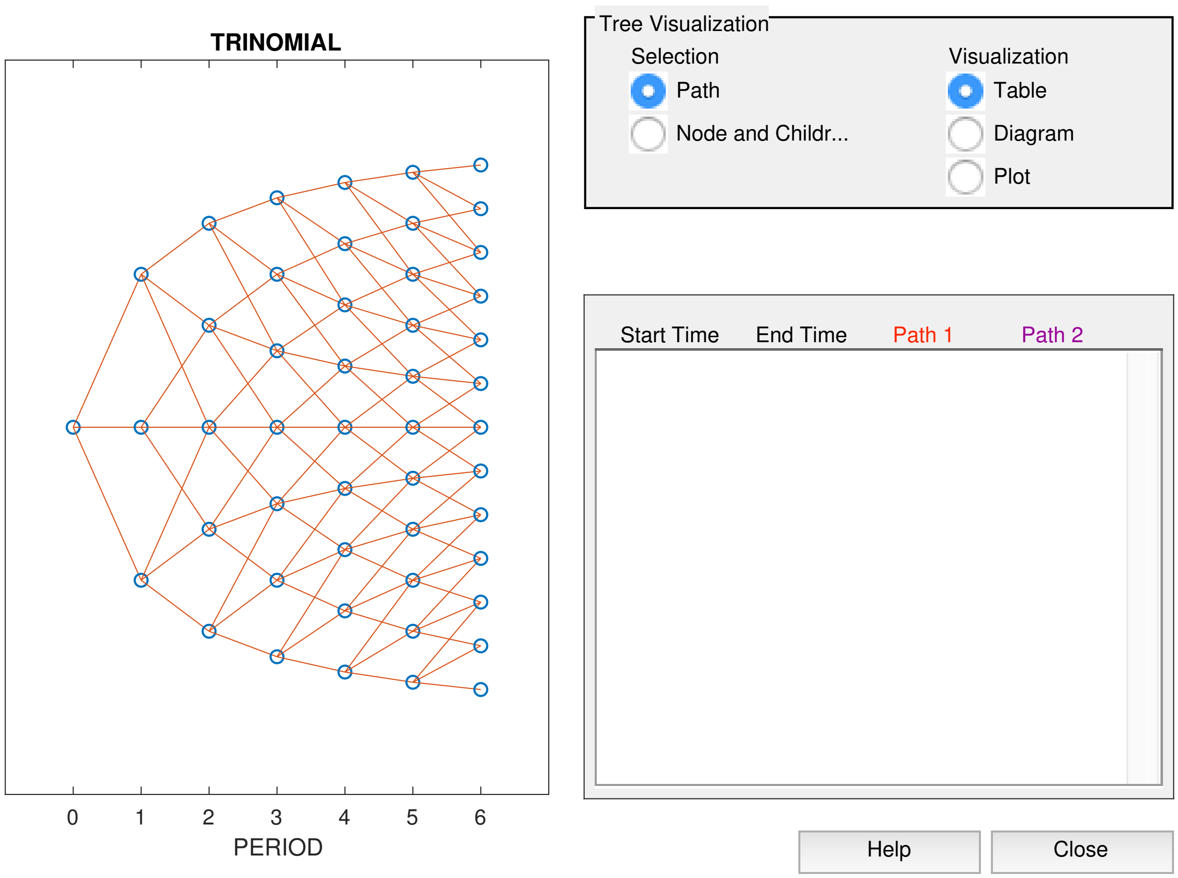
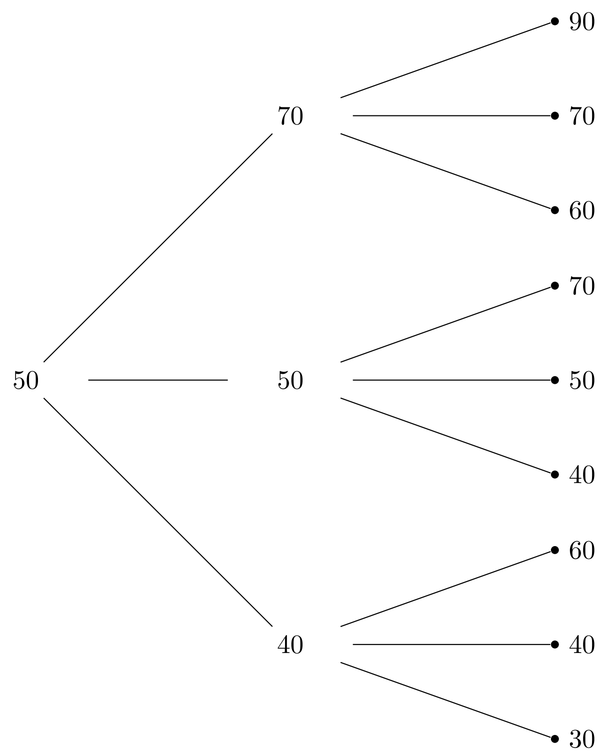


Figure 3.1: Each Node Shows Asset Price Under Trinomial Tree



$$Z = \begin{bmatrix} max(0,\ \sqrt[3]{50*70*90}-55) \\ max(0,\ \sqrt[3]{50*70*70}-55) \\ max(0,\ \sqrt[3]{50*70*60}-55) \\ max(0,\ \sqrt[3]{50*50*70}-55) \\ max(0,\ \sqrt[3]{50*50*50}-55) \\ max(0,\ \sqrt[3]{50*50*40}-55) \\ max(0,\ \sqrt[3]{50*40*60}-55) \\ max(0,\ \sqrt[3]{50*40*40}-55) \\ max(0,\ \sqrt[3]{50*40*30}-55) \end{bmatrix} = \begin{bmatrix} 13.04 \\ 7.57 \\ 4.44 \\ 0.93 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{3.18}$$

The input matrix M for the algorithm containing all strike prices at maturity and the returns of the risk free bonds is

$$M = \begin{bmatrix} 1 & 90 \\ 1 & 70 \\ 1 & 60 \\ 1 & 70 \\ 1 & 50 \\ 1 & 40 \\ 1 & 60 \\ 1 & 40 \\ 1 & 30 \end{bmatrix} \tag{3.19}$$

Using the *sublat.m* function we calculate the corresponding positive basis for M as

$$PB_M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.20}$$

$$\mathcal{S}_M = span\{(1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0),\ (0,\ 1,\ 0,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0),\ \ldots,\ (0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1)\} \tag{3.21}$$

$$Z \notin \mathcal{S}_M \tag{3.22}$$

The approximation using PB_M is determined by

$$\vartheta = (PB'_M PB_M)^{-1} \cdot PB'_M \cdot Z \tag{3.23}$$

$$V^{\vartheta} = PB_M \cdot \vartheta \tag{3.24}$$

with the resulting values

$$\vartheta = \begin{bmatrix} 13.04 \\ 4.25 \\ 2.22 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{3.25}$$

$$V^{\vartheta} = \begin{bmatrix} 13.04 \\ 4.25 \\ 2.22 \\ 4.25 \\ 0 \\ 0 \\ 2.22 \\ 0 \\ 0 \end{bmatrix} \tag{3.26}$$

$$RMSE = 1.8827 \tag{3.27}$$

3.3 Using European Options

The positive base PB_M generated by the algorithm is the basis for all possible payoffs of a portfolio of binary options on the underlying however it is also possible to replicate the same payoffs with European options. Under the same strike price and maturity assumptions both portfolios will separate states at the same paths and can replicate the same payoff space.

Example

To use the previous example, in the binomial case we have the target payoffs Z and the matrix M containing all the asset prices at maturity and the returns of the risk free bonds

$$Z = \begin{bmatrix} 13.04 \\ 4.44 \\ 0 \\ 0 \end{bmatrix} \tag{3.28}$$

$$M = \begin{bmatrix} 1 & 90 \\ 1 & 60 \\ 1 & 60 \\ 1 & 30 \end{bmatrix} \tag{3.29}$$

Using the *sublat.m* function we calculate the corresponding positive basis for M as

$$PB_M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.30}$$

The estimation using PB_M results in the approximation

$$\vartheta = \begin{bmatrix} 13.04 \\ 2.22 \\ 0.00 \end{bmatrix} \tag{3.31}$$

$$V^{\vartheta} = \begin{bmatrix} 13.04 \\ 2.22 \\ 2.22 \\ 0 \end{bmatrix} \tag{3.32}$$

Now we illustrate that the same hedging portfolio V^{ϑ} can be achieved using the returns X of a portfolio of binary or European options. The strike prices $K(i)$ for the options are the unique values from the set of the midpoints between every $S_T(i)$ and $S_T(i+1)$. When $S_T(i+1)$ does not exist for the final node we use a value of 0. The strike prices are the same for both portfolios and are

$$K = [75,\ 45,\ 15] \tag{3.33}$$

For the portfolio of binary options we have

$$X_B = \begin{bmatrix} 100 & 100 & 100 \\ 100 & 100 & 0 \\ 100 & 100 & 0 \\ 100 & 0 & 0 \end{bmatrix} \tag{3.34}$$

$$\vartheta_B = \begin{bmatrix} 0.0000 \\ 0.0222 \\ 0.1082 \end{bmatrix} \tag{3.35}$$

$$V_B^{\vartheta} = \begin{bmatrix} 13.04 \\ 2.22 \\ 2.22 \\ 0 \end{bmatrix} \tag{3.36}$$

For the portfolio of European options we have

$$X_E = \begin{bmatrix} 75 & 45 & 15 \\ 45 & 15 & 0 \\ 45 & 15 & 0 \\ 15 & 0 & 0 \end{bmatrix} \tag{3.37}$$

$$\vartheta_E = \begin{bmatrix} 0.0000 \\ 0.1480 \\ 0.4253 \end{bmatrix} \tag{3.38}$$

$$V_E^{\vartheta} = \begin{bmatrix} 13.04 \\ 2.22 \\ 2.22 \\ 0 \end{bmatrix} \tag{3.39}$$

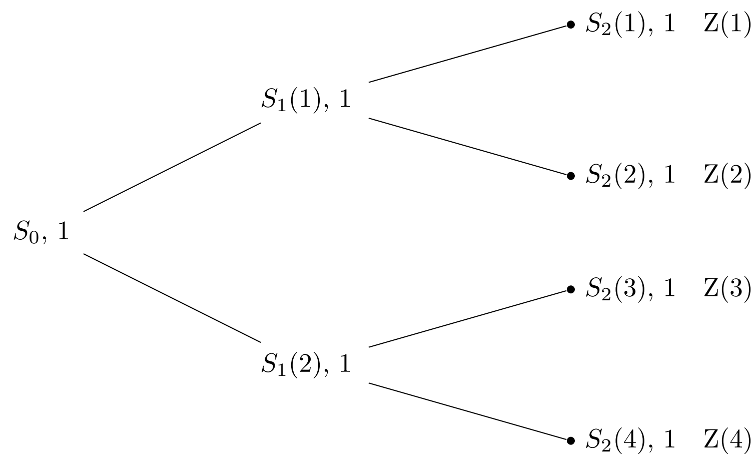
We have shown that using either binary or European options result in the same hedge and both can be used for this purpose. In this paper binary options are considered due to their simplicity as they return $Q \in \{0,\ 100\}$; however in practice European options have many additional benefits such as increased liquidity. As the algorithm is not directly concerned with the options themselves; it is left to the individual to determine the best way to form the hedge according to their needs.

3.4 Delta Hedging

We here present the delta hedging strategy though it has not been applied to analyze the RMSE error.

$$\vartheta_{DH} \ = \ [\vartheta_1,\ \vartheta_2,\ \ldots,\ \vartheta_{T-1}] \tag{3.40}$$

Example



As we are only concerned with the final error at T and there are no restrictions on $[\theta_1,\ \theta_2,\ \dots,\ \theta_{T-1}]$ we only need to calculate the final delta hedging portfolio θ_{T-1}. This is determined by going to every $S_{T-1}(i)$ node and calculating the optimal portfolio to approximate the $Z(i)$ paths from that point. Here we show the calculation for the first portfolio at $S_1(1)$

$$X_{S_1(1)} = \begin{bmatrix} 1 & S_2(1) \\ 1 & S_2(2) \end{bmatrix} \tag{3.41}$$

$$Z_{S_1(1)} = \begin{bmatrix} Z(1) \\ Z(2) \end{bmatrix} \tag{3.42}$$

Using the previous formulas to solve for theta

$$\vartheta_1(1) = (X'_{S_1(1)} \cdot X_{S_1(1)})^{-1} \cdot X'_{S_1(1)} \cdot Z_{S_1(1)} \tag{3.43}$$

$$X_{S_1(1)} \cdot \vartheta_1(1) \approx Z_{S_1(1)} \tag{3.44}$$

In this case this calculation would be done again at $S_1(2)$ and the combined portfolio payoffs at every point $S_T(i)$ would form the delta hedge payoffs. The next section presents simulations of the numerical results.

Chapter 4

Findings and Conclusions

The final chapter simulates the numerical results, followed by conclusions and recommendations. In this chapter, the RMSE errors have been computed and analyzed based on binomial and trinomial model under the static hedging strategy.

4.1 Numerical Results

In this section, we will evaluate root mean square error(RMSR) between payoffs of the target option in each terminal state and payoffs of replicating portfolio. For this objective, we will compute and analyze static hedging errors for twelve months maturity period, and these errors will be used to evaluate hedging performance. Based on the results, an appropriate hedging strategy would be recommended for option type considered in the study. The research consists of European call options, knockout barrier call options, lookback call option (LC), and both Geometric Asian Call(GAC) and Arithmetic Asian call(AAC) options with twelve months maturity and rollover period $N = 6, 12, 14$ under the variation of sigma and strike price. The default values for underlying tree are initial price $S_0 = 50$, strike Price $K = 65$, knockout level $H = 80$, and volatility = 0.35. The tables and plots below present the static RMSE error of the underlying assets under Binomial and Trinomial model.

From tables 4. 1 and 4.2, it is observed that the RMSE of all options increase varying sigma under default option condition. It means that hedging performance declines in the market state with high volatility. The plot 4.1 shows that the RMSE of the options increases as the increase of rollover period . Comparing the average errors of underlying assets, the lookback and arithmetic Asian call(AAC) options have the highest figure as 6.0989, 6.919, 7.0489 and 2.7725, 3.2659, 3.3389 followed by GAC and knockout as 2.4151, 2.8922, 2.9632 and 0.6961, 1.0450, 1.1803 at the respective rollover period $N = 6, 12, 14$ which shows an increase in error as the path moves forward. However; the table 4.3 shows that there is no effect on Euro call option either varying sigma or enhancing tree lengths implying perfect hedging.

Table 4.1: Trinomial Alg RMSE at Different Paths varying σ

σ	GAC		
	$N=6$	$N=12$	$N=14$
0.10	0.0000	0.0079	0.0123
0.15	0.1116	0.2545	0.2744
0.20	0.5873	0.8249	0.8584
0.25	1.2576	1.5915	1.6398
0.30	2.0477	2.4864	2.5502
0.35	2.9234	3.4827	3.5651
0.40	3.8764	4.5747	4.6771
0.45	4.9095	5.7606	5.8883
0.50	6.0228	7.0474	7.2039
RMSE Sum	21.7363	26.0306	26.6695
Mean	2.4151	2.8922	2.9632
S.D.	2.1675	2.5019	2.5526

σ	Knockout		
	$N=6$	$N=12$	$N=14$
0.10	0.0000	0.0638	0.0684
0.15	0.4113	0.6228	0.6489
0.20	0.9696	1.0132	1.2810
0.25	0.9171	1.0653	1.1701
0.30	1.6910	1.8679	1.5301
0.35	0.0000	0.9797	0.6633
0.40	0.2683	1.5806	1.1917
0.45	0.7530	2.2123	1.7451
0.50	1.2552	0.000	2.3247
RMSE Sum	6.2655	9.4057	10.6233
Mean	0.6961	1.0450	1.1803
S.D.	0.5769	0.7554	0.6658

Table 4.2: Trinomial Alg RMSE at Different Paths varying σ

σ	Lookback		
	$N=6$	$N=12$	$N=14$
0.10	0.2138	0.3342	0.3406
0.15	1.0442	1.2694	1.3145
0.20	2.2564	2.5815	2.6703
0.25	3.6182	4.2366	4.2732
0.30	5.3382	5.9892	6.1571
0.35	7.3238	8.0202	8.1867
0.40	9.3736	10.3988	10.5803
0.45	11.5854	13.1431	13.3561
0.50	14.1372	16.2980	16.5614
RMSE Sum	54.8908	62.271	63.4402
Mean	6.0989	6.919	7.0489
S.D.	4.8556	5.5020	5.5855

σ	AAC		
	$N=6$	$N=12$	$N=14$
0.10	0.0000	0.0144	0.0196
0.15	0.1699	0.3093	0.3291
0.20	0.7089	0.9430	0.9759
0.25	1.4559	1.7845	1.8317
0.30	2.3342	2.7718	2.8354
0.35	3.3250	3.8862	3.9693
0.40	4.4173	5.1287	5.2346
0.45	5.6108	6.5095	6.6429
0.50	6.9310	8.0461	8.2122
RMSE Sum	24.953	29.3935	30.0507
Mean	2.7725	3.2659	3.3389
S.D.	2.4749	2.8366	2.8905

Close observation of the tables, we find that there is not only increment in the average errors but also rise in the variation of the errors as rollover path increases. The lookback and AAC show the highest variability in the error indicating more fluctuation in their option prices compared to the errors of GAC and knockout call option, which means latter options have consistency in their price.

Analyzing the plots 4.1, 4.2 and the tables, varying sigma at rollover periods $N = 6, N = 14$, we observe that GAC, AAC and LC errors follow the conic paths creating more errors gap where as knockout errors do not follow the certain rule. The sum at the respective periods of GAC, knockout, LC, AAC are $21.7363, 26.6695$, and $6.2655, 10.6233$, and $54.8908, 63.4402$ and $24.953, 30.0507$. The absolute sum error differences are relatively high in lookback, AAC options as 8.5494 and 5.0977 and in GAC and knockout are as 4.9332, and 4.3578. The tables reveal that for the shorter tree length (period) the options possess

Table 4.3: Trinomial Alg RMSE at Different Paths

σ	Euro		
	$N = 6$	$N = 12$	$N = 14$
0.10	0.0000	0.0000	0.0000
0.15	0.0000	0.0000	0.0000
0.20	0.0000	0.0000	0.0000
0.25	0.0000	0.0000	0.0000
0.30	0.0000	0.0000	0.0000
0.35	0.0000	0.0000	0.0000
0.40	0.0000	0.0000	0.0000
0.45	0.0000	0.0000	0.0000
0.50	0.0000	0.0000	0.0000
RMSE Sum	0.0000	0.0000	0.0000

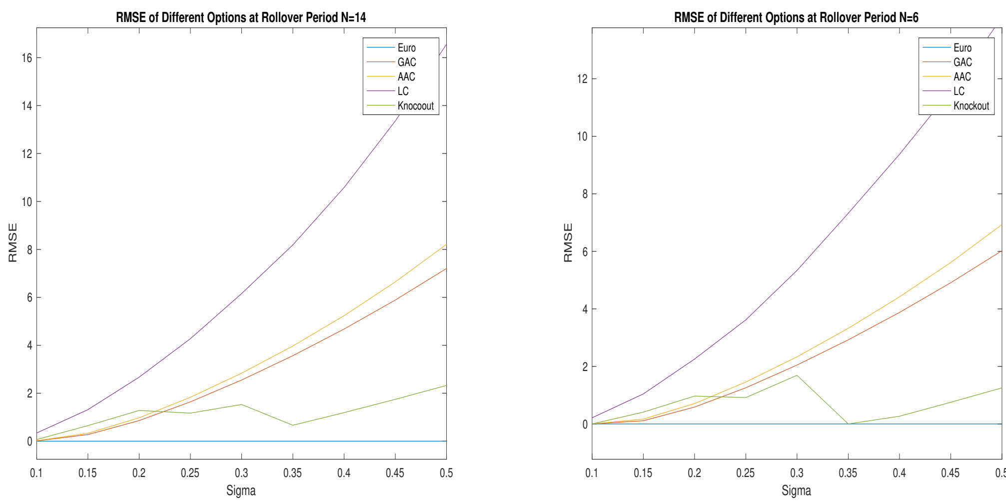


Figure 4.1: RMSE errors of the options while varying σ under Trinomial model .

small absolute error difference. For instance, the options have respective absolute error difference as 0.6389,1.1276 , 1.1692, 0.6572 for the rollover periods $N = 6$ to 12.
These figures are relatively less compared to the errors of the period $N = 6$ to 14. Now, lets look upon theirs standard deviations (sd) errors at the rollover periods $N = 6, N = 14$. The corresponding sd errors are as $2.2675, 2.5526$ and $0.5769, 0.6658$ and $4.8556, 5.5855$ and $2.4749, 2.8905$ and absolute sd errors are as 0.2851 0.0889, 0.7299, 0.4156. These figures implies that knockout and GAC have less effect of the rollover while lookback and AAC show higher rollover influence as compared to knockout and GAC call options. In other words, lookback and AAC call options reflect higher sensitiveness towards the rollover while varying sigma. The plot 4.2 is an example of how the hedging errors sensitize when varying sigma and enhancing rollover paths. In both GAC and lookback options, we see that there exists a wide gap between rollover period $N = 6$, 12 where as there is very small gap between $N = 12$, 14. This means

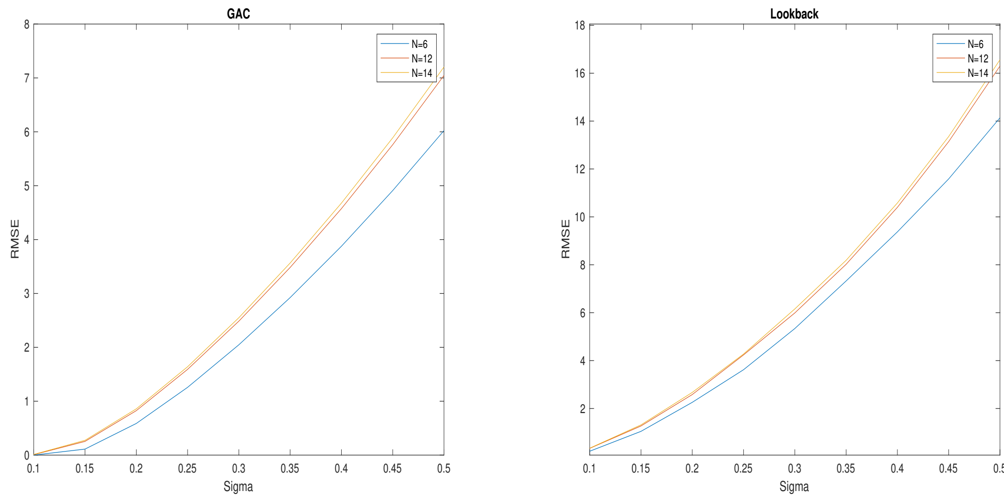


Figure 4.2: Hedging error when varying σ in the Trinomial model .

static hedging performs well relatively to the shorter period.

So far, we have discussed the rollover effect when varying sigma. Now, lets explore on the rollover effect when varying strike prices K. Referring the tables 4.4 and 4.5, the corresponding sum errors of assets GAC, knockout, LC and AAC at strike prices $10, 50, 100$ and rollover period$N = 6$ are $6.3671, 4.9795, 0.0442$,and $12.9278, 2.8513, 0.0000$,and $9.38109.3810, 3.2204,$ and $6.5472, 5.2465, 0.4390$. We can observe the similar pattern at roll over period $N = 12, 14$. This patterns reveals that the RMSE of the underlying assets decreases varying K indicating better approximation of the options price as the exercised prices enhance. However, the average sum error increases as the tree lengths spans showing declining performance of the hedging when rollover period increases.

The table 4.4 and 4.5 shows that the respective sum errors at the rollover period of $N = 6$ and 14 of the assets GAC, knockout, LC and AAC are $37.6954, 39.675, 73.8057, 40.8337$ and 43.7497, 43.6684, 83.2711, 47.5677 , and their corresponding absolute sum errors difference between the period are 6.0543, 3.9934, 9.4654, 6.734 . This shows that LC and AAC have the highest absolute sum error difference. From the tables, we observed the corresponding mean error of the assets at rollover period $N = 6$ and 14 are 3.7695, 3.9675, 7.3805, 4.0833 and 4.3749, 4.3668, 8.3271, 4.7567, yielding absolute mean difference as 0.6054, 0.3993, 0.9466, 0.6734. Like absolute sum difference, the LC and AAC have the highest absolute mean error difference. However, the absolute sd errors difference of the respective assets at the same rollover period are 0.1356, 0.3056, 0.1563, 0.0552 showing the highest error variation in knockout and LC. This implies these assets have more fluctuations in the price error.

Like varying sigma, there is no effect on Euro call option either varying K or increasing tree lengths. In order to avoid the repetition of the table, the table is not included because we get all RMSE value as zero at the rollover periods $N = 6, 12, 14$.

The plot 4.3 shows the gradually decreasing errors of the underlying assets while varying K; however, their(except euro call option) errors are relatively constant at some intervals of the strike prices. This means RMSE errors do not change for some strike prices. The plot 4.4 reveals the rollover effect of lookback and GAC at period $N = 6, 12, 14$ which tells us that small(less) errors for the small(less) rollover period and higher errors for higher rollover period.

Now, let us analyze the rollover effect when varying strike price and sigma between the $N = 6$ and $N = 14$. Varying sigma for the period, we observed the absolute sum error difference of the asset GAC, knockout, LC and AAC are as 4.9332, 4.3578, 8.5494, 5.0977. On the other hand, we got the corresponding absolute sum error difference as 6.0543, 3.9934, 9.4654, 6.734 while varying K. These figures implies that the knockout call option accumulates less error while varying K where as GAC, LC and AAC generate less errors while varying sigma for the same roll over period. It may be concluded that static hedging performs well on Knockout options when strikes price increases. However, the hedging could be appropriate for GAC, LC and AAC while varying sigma.

Varying sigma, the respective absolute mean error difference and the absolute sd error difference of the asset GAC, knockout, LC and AAC between the period $N = 6$ and $N = 14$ are 0.5481, 0.4842, 0.95, 0.5664 and 0.3851, 0.0889, 0.7299, 0.4156. While Varying K, the corresponding absolute mean error difference and the absolute sd error difference of the assets for the same period are 0.6054, 0.4013, 0.9446, 0.6734 and 0.1556, 0.3056, 0.1563, 0.0552. These figures shows that GAC and AAC accumulates higher mean error difference while varying K. On the other hand, Knockout and LC collects greater mean difference while varying sigma. Now, comparing the absolute sd error difference for same period while varying strike price and sigma, knockout shows higher errors fluctuations while varying K where as GAC , LC , AAC shows a greater errors fluctuations while varying sigma.

The researcher observed that Binomial model reflects the same characteristics as trinomial model. Under the binomial model while varying σ, the table 4.6 and 4.7 disclose that lookback and arithmetic Asian call(AAC) options have the highest mean errors as 3.0669,3.5122 ,3.5722 and 1.0978, 1.4274, 1.4738 followed by GAC and knockout as 0.9422, 1.2667,1.3131 and 0.7187, 0.8164,0.5738 at the respective rollover period $N = 6, 12, 14$ which shows the increase in error as the path moves forward. Comparing these figures with trinomial model(referring table 4.1 and 4.2), we found that the lookbackand and AAC also yield the highest mean error as 6.0989, 6.919,7.0489 and 2.7725,3.2659, 3.3389 while GAC and knockout have mean error as 2.4151,2.8922,2.9632 and 0.6961,1.0450,1.1803 showing trinomial model accumulating larger figure. In both model, errors increase varying sigma that implies declining hedging performance with high volatility.

Table 4.4: Trinomial Alg RMSE at Different Paths varying K

K	GAC		
	$N = 6$	$N = 12$	$N = 14$
10	6.3671	7.0032	7.0987
20	6.3671	7.0031	7.0986
30	6.3462	6.9652	7.0580
40	5.9757	6.5586	6.6465
50	4.9795	5.5359	5.6199
60	3.6179	4.1687	4.2518
70	2.2820	2.8444	2.9257
80	1.2160	1.7772	1.8568
90	0.4997	1.0196	1.0946
100	0.0442	0.5334	0.0991
RMSE Sum	37.6954	43.4093	43.7497
Mean	3.7695	4.3409	4.3749
S.D.	2.5774	2.6199	2.7130

K	Knockout		
	$N = 6$	$N = 12$	$N = 14$
10	12.9278	13.7408	13.7885
20	10.2576	11.1080	11.0094
30	7.6370	8.5247	8.2969
40	5.1300	6.0427	5.7161
50	2.8513	3.7580	3.3861
60	0.8713	1.8072	1.4714
70	0.0000	0.1521	0.0000
80	0.0000	0.0000	0.0000
90	0.0000	0.0000	0.0000
100	0.0000	0.0000	0.0000
RMSE Sum	39.675	45.1335	43.6684
Mean	3.9675	4.5133	4.3668
S.D.	4.8127	5.1108	5.1183

Table 4.5: Trinomial Alg RMSE at Different Paths varying K

K	Lookback		
	$N = 6$	$N = 12$	$N = 14$
10	9.3810	10.0370	10.1460
20	9.3810	10.0370	10.1460
30	9.3810	10.0370	10.1460
40	9.3810	10.0370	10.1460
50	9.3810	10.0370	10.1460
60	7.7997	8.8422	8.9402
70	6.4318	7.4778	7.6362
80	5.2998	6.1387	6.3797
90	4.1490	5.0907	5.2229
100	3.2204	4.2899	4.3621
RMSE Sum	73.8057	82.0243	83.2711
Mean	7.3805	8.2024	8.3271
S.D.	2.4304	2.2855	2.2741

K	AAC		
	$N = 6$	$N = 12$	$N = 14$
10	6.5472	7.2021	7.3001
20	6.5472	7.2021	7.3001
30	6.5408	7.1807	7.2762
40	6.2112	6.8107	6.9007
50	5.2465	5.8272	5.9139
60	3.9613	4.5324	4.6164
70	2.7126	3.2818	3.3639
80	1.6999	2.2523	2.3322
90	0.9280	1.4806	1.5568
100	0.4390	0.9372	1.0074
RMSE Sum	40.8337	46.7071	47.5677
Mean	4.0833	4.6707	4.7567
S.D.	2.4692	2.5154	2.5244

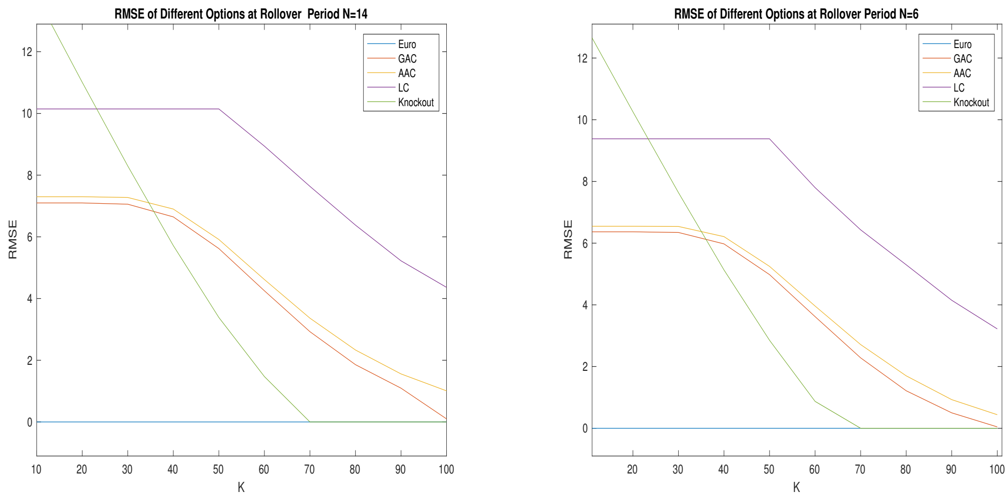


Figure 4.3: Hedging error when varying K in the Trinomial model .

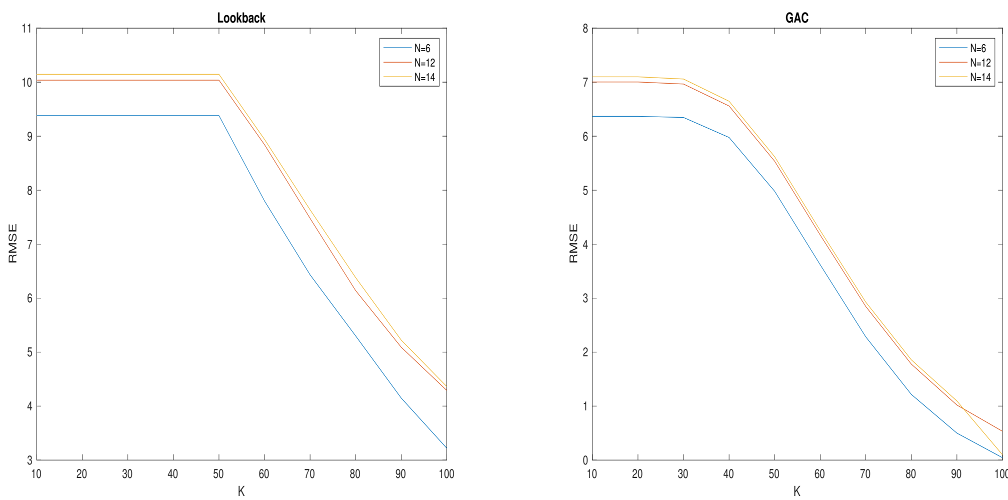


Figure 4.4: Hedging error when varying K in the Trinomial model .

Table 4.6: Binomial Alg RMSE at Different Paths

σ	Lookback		
	$N = 6$	$N = 12$	$N = 14$
0.10	0.0000	0.0394	0.0575
0.15	0.3290	0.4567	0.4334
0.20	0.8475	1.0908	1.1230
0.25	1.6141	2.0051	2.0999
0.30	2.7678	3.0168	3.0772
0.35	3.8193	4.1335	4.2309
0.40	4.8431	5.4660	5.5271
0.45	6.0277	6.9743	7.0126
0.50	7.3538	8.4276	8.6339
RMSE Sum	27.6023	31.6102	32.1955
Mean	3.0669	3.5122	3.5722
S.D.	2.6194	2.9607	3.0068

σ	AAC		
	$N = 6$	$N = 12$	$N = 14$
0.10	0.0000	0.0000	0.0000
0.15	0.0000	0.0019	0.0177
0.20	0.0000	0.1946	0.2168
0.25	0.2138	0.5700	0.6048
0.30	0.7072	1.0795	1.1233
0.35	1.3038	1.6770	1.7352
0.40	1.9324	2.3509	2.4188
0.45	2.5417	3.0883	3.1678
0.50	3.1813	3.8848	3.9802
RMSE Sum	9.8802	12.847	13.2646
Mean	1.0978	1.4274	1.4738
S.D.	1.2100	1.4211	1.4518

Table 4.7: Binomial RMSE at Different Paths varying σ

σ	GAC		
	$N = 6$	$N = 12$	$N = 14$
0.10	0.0000	0.0000	0.0000
0.15	0.0000	0.0000	0.0069
0.20	0.0000	0.1485	0.1722
0.25	0.1067	0.4803	0.5178
0.30	0.5462	0.9429	0.9891
0.35	1.0675	1.4913	1.5462
0.40	1.6454	2.1004	2.1701
0.45	2.2307	2.7637	2.8461
0.50	2.8841	3.4738	3.5700
RMSE Sum	8.4806	11.4009	11.8184
Mean	0.9422	1.2667	1.3131
S.D.	1.0873	1.2767	1.3087

σ	Knockout		
	$N = 6$	$N = 12$	$N = 14$
0.10	0.0000	0.0000	0.0000
0.15	0.0000	0.1802	0.0747
0.20	0.0000	0.7058	0.7593
0.25	1.1601	1.2829	0.6834
0.30	0.0000	0.6695	0.5224
0.35	0.1827	1.7031	1.0316
0.40	1.0153	2.4706	2.0335
0.45	1.6990	0.0000	0.0000
0.50	2.4112	0.3361	0.0595
RMSE Sum	6.4683	7.3482	5.1644
Mean	0.7187	0.8164	0.5738
S.D.	0.8987	0.8472	0.6672

Table 4.8: Binomial Alg RMSE at Different Paths varying K

K	Lookback		
	$N=6$	$N=12$	$N=14$
10	5.9072	6.2310	6.2840
20	5.9072	6.2310	6.2840
30	5.9072	6.2310	6.2840
40	5.9072	6.2310	6.2840
50	5.9072	6.2310	6.2840
60	4.3499	4.9232	5.0382
70	3.0171	3.5100	3.6168
80	1.8565	2.3915	2.5027
90	1.3751	1.7172	1.7541
100	0.2340	1.0143	1.1399
RMSE Sum	40.3687	44.7112	45.4717
Mean	4.0368	4.4711	4.5471
S.D.	2.2343	2.1206	2.1023

K	AAC		
	$N=6$	$N=12$	$N=14$
10	4.4262	4.8511	4.9145
20	4.4262	4.8511	4.9145
30	4.4262	4.8511	4.9142
40	4.2894	4.6714	4.7281
50	3.3571	3.7088	3.7633
60	1.9238	2.3134	2.3667
70	0.6464	1.1562	1.2092
80	0.0000	0.4496	0.5039
90	0.0000	0.1044	0.1632
100	0.0000	0.0000	0.0156
RMSE Sum	23.4953	26.9571	27.4932
Mean	2.3495	2.6957	2.7493
S.D.	2.0373	2.1189	2.1272

Table 4.9: Binomial Alg RMSE at Different Paths varying K

K	GAC		
	$N=6$	$N=12$	$N=14$
10	4.3690	4.7870	4.8494
20	4.3690	4.7870	4.8494
30	4.3690	4.7857	4.8475
40	4.2089	4.5804	4.6366
50	3.2354	3.5946	3.6459
60	1.8064	2.1454	2.2012
70	0.4081	0.9542	1.0135
80	0.0000	0.2823	0.3365
90	0.0000	0.0028	0.0522
100	0.0000	0.0000	0.0000
RMSE sum	22.7658	25.9194	26.4322
Mean	2.2765	2.5919	2.6432
S.D	2.0289	2.1355	2.1459

K	Knockout		
	$N=6$	$N=12$	$N=14$
10	6.8246	13.0029	9.2942
20	5.6170	10.8425	7.7407
30	4.4094	8.6944	6.1928
40	3.2018	6.5708	4.6559
50	1.9942	4.5040	3.1456
60	0.7866	2.5708	1.7058
70	0.0000	0.8394	0.3575
80	0.0000	0.0000	0.0000
90	0.0000	0.0000	0.0000
100	0.0000	0.0000	0.0000
RMSE sum	22.8336	47.0248	33.0925
Mean	2.2833	4.7024	3.3092
S.D.	2.5879	4.8519	3.4920

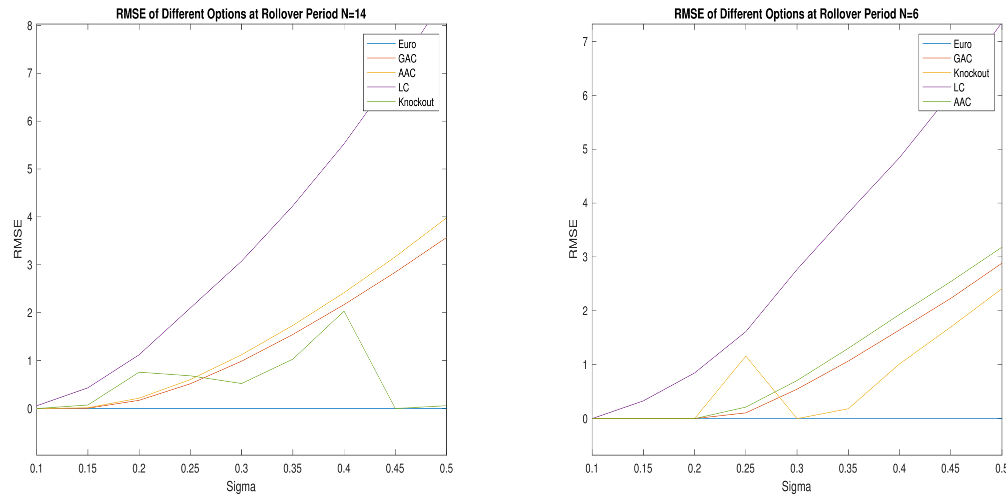


Figure 4.5: Hedging error when varying σ in the Binomial model .

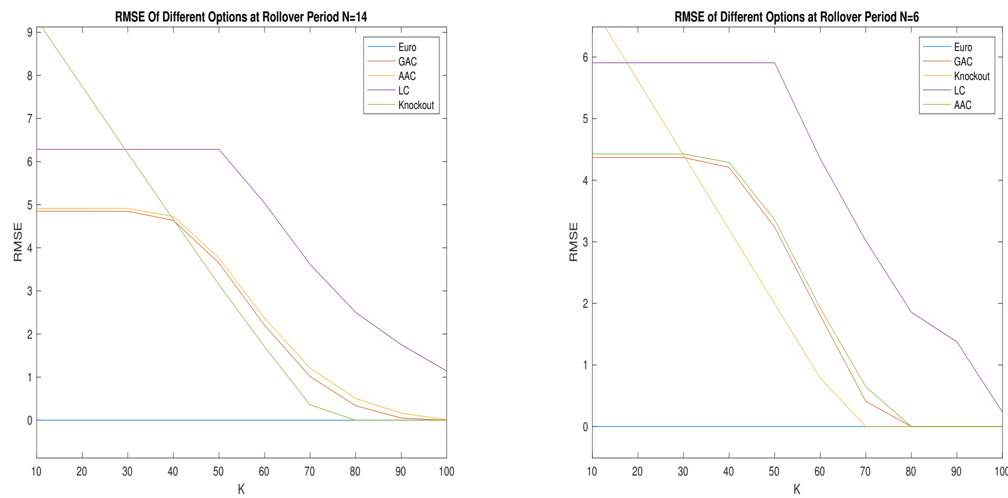


Figure 4.6: Hedging error when varying K in the Binomial model .

4.2 Conclusion and Recommendation

After evaluating and comparing the RMSE errors of call options of the underlying asset under Binomial and Trinomial model, the study divulge that knockout and GAC have shown less inclination towards the rollover compared to the lookback and AAC despite their growth in the mean error and error variation while enhancing the rollover periods. Therefore, we can replicate portfolio which will yield less error in the option pricing. The less error in the pricing will not only motivate the investor to choose an appropriate option but also insure their portfolio at minimum cost or suitable price. The replicated portfolios of GAC and knockout are not necessary to delta hedge as they have shown less rollover effect for the shorter period. This obviously saves the money and time of the investors. However, for the wide gap (rollover periods), the hedging could not be suitable plan for GAC and knockout as it yields the greater mean error and error variation. We can understand that the replicated portfolios and its options price would not escape from the errors. As a result, the investor would be reluctant to invest on the derivatives though the same period could be larger for an investor or smaller for another; it totally depends upon the investor risk taking strategy. Based on the above discussions, we conclude that static hedging could be appropriate strategy for the options Euro, Knockout and GAC.

The following recommendations are purposed for the future and further research plan:

1. To enhance the computational efficiency of calculating positive basis of vector sub-lattice, a proficient algorithm should be developed so that we can replicate the payoffs of bigger size tree under different time modal.

2. To provide a theoretical background to our findings

3. To generalize this study to continuous time model.

Appendix

Matlab Functions

```
function [pb] = sublat(V)
% SUBLAT  Determines the vector sublattice S(V) generated by the input
% matrix V.

if sum(any(V<0)) > 0
   error('Only non-negative input vectors are admissable.')
end

Z = sum(V);
x = size(V);
r = x(1);
c = x(2);
mtrx = zeros(c, r);
i = 1;
j = 1;

if rank(V) < r
    error('Input vectors must be linearly independent.')
end

for i = 1:r
    for j = 1:c
        mtrx(j, i) = V(i, j)/(Z(j) + realmin);
    end
end

```

```
card = length(unique(mtrx, 'rows'));

if r == card
    zcomb = V;
    pb = round(mtrx.' \ V, 10);
    pb( ~any(pb,2), : ) = [];
else
    umtrx = unique(mtrx, 'rows', 'stable');
    X = umtrx.';
    tol = 1e-10;
    if ~nnz(X)
        Xsub=[]; idx=[];
    end
    [Q, R, E] = qr(X,0);
    if ~isvector(R)
        diagr = abs(diag(R));
    else
        diagr = R(1);
    end
    r = find(diagr >= tol*diagr(1), 1, 'last');
    idx=sort(E(1:r));
    Xsub=X(:,idx);
    Xsub = Xsub.';
    enum = [Xsub;umtrx(find(ismember(1:length(umtrx), idx)==0),:)];
    I = r+1:card;
    zk = zeros(length(I), c);
    i = 1;
    j = 1;
    for i = 1:length(I)
        iind = find(ismember(mtrx,enum(I(i),:),'rows'));
        zk(i, iind) = Z(iind);
    end
    zcomb = [V;zk];
    Z2 = sum(zcomb);
    x2 = size(zcomb);
    r2 = x2(1);
    c2 = x2(2);
    mtrx2 = zeros(c2, r2);
    i = 1;
```

```
    j = 1;
    for i = 1:r2
        for j = 1:c2
            mtrx2(j, i) = zcomb(i, j)/(Z2(j) + realmin);
        end
    end
    A = unique(mtrx2, 'rows', 'stable').';
    pb = round(inv(A) * zcomb, 10);
    pb( ~any(pb,2), : ) = [];
    pb=pb~=0;
end
end
```

```
function [err, DHerr, target, estimate, nopt, price] = staticheedge(
    treetype, vary, tp, NumPeriods, defsigma, defP, defK, KO, figc)
% STATICHEDGE Evaluates and graphs the performace of the static hedging
% algorithm in either the binomial or trinomial model under user
% specified parameters. Requires the sublat.m function included in the
% associated paper.

tic

warning('off','MATLAB:singularMatrix')

if tp == 1
    StartDates = 'Jan-01-2018';
    EndDates = 'Jan-31-2018';
elseif tp == 3
    StartDates = 'Jan-01-2018';
    EndDates = 'Mar-31-2018';
elseif tp == 6
    StartDates = 'Jan-01-2018';
    EndDates = 'Jun-30-2018';
elseif tp == 12
    StartDates = 'Jan-01-2018';
    EndDates = 'Dec-31-2018';
else
    error('Time period not recognized')
end

```

```
if strcmp(vary,'sigma')
    ts = round(linspace(0.1, 0.5, 9), 2);
    iter = length(ts);
    tk = repelem(defK, iter);
elseif strcmp(vary,'K')
    tk = linspace(0.2, 2, 10).*defP;
    iter = length(tk);
    ts = repelem(defsigma, iter);
else
    error('Variable type not recognized')
end

opt = {'euro' 'GAC' 'knockout'};

err = zeros(length(ts), length(opt));
DHerr = zeros(length(ts), length(opt));

price = zeros(1, length(opt));

ind=1;
for ind = 1:length(opt)

type = opt{ind};

z = 1;
for z = 1:iter
Sigma=ts(z);
AssetPrice = defP;
K = tk(z);
ValuationDate = StartDates;
Maturity = EndDates;
Compounding = -1;
Rates = 0.02;

RateSpec = intenvset('Compounding',Compounding,'StartDates', StartDates
    ,...
                     'EndDates', EndDates, 'Rates', Rates,'
                        ValuationDate', ValuationDate);
StockSpec = stockspec(Sigma, AssetPrice);
```

```
if strcmp(treetype,'bin') == 1
CRRTimeSpec = crrtimespec(ValuationDate, Maturity, NumPeriods);
CRRTree = c(StockSpec, RateSpec, CRRTimeSpec);
tree = CRRTree.STree;

price(ind) = 0;

[x y] = size(tree);
Ns = zeros(1, y);
i=1;
for i = 1:y
    Ns(i) = length(tree{i});
end

pstates = zeros(NumPeriods+1, NumPeriods);
i=1;
j=1;
for i = 2:NumPeriods+1
    for j =1:Ns(i)
       pstates(j, i-1) = tree{i}(j);
    end
end

endprice = repelem(tree{NumPeriods+1}, diag(fliplr(pascal(NumPeriods+1)
   ))');

npaths = length(endprice);
combos = permn([1 2], NumPeriods);
countc = length(combos);
pathlist = zeros(countc, NumPeriods+1);

i=1;
for i = 1:countc
    PH = treepath(CRRTree.STree, combos(i, :))';
    pathlist(i, :) = PH;
end

pathlist = sortrows(pathlist, fliplr([linspace(1, NumPeriods+1,
```

```
    NumPeriods+1)]), 'descend');

teuro = max(0, endprice-K)';

tGAC = max(0, geomean(pathlist')-K)';
if KO > K
    [row, col] = find(pathlist>=KO);
else
    [row, col] = find(pathlist<=KO);
end

tknockout = max(0, endprice-K)';
tknockout(unique(row)) = 0;

if strcmp(type,'euro') == 1
    target = teuro;
elseif strcmp(type,'GAC') == 1
    target = tGAC;
elseif strcmp(type,'knockout') == 1
    target = tknockout;
else
    error('Option type error.');
end

i=1;
uniq = unique(endprice);
strikes = zeros(1,  length(diff(uniq)));
for i = 1:length(diff(uniq))
    strikes(i) = mean([uniq(i) uniq(i+1)]);
end

strikes = [sort(strikes, 'descend') endprice(end)-1];

states = length(endprice);

pb = sublat([ones(1, length(endprice)); endprice]);

nopt = size(pb, 1);

```

```
theta = mldivide(pb', target);
estimate = pb'*theta;
r=sqrt(sum((target(:)-estimate(:)).^2)/numel(target));
err(z, ind) = r;

elseif strcmp(treetype,'tri') == 1
TimeSpec = stttimespec(ValuationDate,Maturity,NumPeriods);
STTTree = stttree(StockSpec,RateSpec,TimeSpec);
tree = STTTree.STree;

[x y] = size(tree);
Ns = zeros(1, y);
i=1;
for i = 1:y
    Ns(i) = length(tree{i});
end

a = 1;
b = 1;
c = 1;
n = NumPeriods;
soln = [a b c];
storecf = cell(1, n+1);
storecf{1} = 1;
storecf{2} = soln;
i = 1;
j = 1;
for i = 1:n-1
    soln = conv(soln,[a b c]);
    storecf{i+2} = soln;
end

combos = permn([1 2 3], NumPeriods);
countc = length(combos);
pathlist = zeros(countc, NumPeriods+1);

i=1;
for i = 1:countc
    PH = trintreepath(STTTree, combos(i, :))';
```

```
    pathlist(i, :) = PH;
end

endprice = pathlist(:, NumPeriods+1);
npaths = length(endprice);

teuro = max(0, (endprice-K)')';

tGAC = max(0, geomean(pathlist')-K)';

if KO > K
    [row, col] = find(pathlist>=KO);
else
    [row, col] = find(pathlist<=KO);
end

tknockout = max(0, endprice-K);
tknockout(unique(row)) = 0;

if strcmp(type,'euro') == 1
    target = teuro;
elseif strcmp(type,'GAC') == 1
    target = tGAC;
elseif strcmp(type,'knockout') == 1
    target = tknockout;
else
    error('Option type not recognized');
end

i=1;
uniq = unique(endprice);
strikes = zeros(1,  length(diff(uniq)));
for i = 1:length(diff(uniq))
    strikes(i) = mean([uniq(i) uniq(i+1)]);
end

strikes = [sort(strikes, 'descend') endprice(end)-1];

states = length(endprice);
```

```

pb = sublat([ones(length(endprice), 1) endprice]');

nopt = size(pb, 1);

theta = mldivide(pb', target);
estimate = pb'*theta;
r=sqrt(sum((target(:)-estimate(:)).^2)/numel(target));
err(z, ind) = r;

up = STTTree.Probs{1}(1);
neutral = STTTree.Probs{1}(2);
down = STTTree.Probs{1}(3);

probmat = combos;
probmat(probmat == 1) = up;
probmat(probmat == 2) = neutral;
probmat(probmat == 3) = down;

mprob = prod(probmat, 2);
exvalue = cell(1, NumPeriods+1);
exvalue{end} = (target.*mprob)';
i=1;
j=1;
for i = 1:length(exvalue)-1
PHval = zeros(3^(NumPeriods-i) , 1);
for j = 1:length(PHval)
PHval(j) = sum(exvalue{end-i+1}(3*j-2:3*j));
end
exvalue{end-i} = PHval;
end

price(ind) = exvalue{1};

i=1;
dhtheta = zeros(sum(storecf{NumPeriods}), 2);
dhport = [ones(sum(storecf{NumPeriods+1}) ,1) endprice];
estimate = zeros(length(dhport), 1);
for i = 1:length(dhtheta)
```

```
    dhtheta(i, :) = mldivide(dhport([3*i-2 3*i], :), target([3*i-2 3*i]))';
    estimate(3*i-2:3*i, :) =  dhport(3*i-2:3*i, :)*dhtheta(i, :)';
end

r=sqrt(sum((target(:)-estimate(:)).^2)/numel(target));
DHerr(z, ind) = r;
else
    error('Tree type not recognized.')
end
end

figure(figc+ind)
if strcmp(vary,'sigma')
    plot(ts, err(:, ind))
    if strcmp(treetype,'tri') == 1
        hold on
        plot(ts, DHerr(:, ind), '--')
    end
    xlabel('Sigma')
    xticks(ts)
elseif strcmp(vary,'K')
    plot(tk, err(:, ind))
    if strcmp(treetype,'tri') == 1
        hold on
        plot(tk, DHerr(:, ind), '--')
    end
    xlabel('Strike')
    xticks(tk)
else
    error('Variable type not recognized')
end


if sum(err(:, ind)) <= 0.05
    ylim([-0.05 1])
else
    ylim([-0.05 inf])
end
```

```

ylabel('RMSE')
title(opt(ind))
if strcmp(treetype,'tri') == 1
    legend('Algorithm', 'DeltaHedging')
else
    legend('Algorithm')
end

grid on

end

warning('on','MATLAB:singularMatrix')

toc
end
```

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