1-1-2011

EOQ models for deteriorating items with two levels of market

Suborna Paul
Ryerson University

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by

Suborna Paul

B.Sc.(Mechanical Engineering)
Khulna University of Engineering and Technology,
Bangladesh, 2008

A Thesis
presented to Ryerson University
in partial fulfillment of the
requirements for the degree of
Master of Applied Science
in the Program of
Mechanical Engineering

Toronto, Ontario, Canada, 2011

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Suborna Paul

Master of Applied Science, Mechanical Engineering, 2011

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Abstract

This thesis proposes EOQ models for deteriorating items with time–dependent demand, as well as price–dependent demand, for both partial and complete backlogging scenarios. For each type of demand, three different models are developed: (1) items are first sold at the high–end market at a higher price; and then, at a given time, the leftover inventory is transported to the low–end market and sold at a lower price; (2) items are sold only at the high–end market. However, discount on the selling price is offered after a certain time; and (3) items are sold only at the high–end market without any price discount. The proposed models are solved to determine the optimal total profit, optimal order quantity, time at which inventory becomes zero, and optimal backlogged quantity. Finally, numerical examples and a sensitivity analysis are given to illustrate the proposed models.

Acknowledgements

I am deeply indebted to my thesis supervisors, Dr. Mohamed Wahab Mohamed Ismail and Dr. Rongbing Huang, for their valued and generous guidance and encouragement. Their patience has played a fundamental role in the completion of this thesis. I am very grateful for this opportunity to work under their supervision.

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Nomenclature

C	Purchasing cost per unit
P_1	Selling price at the high–end market
P_2	Selling price at the low–end market
P_o	Selling price at the high–end market of the one–market model
P_d	Discounted selling price at the high–end market of the one–market model with discounting
h	Inventory holding cost per unit per unit time
C_1	Deterioration cost per unit
C_2	Backorder cost per unit
C_3	Opportunity cost due to lost sale per unit
C_4	Transportation cost per unit
$\theta(t)$	Deterioration rate at any time t
α	Scale parameter
β	Shape parameter
δ	Time proportional constant for backlogging
$D(t)$	Time dependent demand at time t
$D(P)$	Price dependent demand at price P
S	Initial inventory
B	Backlogged quantity
Q	Order quantity
t_3	Time at which inventory level becomes zero
T	Cycle time
Γ_c	Total cost per unit time
SR	Total sales revenue per unit time
π	Total profit per unit time

Chapter 1

Introduction

This thesis develops EOQ models for deteriorating items considering high–end and low–end markets for two different groups of customers. Deterioration is defined as change, scrap, decay, damage, spoilage, obsolescence, and loss of utility or loss of original value in a commodity (Manna et al. (2009)), and (Wu et al. (2000)). As a result, deterioration leads to the decreasing usefulness of a commodity. For example, the price of seasonal fashion goods (clothes, sweaters, shoes, etc.) is sharply reduced or goods are disposed of after the season is over. High–tech electronics products (e.g., laptops, computers, digital cameras, mobiles, and flash drives) lose their values over time because of rapid changes of technology or the introduction of a new product by competitors. Similarly, spoilage or damage occurs if grocery items such as dairy products, fruits, vegetables, meats, etc. are kept over a period.

In order to minimize the loss due to deterioration, companies dealing with deteriorating items follow a two–market model (high–end market and low–end market) for two groups of customers (Jrgensen and Liddo 2007). The high–end market consists of customers who have the ability to pay a higher price and are willing to buy new, improved, and higher quality products. For example, when a new fashion or technology is launched, high–end customers are always interested in buying it. Therefore, companies keep introducing, innovative, and improved products to attract the customers at the high–end market. On the other hand, the low–end market sells lower price products suitable for customers who are not willing or able to pay higher price.

In marketing literature, simultaneous and consequence strategies to introduce low–end and high–end products are discussed in Moorthy and Png (1992) and Padmanabhan et al. (1997). In this thesis, the focus is on the case where products are first introduced in the high–end market and then the leftover products are sold in the low–end market. For this particular scenario, there are many real–world applications. For example, in China, batteries are first sold in the high–end market (urban customers) and then in the low–end market (rural customers). In Thailand, some bakeries use this strategy to sell their products in the high–end market (urban customers) and low–end market (rural customers). Computers are sold in the high–end market (Canadian domestic market) and then the leftover is sold in China. Jrgensen and Liddo (2007) address the case where a fashion firm sells its products in the high–end market for a limited time and then sells its products in the low–end market to compete with imitators, dispose of leftover inventory, and to enhance the benefit from the original design.

The EOQ model for deteriorating items has been studied by a number of authors since 1953. Different authors focus on two different demand patterns: time–dependent demand and price–dependent demand. Goswami and Chaudhuri (1991), Hariga (1993), Chakrabarti and Chaudhuri (1997), Dye (1999), Wu (2002) developed the EOQ model for deteriorating items considering time–dependent demand, in particular, linear trends. On the other hand, Abad (1996), Abad (2001), Dye (2007), Abad (2008) studied price–dependent demand. However, according to the author's knowledge, there is no study that develops an EOQ model for deteriorating items considering a two–market model.

In this thesis, three different EOQ models are developed for both time–dependent demand and price–dependent demand: (1) a two–market model; (2) a one–market model with discounting; and (3) a one–market mode without any price discounting. In each

EOQ model, two different scenarios are studied: partial backlogging and complete backlogging. In the two–market model, the products are first sold in the high–end market for a period of time and then the unsold products are transported to the low–end market. For both time–dependent demand, and price–dependent demand, the objective of the two–market model is to determine the time at which the inventory level becomes zero, t_3 (see Figure 3.2), and the optimal order quantity, Q, that can result in the maximum total profit. In one–market model with discounting, the products are sold in the high–end market for a period of time and then the price is discounted in the high–end market without transporting to the low–end market. For both time–dependent demand, and price–dependent demand, the objective of the one–market model is to determine the optimal order quantity, Q, the discounted price in the high–end market, P_d, and the time at which the inventory level becomes zero, t_3 (see Figure 3.3) that can result in the same total profit obtained in the two–market model. In the one–market model, the products are only sold in the high–end market without any price discounting. This model is developed to determine the selling price that can be offered to make the same profit as the two–market model. For both time–dependent demand and price–dependent demand, the objective of this one–market model is to determine the optimal order quantity, Q, the price in the high–end market, P_o, and the time at which the inventory level becomes zero, t_3 (see Figure 3.4).

In this thesis, demand rate is assumed to be different for different markets. In the existing literature, inventory level decreases due to market demand and deterioration. However, in the proposed models, the inventory level decreases only due to deterioration during the transportation from the high–end market to the low–end market. Also different selling prices for different markets are introduced, and the optimal selling prices of different markets for the model with price–dependent demand are determined.

The remainder of this thesis is organized as follows: Chapter 2 provides the literature review on EOQ models for deteriorating items. Chapter 3 presents EOQ models for deteriorating items with time–dependent demand and Chapter 4 presents EOQ models with price–dependent demand. Chapters 5 and 6 illustrate several numerical examples by using the proposed models in Chapters 3 and 4, respectively. Lastly, in Chapter 7, the conclusion of this thesis and future work are presented.

Chapter 2

Literature Review

This chapter reviews the relevant literature on EOQ models for deteriorating items.

One of the most important concerns for the inventory management is to decide how much inventory should be ordered each time so that the total cost associated with the inventory system will be minimized. The Economic Order Quantity (EOQ) model is usually used to determine the optimal order quantity that minimizes the total cost. One of the basic assumptions of the EOQ model is the infinite life of products, i.e., the quality of products remains unchanged. However, deteriorating items either become damaged or obsolete during their normal storage periods. As a result, if the rate of deterioration is not sufficiently low, its impact on the modeling of such an inventory system cannot be ignored.

Inventory problems for deteriorating items have been studied extensively by many researchers. Research in this area started with the work of Whitin (1953), who assumed that fashion good's deterioration takes place at the end of the prescribed storage period. But certain commodities may shrink with time by a proportion that can be approximated by a negative exponential function of time. For the first time, Ghare and Schrader (1963) established an EOQ model with exponentially decay, i.e., the constant rate of deterioration over time. However, the rate of deterioration increases with time for a few commodities such as fruits, vegetables, dairy products, etc, and it has been observed that the time of deterioration of those items can be expressed by a Weibull distribution. For the first time, the assumption of the constant deterioration rate was relaxed by Covert and Philip (1973), who used a two–parameter Weibull distribution to represent the distribution of the time to deterioration. Further, Philip (1974) extended this model to a three–parameter Weibull Distribution. Misra (1975) also adopted a two–parameter Weibull distribution deterioration to develop an inventory model with finite rate of replenishment. Then Tadikamalla (1978) examined an EOQ model assuming Gamma distributed deterioration. Fujiwara and Perera (1993) developed an EOQ model for inventory management under the assumption that product value decreases over time according to an exponential distribution. Therefore, a more realistic model is the one that treats the deterioration rate as a time varying function. To avoid complexity, several researchers including Abad (1996), Abad (2001), Dye and Ouyang (2005), Dye et al. (2007), Dye (2007), and Abad (2008) developed inventory models where deterioration depends on time only. However, deterioration does not depend on time only. It can be affected by season, weather, and storage condition as well. According this observation, several studies such as Wu et al. (2000), Giri et al. (2003) considered a two–parameter Weibull distribution to represent the distribution of the time to deterioration.

Demand plays a key role in modeling of inventory deterioration. The demand may be static or dynamic throughout the lifetime of the product. Static demand is of rare occurrence in practice as demand for product often varies with several factors such as time, price, stock, etc. Some deteriorating products are also seasonal in nature and demand for them exhibits different patterns during different seasons. Demand can be categorized into: a) uniform demand, b) time dependent demand c) stock dependent demand and d) price dependent demand and e) stochastic demand. Since this thesis focuses mainly on the time–dependent and price–dependent demand, we review these two topics below.

2.1 Time–Dependent Demand

In the classical EOQ model, it is assumed that the demand rate is constant. However, in the real life situation, the assumption of constant demand is not always suitable. The demand for a few products may rise during the growth phase of their product life cycle. On the other hand, some products' demand may decrease due to the introduction of more attractive products. This phenomenon motivates researchers to develop deterioration models with time–dependent demand pattern. Silver and Meal (1973) proposed a heuristic solution for selecting lot size quantities for the general case of a time–dependent demand pattern, which is known as "Silver Meal Heuristic". Donaldson (1977) probably was the first investigating the classical inventory model with a linearly increasing demand pattern over a known and finite horizon. However, the computational procedure of this model was too complicated. Silver (1979) considered a special case of the linearly increasing demand pattern and applied the "Silver and Meal Heuristic" method to solve the problem raised by Donaldson (1977). Later, McDonald (1979), Ritchie (1980), Ritchie (1984), Ritchie (1985), Mitra et al. (1984), and Goyal (1985) contributed to this direction. However, neither inventory shortages nor backlogging was considered in the above papers. Deb and Chaudhuri (1987b) was the first incorporating shortages into the inventory lot-sizing problem with a linearly increasing time–dependent demand. Goswami and Chaudhuri (1991) considered the inventory replenishment problem over a fixed planning horizon for a linear trend demand with backlogging. Hariga (1993) pointed out some errors in Goswami and Chaudhuri (1991) and provided an alternative simple algorithm to determine the optimal solution. Several researchers including Chakrabarti and Chaudhuri (1997), Dye (1999), Wu et al. (2000), Wu (2002), Teng et al. (2003), and Manna et al. (2009) developed inventory models considering shortages for deteriorating items with a linear trend demand pattern.

2.2 Price–Dependent Demand

In some cases, the retailer's inventory level is affected by the demand, which is price sensitive. It has been seen that lower selling price can generate more selling rate whereas higher selling price has the reverse effect on the selling rate. Therefore, the problem of determining the selling price and the lot size are related to each other.

Cohen (1977) first investigated the pricing problem facing by a retailer who sells a deteriorating product. By assuming that the selling price during the inventory cycle is a constant, he outlined the optimal pricing and ordering policy.

Dynamic pricing and lot-sizing problem for deteriorating products were studied by Abad (1996). It was assumed that the retailers may vary a product's price over the cycle time. The selling price for a given time maximizes only the instantaneous revenue rate and does not depend on the lot size and the cycle time. The problem is solved through two subproblems: at first the optimal price was determined for a given cycle time and then the optimal cycle time was determined for the given optimal price.

Time varying price is very difficult to administer. In some grocery stores, the selling price is held constant over the inventory cycle for administrative convenience. Abad (2001) assumed that the selling price within the inventory cycle is constant and investigated the pricing and lot sizing problems simultaneously.

Chang et al. (2006b), Dye (2007), and Dye et al. (2007) studied the pricing and lot sizing problem for the infinite planning horizon assuming the demand rate to be a convex, decreasing function of the selling price, and the revenue to be a concave function of selling price.

Recently, Abad (2008) developed a model for backlogging case, which assumed that the demand rate is a decreasing function of price and the marginal revenue is an increasing function of price.

When shortages occur, the following cases may arise: (1) all demand is backlogged; (2) all demand is lost due to impatient customers; or (3) a portion of demand is backlogged and the rest is lost. Ghare and Schrader (1963) first investigated an EOQ model without shortage for deteriorating items. Deb and Chaudhuri (1987a) were the pioneer to introduce the shortage into the inventory model with a linear trend demand pattern. They allowed shortages in all cycles except for the last one. Chakrabarti and Chaudhuri (1997) extended the model by allowing shortages for all cycles.

However, all the above models assumed that during a shortage period either all demands are backlogged or all are lost. Padmanabhan and Vrat (1995) developed a model with zero lead time for partial backlogging. The backlogging function depends on the amount of demand backlogged. For some electronics and fashion commodities with short product life cycles, the length of the waiting time for the next replenishment is the major factor affecting the backlogging. During the shortage period, the willingness of a customer waiting until the next replenishment depends on the length of the waiting time. The backlogging rate declines with the length of the waiting time. To reflect this phenomenon Abad (2001) introduced the backlogging rate depending on the time to replenishment. The fraction of backlogging decreases with the time that customers have to wait until the next replenishment. Two different backlogging rates were proposed: time proportional rate and exponential rate. However, since the costs of backlogging and lost sales are hard to estimate, they were not incorporated into the model. Dye and Ouyang (2005) proposed a time proportional backlogging model by adding the costs of backlogging and lost sales. A unique optimal solution was established in which building up inventory has a negative effect on the profit. Later, Chang et al. (2006a) proposed a revision of the previous model and justified that building up inventory is profitable. Dye et al. (2007) amended Abad's exponentially backlogging rate model by adding the costs of backlogging and lost sales.

Chapter 3

Models for Time–Dependent Demand

This chapter presents EOQ models for deteriorating items with time–dependent demand. It is assumed that there is no repair or replacement of deteriorating items during the cycle time. The time varying deterioration rate is considered and the time to deterioration is described by a two–parameter Weibull distribution, which has a probability density function $f(t)$ and a cumulative distribution function $F(t)$: $f(t) = \alpha\beta t^{\beta-1}e^{-\alpha t^{\beta}}$ and $F(t) = \int_0^t f(t)\,dt = 1 - e^{-\alpha t^{\beta}}$, where $\alpha > 0$ and $\beta > 0$ are scale and shape parameters, respectively, and t is the time to deterioration, $t > 0$. The deterioration rate can be obtained from $\theta(t) = \frac{f(t)}{1-F(t)}$, and hence, the instantaneous rate of deterioration of the on-hand inventory is θ(t) $=\alpha\beta t^{\beta-1}$. The Weibull distribution is related to a number of other probability distributions. For example, when the shape parameter $\beta = 1$, it refers to an exponential distribution; when $\beta = 2$, it refers to a Rayleigh distribution. Figure 3.1 indicates, for $\beta = 1$, the deterioration rate is constant with time. When $\beta < 1$, the deterioration rate decreases with time, and when $\beta > 1$, it increases with time. However in this thesis, it is assumed that $\beta > 1$, which means the deterioration rate increases with time.

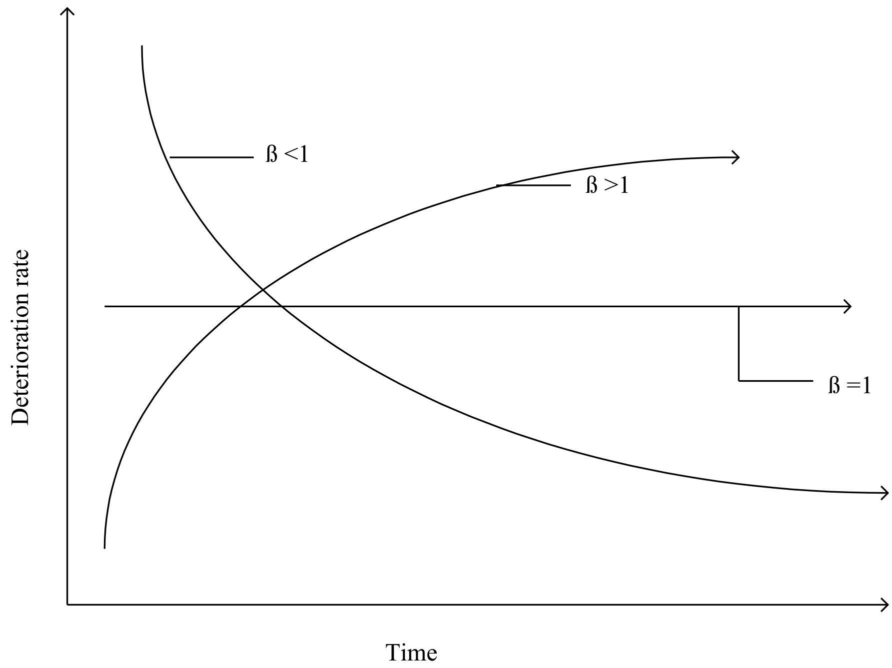

Figure 3.1: Relationship between rate of deterioration and time

In the literature, inventory models with time–dependent demand are considered for both linear and exponential demand with either increasing or decreasing demand. For example, a linear demand is described by $D(t) = a + bt$, where $a > 0$ and $b \neq 0$; and an exponential demand is described by $D(t) = Ae^{\gamma t}$, where $A > 0$ and $\gamma \neq 0$. In this thesis, a general demand function, $D(t)$, is considered in developing the models. In order to investigate the insight of these models, in the numerical section, a demand function, $D(t) = a+bt$, where $a, b > 0$, is considered. However, one can apply any time–dependent demand function in the developed models.

3.1 Two–Market Model

This model focuses on how the given new product is sold in the two markets. It is assumed that a new product, e.g., a fashion product, has only one cycle. It is also assmued that the unmet demand in the low–end market is backlogged and met with the next incoming product. This method has been in practice in many retail stores. If one has the situation where there is no backlog, it can be set to zero.

The two–market model consists of a high–end market and a low–end market for two different levels of customer. The order quantity $Q > 0$ is instantaneously received at the high–end market, and then the inventory level gradually diminishes due to demand and deterioration at the high–end market. At given time t_1, the leftover products are transferred from the high–end market to the low–end market and they arrive at time t_2. The transportation time, $t_2 - t_1$, is assumed to be constant. A certain percentage of the items deteriorates during transportation due to material handling and storage conditions. As soon as the items arrive at the low–end market, the backlogged quantity (B), is fulfilled. The rest of the inventory decreases due to demand and deterioration at the low–end market; and ultimately goes to zero at time t_3. A typical behavior of the inventory system in a cycle is depicted in Figure 3.2.

The next incoming product is replaced in the high–end market at time t_1. If it is assmued that the current product and the next incoming product have the same inventory profile and $t_2 = t_3 - t_2$, then the next incoming product will reach the low–end market at time t_3 and there is no backlog. If $t_2 > t_3 - t_2$, then there will be backlog in the low–end market. However, in this two–market model, we do not focus on the incoming product.

For the two–market model with time–dependent demand, selling prices at both markets are assumed to be given. Customers at the high–end market buy a product at price P_1 and customers at the low–end market buy a product at cheaper price P_2, i.e., $P_1 > P_2$. The price discrimination exists because some items always lose their value over time due to deterioration; and the age of the items also has a negative effect on the price due to loss of customer confidence in the quality. As a result of price discrimination, higher demand is expected at the low–end market. The high–end market demand is denoted by $D_1(t)$ and the low–end market demand is denoted by $D_2(t)$, where $D_2(t) > D_1(t)$.

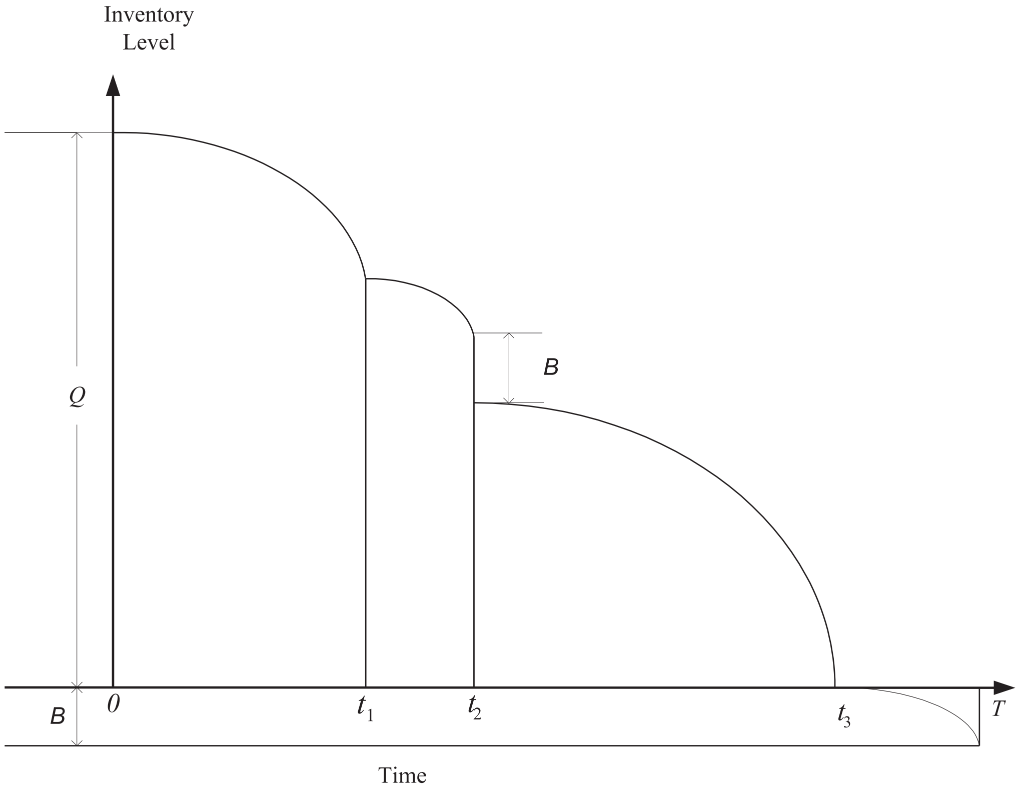

Figure 3.2: Inventory profile for the two–market model

3.1.1 Partial Backlogging

For partial backlogging, during the shortage period, some excess demands are backlogged and the rest of them are lost. That means, when shortages occur, some customers are willing to wait and the others would turn to other suppliers. The backlogging rate depends on the length of the waiting time for the next replenishment. The longer the waiting time is, the smaller the backlogging rate would be. Hence, the proportion of customers who would like to accept backlogging at time t is decreasing with the waiting time, $(T - t)$, for the next replenishment. The backlogging rate is expressed as $\frac{1}{1+\delta(T-t)}$, where $\delta > 0$, which is similar to the one in Abad (2001), Dye and Ouyang (2005), and Chang et al. (2006b).

Let $I_1(t)$, $I_2(t)$, $I_3(t)$, and $I_4(t)$ be the on–hand inventory level at any time $0 \le t \le t_1$, $t_1 \le t \le t_2$, $t_2 \le t \le t_3$, and $t_3 \le t \le T$, respectively. The instantaneous state of inventory level at the high–end market (i.e., $0 \leq t \leq t_1$) is governed by the following differential equation:

$$\frac{dI_1(t)}{dt} = -D_1(t) - \theta(t)I_1(t), \;\; 0 \leq t \leq t_1. \tag{3.1}$$

At the high–end market, the inventory gradually depletes due to two factors: one is demand and the other is deterioration. On the right hand side of Equation (3.1), the first term represents inventory depletion due to market demand and the second term represents inventory depletion due to deterioration. Substituting deterioration rate at any time t, $\theta(t) = \alpha\beta t^{\beta-1}$, in Equation (3.1), it can be written as

$$\frac{dI_1(t)}{dt} + \alpha\beta t^{\beta-1} I_1(t) = -D_1(t). \tag{3.2}$$

This is a first order linear differential equation and its integrating factor is

$$e^{\alpha\beta \int t^{\beta-1}\, dt} = e^{\alpha t^{\beta}}.$$

Multiplying both sides of Equation (3.2) by the integrating factor $e^{\alpha t^{\beta}}$, we obtain

$$\frac{dI_1(t)}{dt} e^{\alpha t^{\beta}} + \alpha\beta t^{\beta-1} I_1(t) e^{\alpha t^{\beta}} = -D_1(t) e^{\alpha t^{\beta}}. \tag{3.3}$$

The left hand side of Equation (3.3) can be simplified by product rule and then it can be expressed as

$$\frac{d}{dt}\{I_1(t) e^{\alpha t^{\beta}}\} = -D_1(t) e^{\alpha t^{\beta}}.$$

Now, integrating both sides of Equation (3.3) with respect to time,

$$I_1(t) = \frac{-\int D_1(t) e^{\alpha t^{\beta}}\, dt + k_1}{e^{\alpha t^{\beta}}}.$$

At time $t = 0$, inventory level is the maximum. Applying this boundary condition, $I_1(0) = Q$,

$$I_1(t) = \frac{Q - \int_0^t D_1(x) e^{\alpha x^{\beta}}\, dx}{e^{\alpha t^{\beta}}}. \tag{3.4}$$

At time t_1, the leftover inventory is transferred from the high–end market to the low–end market. During transportation time, inventory level depletes only due to deterioration. The instantaneous state of inventory level during transportation is given by

$$\frac{dI_2(t)}{dt} = -\theta(t)I_2(t), \quad t_1 \leq t \leq t_2. \tag{3.5}$$

Now, substituting $\theta(t) = \alpha\beta t^{\beta-1}$ and dividing both sides of the above equation by $I_2(t)$, we get,

$$\frac{dI_2(t)}{I_2(t)} = -\alpha\beta t^{\beta-1}\, dt.$$

Integrating both sides of the above equation, it can be written as

$$I_2(t) = k_2 e^{-\alpha t^{\beta}}. \tag{3.6}$$

The value of k_2 can be found by applying the boundary condition that $I_1(t_1) = I_2(t_1)$. Hence,

$$\frac{Q - \int_0^{t_1} D_1(x)e^{\alpha x^{\beta}}\, dx}{e^{\alpha t_1^{\beta}}} = k_2 e^{-\alpha t_1^{\beta}}.$$

$$k_2 = Q - \int_0^{t_1} D_1(x)e^{\alpha x^{\beta}} dx.$$

Substituting the value of k_2 in Equation (3.6), it can be written as

$$I_2(t) = \frac{Q - \int_0^{t_1} D_1(x)e^{\alpha x^{\beta}} dx}{e^{\alpha t^{\beta}}}. \tag{3.7}$$

At time t_2, transported items arrive at the low–end market and then B amount of inventory is used to fulfill the backlogged demand. After fulfilling the backlogged demand, inventory level decreases due to demand and deterioration in the low-end market. The instantaneous state of inventory level in the low–end market is given by

$$\frac{dI_3(t)}{dt} = -D_2(t) - \theta(t)I_3(t), \quad t_2 \leq t \leq t_3. \tag{3.8}$$

Now, substituting $\theta(t) = \alpha\beta t^{\beta-1}$ and integrating both sides of the above equation with respect to time,

$$I_3(t) = \frac{-\int D_2(t)e^{\alpha t^{\beta}}\, dt + k_3}{e^{\alpha t^{\beta}}}.$$

Using the boundary condition $I_2(t_2) - B = I_3(t_2)$, we obtain

$$I_3(t) = \frac{Q - \int_0^{t_1} D_1(x)e^{\alpha x^\beta}\,dx - Be^{\alpha t_2^\beta} - \int_{t_2}^{t} D_2(x)e^{\alpha x^\beta}\,dx}{e^{\alpha t^\beta}}. \tag{3.9}$$

At time t_3, the inventory level goes to zero and shortage occurs. A percentage of shortages is backlogged and the rest is lost. Only the backlogged units are replaced by the next replenishment. The inventory level is governed by the following equation:

$$\frac{dI_4(t)}{dt} = -\frac{D_2(t)}{1+\delta(T-t)}, \quad t_3 \le t \le T. \tag{3.10}$$

With the boundary condition $I_4(t_3) = 0$, solving Equation (3.10), we obtain the following

$$I_4(t) = -\int_{t_3}^{t} \frac{D_2(x)}{1+\delta(T-x)}dx. \tag{3.11}$$

Hence, the backlogged quantity at time T is

$$B = \int_{t_3}^{T} \frac{D_2(x)}{1+\delta(T-x)}dx. \tag{3.12}$$

Finally, by applying the boundary condition $I_3(t_3) = 0$ in Equation (3.9), the total order quantity Q can be expressed as

$$Q = \int_0^{t_1} D_1(x)e^{\alpha x^\beta}\,dx + \int_{t_2}^{t_3} D_2(x)e^{\alpha x^\beta}\,dx + e^{\alpha t_2^\beta}\int_{t_3}^{T} \frac{D_2(x)}{1+\delta(T-x)}dx. \tag{3.13}$$

Now, substituting the value of Q in Equations (3.4), (3.7), and (3.9), we obtain,

$$\begin{aligned} I_1(t) = {} & e^{-\alpha t^\beta}\int_t^{t_1} D_1(x)e^{\alpha x^\beta}dx + e^{-\alpha t^\beta}\int_{t_2}^{t_3} D_2(x)e^{\alpha x^\beta}dx \\ & + e^{-\alpha t^\beta}e^{\alpha t_2^\beta}\int_{t_3}^{T}\frac{D_2(x)}{1+\delta(T-x)}dx. \end{aligned} \tag{3.14}$$

$$I_2(t) = e^{-\alpha t^\beta}\int_{t_2}^{t_3} D_2(x)e^{\alpha x^\beta}dx + e^{-\alpha t^\beta}e^{\alpha t_2^\beta}\int_{t_3}^{T}\frac{D_2(x)}{1+\delta(T-x)}dx. \tag{3.15}$$

$$I_3(t) = e^{-\alpha t^\beta}\int_t^{t_3} D_2(x)e^{\alpha x^\beta}dx. \tag{3.16}$$

The total inventory during a cycle is

$$\int_0^{t_1} I_1(t)dt + \int_{t_1}^{t_2} I_2(t)dt + \int_{t_2}^{t_3} I_3(t)dt.$$

The total holding cost per cycle can be determined by

$$H_c = h\int_0^{t_1} I_1(t)dt + h\int_{t_1}^{t_2} I_2(t)dt + h\int_{t_2}^{t_3} I_3(t)dt, \tag{3.17}$$

where h is the holding cost per unit per period. Substituting the value of $I_1(t)$, $I_2(t)$, and $I_3(t)$ in Equation (3.17), the total holding cost per cycle can be expressed as

$$\begin{aligned} H_c = {} & h\int_0^{t_1} e^{-\alpha t^\beta}[\int_t^{t_1} D_1(x)e^{\alpha x^\beta}\,dx]\,dt \\ & + h\int_0^{t_2} e^{-\alpha t^\beta}[\int_{t_2}^{t_3} D_2(x)e^{\alpha x^\beta}\,dx]\,dt \\ & + h\int_0^{t_2} e^{-\alpha t^\beta}[e^{\alpha t_2^\beta}\int_{t_3}^{T} \frac{D_2(x)}{1+\delta(T-x)}\,dx]\,dt \\ & + h\int_{t_2}^{t_3} e^{-\alpha t^\beta}[\int_t^{t_3} D_2(x)e^{\alpha x^\beta}\,dx]\,dt. \end{aligned} \tag{3.18}$$

From t_3 to T, demand is partially backlogged. Since shortages are negative inventory, the total demand backlogged can be determined as follows:

$$-\int_{t_3}^{T} I_4(t)dt = \int_{t_3}^{T}\int_{t_3}^{t} \frac{D_2(x)}{1+\delta(T-x)}dxdt.$$

Changing the order of integration, it can be written as

$$= \int_{t_3}^{T} \frac{D_2(x)(T-x)}{1+\delta(T-x)}dx.$$

Backorder cost is assumed to be directly proportional to the number of units backlogged. C_2 is the backorder cost per unit. The total backorder cost per cycle is

$$B_c = C_2\int_{t_3}^{T} \frac{D_2(x)(T-x)}{1+\delta(T-x)}dx. \tag{3.19}$$

During the shortage period, it is necessary to distinguish between backlog and lost sale. Lost sale units are satisfied by the competitors, therefore this is considered as a loss of profit. The amount of lost sales during the interval $[t_3, T]$ is

$$\begin{aligned} L_c &= \int_{t_3}^{T} [1 - \frac{1}{1+\delta(T-x)}]D_2(x)\,dx \\ &= \delta\int_{t_3}^{T} \frac{D_2(x)(T-x)}{1+\delta(T-x)}dx. \end{aligned}$$

The opportunity cost due to lost sale is defined as the sum of the gross profit margin and loss of goodwill (Dye and Ouyang 2005). Assuming C_3 is the opportunity cost per unit, the total opportunity cost per cycle can be determined as follows:

$$L_c = C_3 \delta \int_{t_3}^{T} \frac{D_2(x)(T-x)}{1+\delta(T-x)} dx. \tag{3.20}$$

Some products become unuseable or obsolete during storage or transportation and this loss should be taken into account in the total cost. The total amount of deteriorated items during a cycle is

$$U_1 = Q - \int_0^{t_1} D_1(x)\, dx - \int_{t_2}^{t_3} D_2(x)\, dx - \int_{t_3}^{T} \frac{D_2(x)}{1+\delta(T-x)}\, dx.$$

Now, substituting the value Q, it can be written as

$$\begin{aligned} U_1 = &\int_0^{t_1} D_1(x) e^{\alpha x^{\beta}}\, dx + \int_{t_2}^{t_3} D_2(x) e^{\alpha x^{\beta}}\, dx + e^{\alpha t_2^{\beta}} \int_{t_3}^{T} \frac{D_2(x)}{1+\delta(T-x)}\, dx \\ &- \int_0^{t_1} D_1(x)\, dx - \int_{t_2}^{t_3} D_2(x)\, dx - \int_{t_3}^{T} \frac{D_2(x)}{1+\delta(T-x)}\, dx. \end{aligned}$$

Assuming C_1 is the cost per deteriorated unit, the total deterioration cost per cycle can be expressed as,

$$U_c = U_1 C_1. \tag{3.21}$$

The total amount of transported units from the high–end market to the low–end market is $I(t_1)$. Hence, subsituting $t = t_1$ in Equation (3.14) results in

$$I_1(t_1) = e^{-\alpha t_1^{\beta}} \int_{t_2}^{t_3} D_2(x) e^{\alpha_2 x^{\beta_2}}\, dx + e^{-\alpha t_1^{\beta}} e^{\alpha t_2^{\beta}} \int_{t_3}^{T} \frac{D_2(x)}{1+\delta(T-x)}\, dx.$$

Transportation cost is assumed to be directly proportional to the number of transported units. C_4 is the transportation cost per unit. Hence, the total transportation cost per cycle is

$$M_c = C_4 \{ e^{-\alpha t_1^{\beta}} \int_{t_2}^{t_3} D_2(x) e^{\alpha x^{\beta}}\, dx + e^{-\alpha t_1^{\beta}} e^{\alpha t_2^{\beta}} \int_{t_3}^{T} \frac{D_2(x)}{1+\delta(T-x)}\, dx \}. \tag{3.22}$$

The total amount of ordering units including backorder quantities is:

$$Q = \int_0^{t_1} D_1(x) e^{\alpha x^{\beta}}\, dx + \int_{t_2}^{t_3} D_2(x) e^{\alpha x^{\beta}}\, dx + e^{\alpha t_2^{\beta}} \int_{t_3}^{T} \frac{D_2(x)}{1+\delta(T-x)} dx.$$

Purchase cost is directly proportional to the number of purchase units. It also includes the cost of placing an order, the cost of processing the receipt, incoming inspection, and invoice processing. Assuming C is the purchase cost per unit, the total purchase cost per cycle is

$$P_c = C\{\int_0^{t_1} D_1(x)e^{\alpha x^\beta}\,dx + \int_{t_2}^{t_3} D_2(x)e^{\alpha x^\beta}\,dx + e^{\alpha t_2^\beta}\int_{t_3}^{T} \frac{D_2(x)}{1+\delta(T-x)}\,dx\}. \tag{3.23}$$

Finally, the total cost per time is the summation of the purchase cost (P_c), holding cost (H_c), deterioration cost (U_c), transportation cost (M_c), backorder cost (B_c), and opportunity cost due to lost sale (L_c) divided by the cycle time T:

$$\begin{aligned}
\Gamma_c(t_3) = {} & \frac{1}{T}[C\{\int_0^{t_1} D_1(x)e^{\alpha x^\beta}\,dx + \int_{t_2}^{t_3} D_2(x)e^{\alpha x^\beta}\,dx + e^{\alpha t_2^\beta}\int_{t_3}^{T} \frac{D_2(x)}{1+\delta(T-x)}\,dx\} \\
& + h\int_0^{t_1} e^{-\alpha t^\beta}[\int_t^{t_1} D_1(x)e^{\alpha x^\beta}\,dx]\,dt + h\int_0^{t_2} e^{-\alpha t^\beta}[\int_{t_2}^{t_3} D_2(x)e^{\alpha x^\beta}\,dx]\,dt \\
& + h\int_0^{t_2} e^{-\alpha t^\beta}[e^{\alpha t_2^\beta}\int_{t_3}^{T} \frac{D_2(x)}{1+\delta(T-x)}\,dx]\,dt + h\int_{t_2}^{t_3} e^{-\alpha t^\beta}[\int_t^{t_3} D_2(x)e^{\alpha x^\beta}\,dx]\,dt \\
& + C_1\{\int_0^{t_1} D_1(x)e^{\alpha x^\beta}\,dx + \int_{t_2}^{t_3} D_2(x)e^{\alpha x^\beta}\,dx + e^{\alpha t_2^\beta}\int_{t_3}^{T} \frac{D_2(x)}{1+\delta(T-x)}\,dx \\
& - \int_0^{t_1} D_1(x)\,dx - \int_{t_2}^{t_3} D_2(x)\,dx - \int_{t_3}^{T} \frac{D_2(x)}{1+\delta(T-x)}\,dx\} \\
& + C_4\{e^{-\alpha t_1^\beta}\int_{t_2}^{t_3} D_2(x)e^{\alpha x^\beta}\,dx + e^{-\alpha t_1^\beta}e^{\alpha t_2^\beta}\int_{t_3}^{T} \frac{D_2(x)}{1+\delta(T-x)}\,dx\} \\
& + (C_2 + C_3\delta)\int_{t_3}^{T} \frac{D_2(x)(T-x)}{1+\delta(T-x)}\,dx].
\end{aligned} \tag{3.24}$$

Assuming P_1 and P_2 are the selling prices at high–end and low–end markets, respectively, the total sales revenue per unit time can be expressed as

$$SR = \frac{1}{T}[P_1\int_0^{t_1} D_1(x)\,dx + P_2\int_{t_2}^{t_3} D_2(x)\,dx + P_2\int_{t_3}^{T} \frac{D_2(x)}{1+\delta(T-x)}\,dx]. \tag{3.25}$$

The profit per unit time is the total sales revenue per unit time minus the total cost per unit time. The total profit per unit time can be expressed from Equations (3.25) and

(3.24) as

$$\pi(t_3) = \frac{1}{T}[P_1 \int_0^{t_1} D_1(x)\,dx + P_2 \int_{t_2}^{t_3} D_2(x)\,dx + P_2 \int_{t_3}^{T} \frac{D_2(x)}{1+\delta(T-x)}\,dx$$
$$- C\{\int_0^{t_1} D_1(x)e^{\alpha x^\beta}\,dx + \int_{t_2}^{t_3} D_2(x)e^{\alpha x^\beta}\,dx + e^{\alpha t_2^\beta} \int_{t_3}^{T} \frac{D_2(x)}{1+\delta(T-x)}\,dx\}$$
$$- h\int_0^{t_1} e^{-\alpha t^\beta}[\int_t^{t_1} D_1(x)e^{\alpha x^\beta}\,dx]\,dt - h\int_0^{t_2} e^{-\alpha t^\beta}[\int_{t_2}^{t_3} D_2(x)e^{\alpha x^\beta}\,dx]\,dt$$
$$- h\int_0^{t_2} e^{-\alpha t^\beta}[e^{\alpha t_2^\beta} \int_{t_3}^{T} \frac{D_2(x)}{1+\delta(T-x)}\,dx]\,dt - h\int_{t_2}^{t_3} e^{-\alpha t^\beta}[\int_t^{t_3} D_2(x)e^{\alpha x^\beta}\,dx]\,dt$$
$$- C_1\{\int_0^{t_1} D_1(x)e^{\alpha x^\beta}\,dx + \int_{t_2}^{t_3} D_2(x)e^{\alpha x^\beta}\,dx + e^{\alpha t_2^\beta} \int_{t_3}^{T} \frac{D_2(x)}{1+\delta(T-x)}\,dx$$
$$- \int_0^{t_1} D_1(x)\,dx - \int_{t_2}^{t_3} D_2(x)\,dx - \int_{t_3}^{T} \frac{D_2(x)}{1+\delta(T-x)}\,dx\}$$
$$- C_4\{e^{-\alpha t_1^\beta} \int_{t_2}^{t_3} D_2(x)e^{\alpha x^\beta}\,dx + e^{-\alpha t_1^\beta} e^{\alpha t_2^\beta} \int_{t_3}^{T} \frac{D_2(x)}{1+\delta(T-x)}\,dx\}$$
$$- (C_2 + C_3\delta) \int_{t_3}^{T} \frac{D_2(x)(T-x)}{1+\delta(T-x)}\,dx]. \tag{3.26}$$

Now, the objective is to determine, for a given time t_1, the optimal time t_3^* that maximizes the total profit $\pi(t_3)$. Consequently, for the two–market model, we can find the optimal time at which inventory becomes zero at the low–end market, t_3^*, the optimal order quantity, Q^*, the optimal profit, π^*, and the optimal level of backlogging, B^*.

The necessary condition for $\pi(t_3)$ to be maximum is

$$\frac{d\pi(t_3)}{dt_3}|_{t_3^*} = 0$$

The first derivative of $\pi(t_3)$ with respect to t_3 is as follows:

$$\frac{d\pi(t_3)}{dt_3} = \frac{D_2(t_3)}{T}[P_2 - \frac{P_2}{1+\delta(T-t_3)} - Ce^{\alpha t_3^\beta} + \frac{Ce^{\alpha t_2^\beta}}{1+\delta(T-t_3)}$$
$$- he^{\alpha t_3^\beta} \int_0^{t_3} e^{-\alpha t^\beta}\,dt + \frac{he^{\alpha t_2^\beta}}{\beta+1}\{\frac{t_2(\beta+1) - \alpha t_2^{\beta+1}}{1+\delta(T-t_3)}\} - C_1\{e^{\alpha t_3^\beta} - 1\}$$
$$+ C_1\{\frac{e^{\alpha t_2^\beta}}{1+\delta(T-t_3)} - \frac{1}{1+\delta(T-t_3)}\} - C_4 e^{-\alpha t_1^\beta} e^{\alpha t_3^\beta}$$
$$+ \frac{C_4 e^{-\alpha t_1^\beta} e^{\alpha t_2^\beta}}{1+\delta(T-t_3)} + \frac{(C_2 + C_3\delta)(T-t_3)}{1+\delta(T-t_3)}]. \tag{3.27}$$

Setting the right hand side of Equation (3.27) equal to zero, we can obtain the following:

$$
\begin{aligned}
&\frac{D_2(t_3)}{T}[P_2 - \frac{P_2}{1+\delta(T-t_3)} - Ce^{\alpha t_3^\beta} + \frac{Ce^{\alpha t_2^\beta}}{1+\delta(T-t_3)} \\
&- he^{\alpha t_3^\beta}\int_0^{t_3} e^{-\alpha t^\beta}\,dt + \frac{he^{\alpha t_2^\beta}}{\beta+1}\{\frac{t_2(\beta+1)-\alpha t_2^{\beta+1}}{1+\delta(T-t_3)}\} - C_1\{e^{\alpha t_3^\beta}-1\} \\
&+ C_1\{\frac{e^{\alpha t_2^\beta}}{1+\delta(T-t_3)} - \frac{1}{1+\delta(T-t_3)}\} - C_4e^{-\alpha t_1^\beta}e^{\alpha t_3^\beta} \\
&+ \frac{C_4e^{-\alpha t_1^\beta}e^{\alpha t_2^\beta}}{1+\delta(T-t_3)} + \frac{(C_2+C_3\delta)(T-t_3)}{1+\delta(T-t_3)}] = 0.
\end{aligned}
$$

Since $\frac{D_2(t_3)}{T} \neq 0$, let

$$
\begin{aligned}
f(t_3) = {} & P_2 - \frac{P_2}{1+\delta(T-t_3)} - Ce^{\alpha t_3^\beta} + \frac{Ce^{\alpha t_2^\beta}}{1+\delta(T-t_3)} \\
&- he^{\alpha t_3^\beta}\int_0^{t_3} e^{-\alpha t^\beta}\,dt + \frac{he^{\alpha t_2^\beta}}{\beta+1}\{\frac{t_2(\beta+1)-\alpha t_2^{\beta+1}}{1+\delta(T-t_3)}\} - C_1\{e^{\alpha t_3^\beta}-1\} \\
&+ C_1\{\frac{e^{\alpha t_2^\beta}}{1+\delta(T-t_3)} - \frac{1}{1+\delta(T-t_3)}\} - C_4e^{-\alpha t_1^\beta}e^{\alpha t_3^\beta} \\
&+ \frac{C_4e^{-\alpha t_1^\beta}e^{\alpha t_2^\beta}}{1+\delta(T-t_3)} + \frac{(C_2+C_3\delta)(T-t_3)}{1+\delta(T-t_3)} = 0.
\end{aligned} \tag{3.28}
$$

Equation (3.28) implies that the optimal t_3^* is independent of the high–end market selling price. One can find t_3^* by using an iterative method from Equation (3.28).

In addition, the sufficient condition is that total profit function needs to be concave and must satisfy

$$\frac{d^2\pi(t_3)}{dt_3^2}|_{t=t_3^*} < 0, \quad \forall\, t_3^* > 0.$$

Therefore, differentiating Equation (3.27) with respect to t_3, we get

$$
\begin{aligned}
\frac{d^2\pi(t_3)}{dt_3^2} = {} & \frac{D_2'(t_3^*)}{T}[P_2 - \frac{P_2}{1+\delta(T-t_3^*)} - Ce^{\alpha t_3^{*\beta}} + \frac{Ce^{\alpha t_2^\beta}}{1+\delta(T-t_3^*)} \\
&- he^{\alpha t_3^{*\beta}}\int_0^{t_3^*} e^{-\alpha t^\beta}\,dt + \frac{he^{\alpha t_2^\beta}}{\beta+1}\{\frac{t_2(\beta+1)-\alpha t_2^{\beta+1}}{1+\delta(T-t_3^*)}\} - C_1\{e^{\alpha t_3^{*\beta}}-1\}
\end{aligned}
$$

$$
\begin{aligned}
&+ C_1\{\frac{e^{\alpha t_2^\beta}}{1+\delta(T-t_3^*)} - \frac{1}{1+\delta(T-t_3^*)}\} - C_4 e^{-\alpha t_1^\beta} e^{\alpha t_3^{*\beta}} + \frac{C_4 e^{-\alpha t_1^\beta} e^{\alpha t_2^\beta}}{1+\delta(T-t_3^*)} \\
&+ \frac{(C_2+C_3\delta)(T-t_3^*)}{1+\delta(T-t_3^*)}] + \frac{D_2(t_3^*)}{T}[\frac{-P_2\delta}{(1+\delta(T-t_3^*))^2} - C\alpha\beta t_3^{*\beta-1} e^{\alpha t_3^{*\beta}} \\
&+ \frac{C e^{\alpha t_2^\beta}\delta}{(1+\delta(T-t_3^*))^2} - h\alpha\beta t_3^{*\beta-1} e^{\alpha t_3^{*\beta}} \int_0^{t_3^*} e^{-\alpha t^\beta}\, dt - h + \frac{h e^{\alpha t_2^\beta}\delta}{\beta+1}\{\frac{t_2(\beta+1)-\alpha t_2^{\beta+1}}{1+\delta(T-t_3^*)^2}\} \\
&- C_4\alpha\beta t_3^{*\beta-1} e^{-\alpha t_1^\beta} e^{\alpha t_3^{*\beta}} + \frac{C_4 e^{-\alpha t_1^\beta} e^{\alpha t_2^\beta}\delta}{1+\delta(T-t_3^*)^2} - C_1\alpha\beta t_3^{*\beta-1} e^{\alpha t_3^{*\beta}} \\
&+ C_1\{\frac{e^{\alpha t_2^\beta}\delta}{1+\delta(T-t_3^*)^2} - \frac{\delta}{1+\delta(T-t_3^*)^2}\} - \frac{(C_2+C_3\delta)}{(1+\delta(T-t_3^*))^2}]. \qquad (3.29)
\end{aligned}
$$

From Equation (3.28), we know $f(t_3^*) = 0$, which is

$$
\begin{aligned}
&P_2 - \frac{P_2}{1+\delta(T-t_3^*)} - C e^{\alpha t_3^{*\beta}} + \frac{C e^{\alpha t_2^\beta}}{1+\delta(T-t_3^*)} \\
&- h e^{\alpha t_3^{*\beta}} \int_0^{t_3^*} e^{-\alpha t^\beta}\, dt + \frac{h e^{\alpha t_2^\beta}}{\beta+1}\{\frac{t_2(\beta+1)-\alpha t_2^{\beta+1}}{1+\delta(T-t_3^*)}\} - C_1\{e^{\alpha t_3^{*\beta}} - 1\} \\
&+ C_1\{\frac{e^{\alpha t_2^\beta}}{1+\delta(T-t_3^*)} - \frac{1}{1+\delta(T-t_3^*)}\} - C_4 e^{-\alpha t_1^\beta} e^{\alpha t_3^{*\beta}} \\
&+ \frac{C_4 e^{-\alpha t_1^\beta} e^{\alpha t_2^\beta}}{1+\delta(T-t_3^*)} + \frac{(C_2+C_3\delta)(T-t_3^*)}{1+\delta(T-t_3^*)} = 0.
\end{aligned}
$$

Hence, $\frac{d^2\pi(t_3)}{dt_3^2}|_{t=t_3^*}$ can be expressed as

$$
\begin{aligned}
\frac{d^2\pi(t_3)}{dt_3^2}|_{t=t_3^*} = {}& \frac{D_2(t_3^*)}{T}[\frac{-P_2\delta}{(1+\delta(T-t_3^*))^2} + \frac{C e^{\alpha t_2^\beta}\delta}{(1+\delta(T-t_3^*))^2} - C\alpha\beta t_3^{*\beta-1} e^{\alpha t_3^{*\beta}} \\
&- h\alpha\beta t_3^{*\beta-1} e^{\alpha t_3^{*\beta}} \int_0^{t_3^*} e^{-\alpha t^\beta}\, dt - h + \frac{h e^{\alpha t_2^\beta}\delta}{\beta+1}\{\frac{t_2(\beta+1)-\alpha t_2^{\beta+1}}{1+\delta(T-t_3^*)^2}\} \\
&- C_4\alpha\beta t_3^{*\beta-1} e^{-\alpha t_1^\beta} e^{\alpha t_3^{*\beta}} + \frac{C_4 e^{-\alpha t_1^\beta} e^{\alpha t_2^\beta}\delta}{1+\delta(T-t_3^*)^2} - C_1\alpha\beta t_3^{*\beta-1} e^{\alpha t_3^{*\beta}} \\
&+ C_1\{\frac{e^{\alpha t_2^\beta}\delta}{1+\delta(T-t_3^*)^2} - \frac{\delta}{1+\delta(T-t_3^*)^2}\} - \frac{(C_2+C_3\delta)}{(1+\delta(T-t_3^*))^2}] < 0. \qquad (3.30)
\end{aligned}
$$

In Equation (3.30), P_2 and C are very influential and $P_2 > C$. Hence, the sum of the first two terms is negative; and regarding the rest of the terms, the value of the negative terms is larger than that of the positive terms. Consequently, $\frac{d^2\pi(t_3)}{dt_3^2}|_{t=t_3^*} < 0$ indicates that the total profit function is concave at $t = t_3^*$.

Substituting $t_3 = t_3^*$ in Equations (3.26) and (3.13), we can find the optimal total profit, $\pi(t_3^*)$, and optimal ordering quantity, Q^*, respectively, as follows:

$$\begin{aligned}
\pi(t_3^*) &= \frac{1}{T}[P_1 \int_0^{t_1} D_1(x)\,dx + P_2 \int_{t_2}^{t_3^*} D_2(x)\,dx + P_2 \int_{t_3^*}^{T} \frac{D_2(x)}{1+\delta(T-x)}\,dx \\
&- C\{\int_0^{t_1} D_1(x)e^{\alpha x^\beta}\,dx + \int_{t_2}^{t_3^*} D_2(x)e^{\alpha x^\beta}\,dx + e^{\alpha t_2^\beta} \int_{t_3^*}^{T} \frac{D_2(x)}{1+\delta(T-x)}\,dx\} \\
&- h\int_0^{t_1} e^{-\alpha t^\beta}[\int_t^{t_1} D_1(x)e^{\alpha x^\beta}\,dx]\,dt - h\int_0^{t_2} e^{-\alpha t^\beta}[\int_{t_2}^{t_3^*} D_2(x)e^{\alpha x^\beta}\,dx]\,dt \\
&- h\int_0^{t_2} e^{-\alpha t^\beta}[e^{\alpha t_2^\beta}\int_{t_3^*}^{T} \frac{D_2(x)}{1+\delta(T-x)}\,dx]\,dt - h\int_{t_2}^{t_3^*} e^{-\alpha t^\beta}[\int_t^{t_3^*} D_2(x)e^{\alpha x^\beta}\,dx]\,dt \\
&- C_1\{\int_0^{t_1} D_1(x)e^{\alpha x^\beta}\,dx + \int_{t_2}^{t_3^*} D_2(x)e^{\alpha x^\beta}\,dx + e^{\alpha t_2^\beta} \int_{t_3^*}^{T} \frac{D_2(x)}{1+\delta(T-x)}\,dx \\
&- \int_0^{t_1} D_1(x)\,dx - \int_{t_2}^{t_3^*} D_2(x)\,dx - \int_{t_3^*}^{T} \frac{D_2(x)}{1+\delta(T-x)}\,dx\} \\
&- C_4\{e^{-\alpha t_1^\beta} \int_{t_2}^{t_3^*} D_2(x)e^{\alpha x^\beta}\,dx + e^{-\alpha t_1^\beta} e^{\alpha t_2^\beta} \int_{t_3^*}^{T} \frac{D_2(x)}{1+\delta(T-x)}\,dx\} \\
&- (C_2 + C_3\delta) \int_{t_3^*}^{T} \frac{D_2(x)(T-x)}{1+\delta(T-x)}\,dx],
\end{aligned} \tag{3.31}$$

and

$$Q^* = \int_0^{t_1} D_1(x)e^{\alpha x^\beta}\,dx + \int_{t_2}^{t_3^*} D_2(x)e^{\alpha x^\beta}\,dx + e^{\alpha t_2^\beta} \int_{t_3^*}^{T} \frac{D_2(x)}{1+\delta(T-x)}\,dx. \tag{3.32}$$

3.1.2 Complete Backlogging

In a complete backlogging case, all shortages in the low–end market are backlogged and they are replaced by the next replenishment. In this case, the average total cost per unit time is the summation of the purchase cost, holding cost, deterioration cost, transportation cost, and backlogging cost. However, there is no opportunity cost, because all shortages are backlogged. Therefore, substituting $\delta = 0$ in Equation (3.26), we get the total profit of the complete backlogging case as follows:

$$\begin{aligned}
\pi(t_3) = & \frac{1}{T}[P_1 \int_0^{t_1} D_1(x)\,dx + P_2 \int_{t_2}^{t_3} D_2(x)\,dx + P_2 \int_{t_3}^{T} D_2(x)\,dx \\
& - C\{\int_0^{t_1} D_1(x)e^{\alpha x^\beta}\,dx + \int_{t_2}^{t_3} D_2(x)e^{\alpha x^\beta}\,dx + e^{\alpha t_2^\beta}\int_{t_3}^{T} D_2(x)\,dx\} \\
& - h\int_0^{t_1} e^{-\alpha t^\beta}[\int_t^{t_1} D_1(x)e^{\alpha x^\beta}\,dx]\,dt - h\int_0^{t_2} e^{-\alpha t^\beta}[\int_{t_2}^{t_3} D_2(x)e^{\alpha x^\beta}\,dx]\,dt \\
& - h\int_0^{t_2} e^{-\alpha t^\beta}[e^{\alpha t_2^\beta}\int_{t_3}^{T} D_2(x)\,dx]\,dt - h\int_{t_2}^{t_3} e^{-\alpha t^\beta}[\int_t^{t_3} D_2(x)e^{\alpha x^\beta}\,dx]\,dt \\
& - C_1\{\int_0^{t_1} D_1(x)e^{\alpha x^\beta}\,dx + \int_{t_2}^{t_3} D_2(x)e^{\alpha x^\beta}\,dx + e^{\alpha t_2^\beta}\int_{t_3}^{T} D_2(x)\,dx \\
& - \int_0^{t_1} D_1(x)\,dx - \int_{t_2}^{t_3} D_2(x)\,dx - \int_{t_3}^{T} D_2(x)\,dx\} \\
& - C_4\{e^{-\alpha t_1^\beta}\int_{t_2}^{t_3} D_2(x)e^{\alpha x^\beta}\,dx + e^{-\alpha t_1^\beta}e^{\alpha t_2^\beta}\int_{t_3}^{T} D_2(x)\,dx\} \\
& - C_2\int_{t_3}^{T} D_2(x)(T-x)\,dx].
\end{aligned} \tag{3.33}$$

The concavity of the profit function and the optimal value of t_3 can be simply obtained from the partial backlogging case by substituting $\delta = 0$ in appropriate equations. Then, the optimal total profit can be calculated by substituting optimal t_3^* in Equation (3.33) and the optimal order quantity Q^* can be obtained by substituting $\delta = 0$ in Equation (3.32) as follows:

$$Q^* = \int_0^{t_1} D_1(x)e^{\alpha x^\beta}\,dx + \int_{t_2}^{t_3^*} D_2(x)e^{\alpha x^\beta}\,dx + e^{\alpha t_2^\beta}\int_{t_3^*}^{T} D_2(x)\,dx. \tag{3.34}$$

3.2 One–Market Model with Discounting

The deterioration rate of deteriorating items usually increases with time. In the marketplace, in order to reduce the loss due to deterioration and to increase the market demand rate, one of the strategies is discounting on the selling price. Hence, our aim is to develop a one–market model, where order quantity is received instantaneously at the high–end market and backlogged demand is satisfied, if there is any; and then the items are sold at a higher price for a given period, and after that the price is discounted at the high–end market without transporting the items to the low–end market.

The inventory profile for this model is given in Figure 3.3, where Q is the total amount of order quantity at the beginning of each cycle. After fulfilling the backlogged demand, S is the maximum inventory level at time $t = 0$. As time progresses, the inventory level decreases due to demand and deterioration. At fixed time t_1, a discount on unit selling price is offered to increase the demand. $D_1(t)$ is the demand before discounting, $D_d(t)$ is the demand after discounting, and usually $D_d(t) > D_1(t)$. The inventory level decreases due to deterioration and demand; and ultimately it reaches zero at time t_3. Shortages occur from time t_3 to T and this could be either partial backlogging or complete backlogging. It is assumed that the initial price (higher price) from time 0 to t_1 in this one–market model is the same as the price from time 0 to t_1 in the two–market model, but the discounted price from time t_1 to t_3 is determined such that the total profit obtained from this one–market model is the same as that of the two–market model. In addition, similar to the two–market model, optimal time at which the inventory becomes zero, t_3^*, the optimal order quantity, Q^*, and the optimal backlogged demand, B^*, are also optimally determined.

3.2.1 Partial Backlogging

In this case, during the shortage period, some excess demands are backlogged and the rest of them are lost.

Let $I_1(t)$, $I_2(t)$, and $I_3(t)$ be the on–hand inventory level at any time $0 \leq t \leq t_1$, $t_1 \leq t \leq t_3$, and $t_3 \leq t \leq T$, respectively. The instantaneous state of inventory levels in the interval $[0, T]$ are governed by the following differential equations:

$$\frac{dI_1(t)}{dt} = -D_1(t) - \theta(t)I_1(t),\ 0 \leq t \leq t_1. \tag{3.35}$$

$$\frac{dI_2(t)}{dt} = D_d(t) - \theta(t)I_2(t),\ t_1 \leq t \leq t_3. \tag{3.36}$$

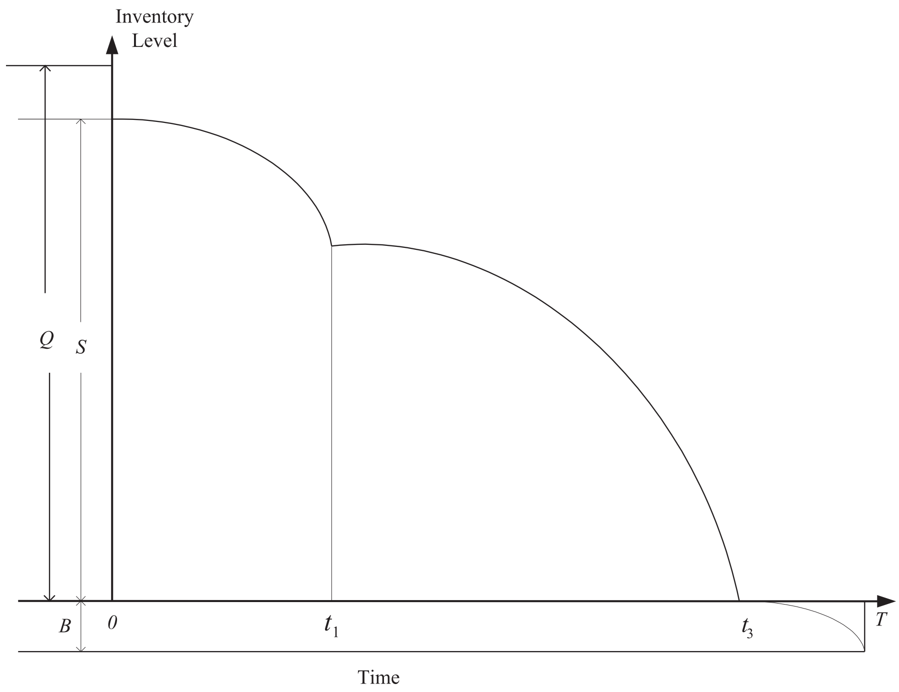

Figure 3.3: Inventory profile for the one–market model without discounting with

$$\frac{dI_3(t)}{dt} = -\frac{D_d(t)}{1+\delta(T-t)},\ t_3 \le t \le T. \tag{3.37}$$

The solutions of the above differential equations (3.35), (3.36), (3.37) are obtained by applying the boundary conditions $I_1(0) = S,\ I_2(t_3) = 0$, and $I_3(t_3) = 0$, respectively.

Hence, the inventory level during the interval $[0, t_1]$ is

$$I_1(t) = \frac{S - \int_0^t D_1(x)e^{\alpha x^\beta}\, dx}{e^{\alpha t^\beta}}. \tag{3.38}$$

The inventory level during $[t_1, t_3]$ is

$$I_2(t) = \frac{S - \int_0^{t_1} D_1(x)e^{\alpha x^\beta}\, dx - \int_{t_1}^t D_d(x)e^{\alpha x^\beta}}{e^{\alpha t^\beta}}. \tag{3.39}$$

And the inventory level during $[t_3, T]$ is

$$I_3(t) = -\int_{t_3}^t \frac{D_d(x)}{1+\delta(T-x)}\, dx. \tag{3.40}$$

The total order quantity including backlogged quantity can be expressed as

$$Q = \int_0^{t_1} D_1(x)e^{\alpha x^\beta}\, dx + \int_{t_1}^{t_3} D_d(x)e^{\alpha x^\beta}\, dx + \int_{t_3}^T \frac{D_d(x)}{1+\delta(T-x)}\, dx. \tag{3.41}$$

To calculate the cost function including purchase cost, holding cost, backorder cost, opportunity cost, and deterioration cost, we follow the same process applied in the previous two–market model with the partial backlogging case.

Therefore, the total cost per unit time is

$$\begin{aligned}
\Gamma_c(t_3) = {} & \frac{1}{T}[C\{\int_0^{t_1} D_1(x)e^{\alpha x^\beta}\,dx + \int_{t_1}^{t_3} D_d(x)e^{\alpha x^\beta}\,dx + \int_{t_3}^{T} \frac{D_d(x)}{1+\delta(T-x)}\,dx\} \\
& + h\int_0^{t_1} e^{-\alpha t^\beta}[\int_t^{t_1} D_1(x)e^{\alpha x^\beta}\,dx]\,dt + h\int_0^{t_1} e^{-\alpha t^\beta}[\int_{t_1}^{t_3} D_d(x)e^{\alpha x^\beta}\,dx]\,dt \\
& + h\int_{t_1}^{t_3} e^{-\alpha t^\beta}[\int_t^{t_3} D_d(x)e^{\alpha x^\beta}\,dx]\,dt + (C_2+C_3\delta)\int_{t_3}^{T} \frac{D_d(x)(T-x)}{1+\delta(T-x)}\,dx \\
& + C_1\{\int_0^{t_1} D_1(x)e^{\alpha x^\beta}\,dx + \int_{t_1}^{t_3} D_d(x)e^{\alpha x^{\beta_2}}\,dx - \int_0^{t_1} D_1(x)\,dx - \int_{t_1}^{t_3} D_d(x)\,dx\}].
\end{aligned} \tag{3.42}$$

P_1 is the same selling price as in the two–market model during $[0, t_1]$ and P_d is the discounted selling price. Therefore, the total sales revenue per unit time can be expressed as

$$SR = \frac{1}{T}P_1\int_0^{t_1} D_1(x)\,dx + P_d\int_{t_1}^{t_3} D_d(x)\,dx + P_d\int_{t_3}^{T} \frac{D_d(x)}{1+\delta(T-x)}\,dx. \tag{3.43}$$

Then, the total profit per unit time is

$$\begin{aligned}
\pi(t_3) = {} & \frac{1}{T}[P_1\int_0^{t_1} D_1(x)\,dx + P_d\int_{t_1}^{t_3} D_d(x)\,dx + P_d\int_{t_3}^{T} \frac{D_d(x)}{1+\delta(T-x)}\,dx \\
& - C\{\int_0^{t_1} D_1(x)e^{\alpha x^\beta}\,dx + \int_{t_1}^{t_3} D_d(x)e^{\alpha x^\beta}\,dx + \int_{t_3}^{T} \frac{D_d(x)}{1+\delta(T-x)}\,dx\} \\
& - h\int_0^{t_1} e^{-\alpha t^\beta}[\int_t^{t_1} D_1(x)e^{\alpha x^\beta}\,dx]\,dt - h\int_0^{t_1} e^{-\alpha t^\beta}[\int_{t_1}^{t_3} D_d(x)e^{\alpha x^\beta}\,dx]\,dt \\
& - h\int_{t_1}^{t_3} e^{-\alpha t^\beta}[\int_t^{t_3} D_d(x)e^{\alpha x^\beta}\,dx]\,dt - (C_2+C_3\delta)\int_{t_3}^{T} \frac{D_d(x)(T-x)}{1+\delta(T-x)}\,dx \\
& - C_1\{\int_0^{t_1} D_1(x)e^{\alpha x^\beta}\,dx + \int_{t_1}^{t_3} D_d(x)e^{\alpha x^{\beta_2}}\,dx - \int_0^{t_1} D_1(x)\,dx - \int_{t_1}^{t_3} D_d(x)\,dx\}].
\end{aligned} \tag{3.44}$$

Now, the objective is to determine the optimal time t_3^* and the discounted selling price P_d that can result in the same profit as in the two–market model. In order to determine optimal time t_3^*, $\pi(t_3)$ is differentiated with respect to t_3 and set to zero as follows:

$$\begin{aligned} f(t_3) = P_d - \frac{P_d}{1+\delta(T-t_3)} & - Ce^{\alpha t_3^\beta} + \frac{C}{1+\delta(T-t_3)} \\ - he^{\alpha t_3^\beta}\int_0^{t_3} e^{-\alpha t^\beta}\,dt + \frac{(C_2+C_3\delta)(T-t_3)}{1+\delta(T-t_3)} & - C_1 e^{\alpha t_3^\beta} + C_1] = 0. \end{aligned} \tag{3.45}$$

In Equation (3.45) P_d and t_3 are two unknown variables. Therefore, for a given P_d ($C < P_d < P_1$), Equation (3.45) is solved to determine optimal t_3^*. In addition, the sufficient condition is that the total profit function needs to be concave and must satisfy $\frac{d^2\pi(t_3)}{dt_3^2}|_{t=t_3^*} < 0,\ \ \forall\, t_3^* > 0$. Consequently, the derivative of Equation (3.45) with respect to t_3 can be expressed as:

$$\begin{aligned} \frac{d^2\pi(t_3)}{dt_3^2}|_{t=t_3^*} = \frac{D_d(t_3^*)}{T}[\frac{-P_d\delta}{(1+\delta(T-t_3^*))^2} - C\alpha\beta t_3^{*\beta-1} e^{\alpha t_3^{*\beta}} + \frac{C\delta}{(1+\delta(T-t_3^*))^2} \\ - h\alpha\beta t_3^{*\beta-1} e^{\alpha t_3^{*\beta}}\int_0^{t_3^*} e^{-\alpha t^\beta}\,dt - h - \frac{(C_2+C_3\delta)}{(1+\delta(T-t_3^*))^2} - C_1\{\alpha\beta t_3^{*\beta-1} e^{\alpha t_3^{*\beta}}\}] < 0. \end{aligned} \tag{3.46}$$

It is clear from Equation (3.46), that $\frac{d^2\pi(t_3)}{dt_3^2}|_{t=t_3^*}$ always less than zero for the given P_d ($C < P_d < P_1$). Then, by solving the Equations (3.45) and (3.44) simultaneously, one can determine the discounted selling price P_d, which can result in the same profit as the two–market model.

3.2.2 Complete Backlogging

In this case, all shortages are backlogged and they are satisfied by the next replenishment. Substituting $\delta = 0$ in Equations (3.44) and (3.41), we get the total profit per unit time and total order quantity for the complete backlogging case.

The total profit per unit time is

$$\pi(t_3) = \frac{1}{T}[P_1 \int_0^{t_1} D_1(x)\,dx + P_d \int_{t_1}^{t_3} D_d(x)\,dx + P_d \int_{t_3}^{T} D_d(x)\,dx$$
$$-C\{\int_0^{t_1} D_1(x)e^{\alpha x^\beta}\,dx + \int_{t_1}^{t_3} D_d(x)e^{\alpha x^\beta}\,dx + \int_{t_3}^{T} D_d(x)\,dx\}$$
$$-h\int_0^{t_1} e^{-\alpha t^\beta}[\int_t^{t_1} D_1(x)e^{\alpha x^\beta}\,dx]\,dt - h\int_0^{t_1} e^{-\alpha t^\beta}[\int_{t_1}^{t_3} D_d(x)e^{\alpha x^\beta}\,dx]\,dt$$
$$-h\int_{t_1}^{t_3} e^{-\alpha t^\beta}[\int_t^{t_3} D_d(x)e^{\alpha x^\beta}\,dx]\,dt - C_2\int_{t_3}^{T} D_d(x)(T-x)\,dx$$
$$-C_1\{\int_0^{t_1} D_1(x)e^{\alpha x^\beta}\,dx + \int_{t_1}^{t_3} D_d(x)e^{\alpha x^{\beta_2}}\,dx - \int_0^{t_1} D_1(x)\,dx - \int_{t_1}^{t_3} D_d(x)\,dx\}], \quad (3.47)$$

and the total order quantity is

$$Q = \int_0^{t_1} D_1(x)e^{\alpha x^\beta}\,dx + \int_{t_1}^{t_3} D_d(x)e^{\alpha x^\beta}\,dx + \int_{t_3}^{T} D_d(x)\,dx. \quad (3.48)$$

The optimal value of t_3^* can be obtained from the partial backlogging case by substituting $\delta = 0$ in Equation (3.45) and the concavity of the profit function can also be shown by substituting $\delta = 0$ in appropriate equations. After substituting t_3^* in Equation (3.47), one can determine the discounted selling price P_d that can result in the same total profit obtained in the two–market model.

3.3 One–Market Model

In this section, the aim is to develop a one–market model, where order quantity is received instantaneously at the high–end market and backlogged demand is satisfied, if there is any, and then the items are sold at a fixed price, P_0, at the high–end market without transporting the items to the low–end market.

The inventory profile for this system is depicted in Figure 3.4. Replenishment is made and backlogged demand is satisfied at time $t = 0$, and then the maximum inventory level is S. The inventory level decreases due to customer demand and deterioration and it becomes zero at time t_3. Let $D_o(t)$ be the demand rate at any time for the one–market model. Shortages occur from t_3 to T and this could be partially or completely backlogged.

The fixed price per unit, P_o, is determined such that the total profit obtained from this one–market model is the same as that of the two–market model. Moreover, similar to the two–market model, the optimal time at which the inventory becomes zero, t_3^*, the optimal order quantity, Q^*, and the optimal backlogged demand, B^*, are optimally determined.

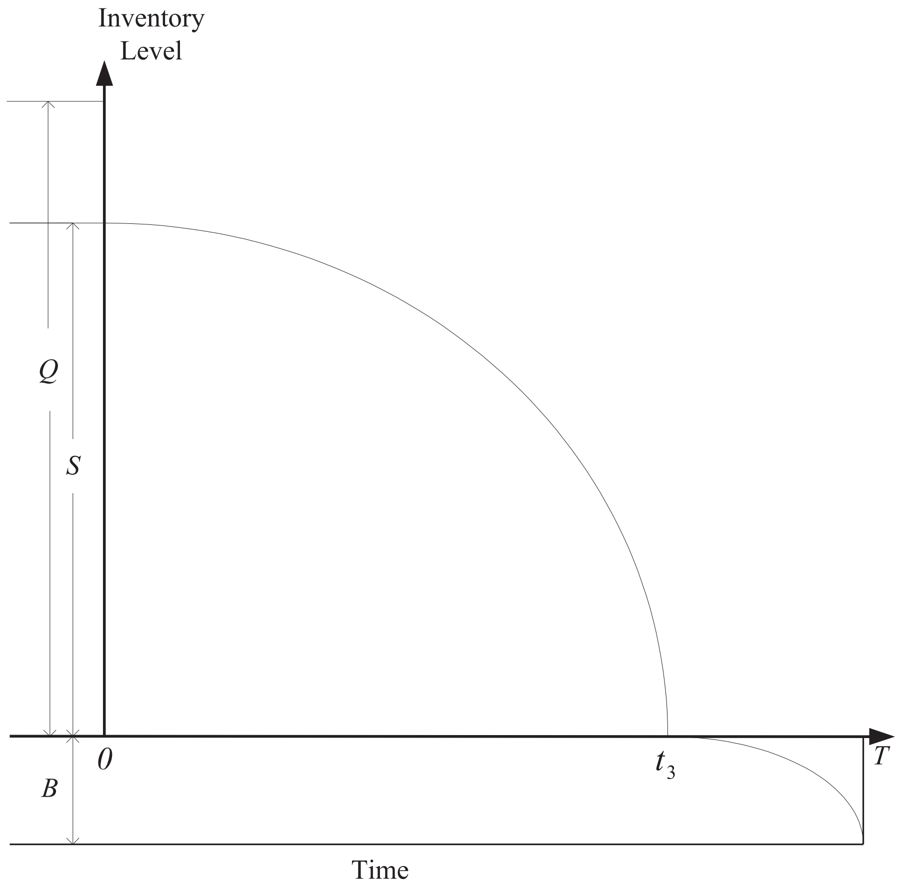

Figure 3.4: Inventory profile for the one–market model

3.3.1 Partial Backlogging

In this case, during the shortage period, some excess demands are backlogged and the rest of them are lost.

Let $I_1(t)$ be the inventory level at any time $t(0 \leq t \leq t_3)$ and $I_2(t)$ be the inventory level during shortage period (t_3, T). The behaviour of the inventory level is governed by the following differential equations:

$$\frac{dI_1(t)}{dt} = -D_o(t) - \theta(t)I_1(t), \ 0 \leq t \leq t_3. \tag{3.49}$$

$$\frac{dI_2(t)}{dt} = -\frac{D_o(t)}{1+\delta(T-t)}, \ t_3 \leq t \leq T. \tag{3.50}$$

Applying the boundary condition $I_1(0) = S$ in Equation (3.49) and $I_2(t_3) = 0$ in Equation (3.50) we get

$$I_1(t) = \frac{S - \int_0^t D_o(x)e^{\alpha x^\beta}\, dx}{e^{\alpha t^\beta}}. \tag{3.51}$$

$$I_2(t) = -\int_{t_3}^{t} \frac{D_o(x)}{1+\delta(T-t)}\, dx. \tag{3.52}$$

The total order quantity is

$$Q = \int_0^{t_3} D_o(x)e^{\alpha x^\beta}\, dx + \int_{t_3}^{T} \frac{D_o(x)}{1+\delta(T-x)}\, dx. \tag{3.53}$$

The total cost per unit time can be expressed as:

$$\begin{aligned}\Gamma_c(t_3) = &\frac{1}{T}[C\int_0^{t_3} D_o(x)e^{\alpha x^\beta}\, dx + C\int_{t_3}^{T} \frac{D_o(x)}{1+\delta(T-x)}\, dx \\ &+ h\int_0^{t_3} e^{-\alpha t^\beta}[\int_t^{t_3} D_o(x)e^{\alpha x^\beta}\, dx]\, dt + C_1\int_0^{t_3} D_o(x)e^{\alpha x^\beta}\, dx \\ &- C_1\int_0^{t_3} D_o(x)\, dx + (C_2 + C_3\delta)\int_{t_3}^{T} \frac{D_0(x)(T-x)}{1+\delta(T-x)}\, dx.\end{aligned} \tag{3.54}$$

Since P_o is the selling price of this one–market model, the total sales revenue per unit time is:

$$SR = \frac{1}{T}[P_o\int_0^{t_3} D_o(x)\, dx + P_o\int_{t_3}^{T} \frac{D_o(x)}{1+\delta(T-x)}\, dx]. \tag{3.55}$$

The total profit per unit time is as follows:

$$\begin{aligned}\pi(t_3) = &\frac{1}{T}[P_o\int_0^{t_3} D_o(x)\, dx + P_o\int_{t_3}^{T} \frac{D_o(x)}{1+\delta(T-x)}\, dx - C\int_0^{t_3} D_o(x)e^{\alpha x^\beta}\, dx \\ &- C\int_{t_3}^{T} \frac{D_o(x)}{1+\delta(T-x)}\, dx \ - h\int_0^{t_3} e^{-\alpha t^\beta}[\int_t^{t_3} D_o(x)e^{\alpha x^\beta}\, dx]\, dt \\ &- C_1\int_0^{t_3} D_o(x)\{e^{\alpha x^\beta}\, dx - 1\}\, dx - (C_2 + C_3\delta)\int_{t_3}^{T} \frac{D_o(x)(T-x)}{1+\delta(T-x)}\, dx].\end{aligned} \tag{3.56}$$

Now, the objective is to determine the optimal value of t_3^* and the selling price P_o. At first, in order to determine optimal time t_3^*, taking the first derivative of $\pi(t_3)$ with respect to t_3 and setting it to zero will give the following:

$$P_o - \frac{P_o}{1+\delta(T-t_3)} - Ce^{\alpha t_3^\beta} + \frac{C}{1+\delta(T-t_3)}$$
$$- he^{\alpha t_3^\beta}\int_0^{t_3} e^{-\alpha t^\beta}\,dt + \frac{(C_2+C_3\delta)(T-t_3)}{1+\delta(T-t_3)} - C_1 e^{\alpha t_3^\beta} + C_1] = 0. \tag{3.57}$$

In Equation (3.57) P_o and t_3 are two unknown variables. Hence, for a given P_o ($P_o > C$), Equation (3.57) is solved to determine the optimal value of t_3^*. In addition, for a given P_o, the total profit function needs to be concave and must satisfy $\frac{d^2\pi(t_3)}{dt_3^2}|_{t=t_3^*} < 0$. Consequently, taking the derivative of Equation (3.57) with respect to t_3^* yields the following:

$$\frac{d^2\pi(t_3)}{dt_3^2}|_{t=t_3^*} = \frac{-P_o\delta}{(1+\delta(T-t_3^*))^2} - C\alpha\beta t_3^{*\beta-1}e^{\alpha t_3^{*\beta}} + \frac{C\delta}{(1+\delta(T-t_3^*))^2}$$
$$- h\alpha\beta t_3^{*\beta-1}e^{\alpha t_3^{*\beta}}\int_0^{t_3^*} e^{-\alpha t^\beta}\,dt - h - \frac{(C_2+C_3\delta)}{(1+\delta(T-t_3^*))^2} - C_1\alpha\beta t_3^{*\beta-1}e^{\alpha t_3^{*\beta}} < 0. \tag{3.58}$$

Equation (3.58) shows, for a given P_o ($P_o > C$), $\frac{d^2\pi(t_3)}{dt_3^2}|_{t=t_3^*} < 0$. Then, solving Equations (3.56) and (3.57) simultaneously, one can determine a selling price P_o for which the total profit of this one–market model will be the same as the total profit of the two–market model.

3.3.2 Complete Backlogging

In this case, all shortages are backlogged and they are replaced by the next replenishment. Substituting $\delta = 0$ in Equation (3.56), we get the total profit per unit time for complete backlogging case as:

$$\pi(t_3) = \frac{1}{T}[P_o\int_0^{t_3} D_o(x)\,dx + P_o\int_{t_3}^{T} D_o(x)\,dx - C\int_0^{t_3} D_o(x)e^{\alpha x^\beta}\,dx$$
$$- C\int_{t_3}^{T} D_o(x)\,dx - h\int_0^{t_3} e^{-\alpha t^\beta}[\int_t^{t_3} D_o(x)e^{\alpha x^\beta}\,dx]\,dt$$
$$- C_2\int_{t_3}^{T} D_o(x)(T-x)\,dx - C_1\{\int_0^{t_3} D_o(x)e^{\alpha x^\beta}\,dx + \int_0^{t_3} D(x)\,dx\}]. \tag{3.59}$$

The optimal value of t_3^* and the concavity of the profit function can be shown by substituting $\delta = 0$ in the corresponding equations of the partial backlogging case. Then, using this t_3^*, one can determine a selling price P_o from Equation (3.59) by applying the iterative method, that can result in the same profit obtained in the two–market model.

Chapter 4

Models for Price–Dependent Demand

In reality, the demand depends on the product's price. In the literature, several types of price dependent demand pattern were studied by several researchers. For example, Dye et al. (2007) assumed demand rate as $D(P) = a - bP$ and $D(P) = aP^{-b}$; Abad (2001) described the demand by the constant price elasticity function, $D(P) = ae^{-bP}$.

The following assumptions are made for demand function:

1. $D(P) > 0$ for $P > 0$.

2. $D(P)$ is a continusous, convex, decreasing function of the selling price.

All inventory equations and cost functions for two–market model, one–market model with discounting, and one–market model are calculated as same as the time–dependent demand model in previous chapter.

4.1 Two–Market Model

The inventory level follows the pattern which is shown in Figure 3.2. The high–end market demand is denoted by $D_1(P_1)$ and the low–end market demand is denoted by $D_2(P_2)$, where P_1 and P_2 are the high–end and low–end market selling prices respectively. It is assumed that $D_2(P_2) > D_1(P_1)$. In order to maximize the total profit per unit time, we need to determine the optimal selling price of the high–end market P_1^*, the optimal selling price of the low–end market P_2^*, and the optimal time at which the inventory level becomes zero, t_3^*. As a result, the optimal order quantity Q^* and the optimal backlogged quantity B^* are also determined.

4.1.1 Partial Backlogging

Using the same argument as deriving (3.26), we can obtain the total profit $\pi(t_3, P_1, P_2)$ per unit time for the partial backlogging case when the demand depends on the selling price.

$$
\begin{aligned}
\pi(t_3,P_1,P_2) = &\frac{1}{T}[P_1D_1(P_1)\int_0^{t_1} dt + P_2D_2(P_2)\int_{t_2}^{t_3} dt + P_2\int_{t_3}^{T}\frac{D_2(P_2)}{1+\delta(T-x)}\,dx \\
&- C\{D_1(P_1)\int_0^{t_1} e^{\alpha x^\beta}\,dx + D_2(P_2)\int_{t_2}^{t_3} e^{\alpha x^\beta}\,dx + D_2(P_2)e^{\alpha t_2^\beta}\int_{t_3}^{T}\frac{1}{1+\delta(T-x)}\,dx\} \\
&- hD_1(P_1)\int_0^{t_1} e^{-\alpha t^\beta}[\int_t^{t_1} e^{\alpha x^\beta}\,dx]\,dt - hD_2(P_2)\int_0^{t_2} e^{-\alpha_1 t^{\beta_1}}[\int_{t_2}^{t_3} e^{\alpha x^\beta}\,dx]\,dt \\
&- he^{\alpha t_2^\beta}\int_0^{t_2} e^{-\alpha t^\beta}[\int_{t_3}^{T}\frac{D_2(P_2)}{1+\delta(T-x)}\,dx]\,dt - hD_2(P_2)\int_{t_2}^{t_3} e^{-\alpha t^\beta}[\int_t^{t_3} e^{\alpha x^\beta}\,dx]\,dt \\
&- C_1\{D_1(P_1)\int_0^{t_1} e^{\alpha x^\beta}\,dx + D_2(P_2)\int_{t_2}^{t_3} e^{\alpha x^\beta}\,dx + e^{\alpha t_2^\beta}D_2(P_2)\int_{t_3}^{T}\frac{1}{1+\delta(T-x)}\,dx \\
&- D_1(P_1)t_1 - D_2(P_2)(t_3-t_2) - D_2(P_2)\int_{t_3}^{T}\frac{1}{1+\delta(T-x)}\,dx\} \\
&- C_4D_2(P_2)e^{-\alpha t_1^\beta}\int_{t_2}^{t_3} e^{\alpha x^\beta}\,dx - C_4e^{-\alpha t_1^\beta}e^{\alpha t_2^\beta}D_2(P_2)\int_{t_3}^{T}\frac{1}{1+\delta(T-x)}\,dx \\
&- (C_2+C_3\delta)D_2(P_2)\int_{t_3}^{T}\frac{(T-x)}{1+\delta(T-x)}\,dx]. \qquad (4.1)
\end{aligned}
$$

In this function, t_3, P_1, and P_2 are unknown variables. The objective is to determine the optimal t_3^*, P_1^*, and P_2^* which maximize the total profit $\pi(t_3, P_1, P_2)$. Apparently, we can separate P_1 from P_2 and t_3 in the profit function. It is easy to see that the profit function is a concave function of P_1. The necessary conditions for maximizing $\pi(t_3, P_1, P_2)$ are

$\frac{d\pi(t_3,P_1,P_2)}{dt_3}|_{t_3^*} = 0$, $\frac{d\pi(t_3,P_1,P_2)}{dP_1}|_{P_1^*} = 0$, and $\frac{d\pi(t_3,P_1,P_2)}{dP_2}|_{P_2^*} = 0$ respectively.

At first, the first derivative of $\pi(t_3, P_1, P_2)$ with respect to P_1 is

$$\frac{d\pi(t_3, P_1, P_2)}{dP_1} = \frac{1}{T}[D_1(P_1)t_1 + P_1 D_1'(P_1)t_1 - CD_1'(P_1)\int_0^{t_1} e^{\alpha x^\beta}\, dx$$
$$- hD_1'(P_1)\int_0^{t_1} e^{-\alpha t^\beta}[\int_t^{t_1} e^{\alpha x^\beta}\, dx]\, dt - C_1 D_1'(P_1)\int_0^{t_1} e^{\alpha x^\beta}\, dx + C_1 D_1'(P_1)t_1]. \tag{4.2}$$

Define $f(P_1) = \frac{d\pi(t_3,P_1,P_2)}{dP_1}$. After rearranging the right-hand side of the above equation, we have

$$f(P_1) =\{D_1(P_1)t_1 - CD_1'(P_1)\int_0^{t_1} e^{\alpha x^\beta}\, dx - hD_1'(P_1)\int_0^{t_1} e^{-\alpha t^\beta}[\int_t^{t_1} e^{\alpha x^\beta}\, dx]\, dt$$
$$- C_1 D_1'(P_1)\int_0^{t_1} e^{\alpha x^\beta}\, dx\} + \{C_1 D_1'(P_1)t_1 + P_1 D_1'(P_1)t_1\} = 0. \tag{4.3}$$

On the right hand side of Equation (4.3), $D_1(P_1) > 0$ and $D_1'(P_1) < 0$. Therefore, the components in the first braces are always positive and the components in the second braces are always negative. By the iteration method, one can find out the value of P_1^* from Equation (4.3), which satisfies $f(P_1^*) = 0$.

In addition, the sufficient condition is that the total profit function needs to be concave and must satisfy $\frac{d^2\pi(P_1,t_3,P_2)}{dP_1^2}|_{P_1=P_1^*} < 0$. So taking the second derivative of $\pi(P_1, t_3, P_2)$ with respect to P_1 and substituting $P_1 = P_1^*$.

$$\frac{d^2\pi(P_1, t_3, P_2)}{dP_1^2}|_{P_1=P_1^*}$$
$$\frac{1}{T}[\{D_1'(P_1^*)t_1 + D_1'(P_1^*)t_1 - CD_1''(P_1^*)\int_0^{t_1} e^{\alpha x^\beta}\, dx - C_1 D_1''(P_1^*)\{\int_0^{t_1} e^{\alpha x^\beta}\, dx$$
$$- h\{D_1''(P_1^*)\int_0^{t_1} e^{-\alpha t^\beta}[\int_t^{t_1} e^{\alpha x^\beta}\, dx]\, dt\} + \{P_1^* D_1''(P_1^*)t_1 + C_1 D_1''(P_1^*)t_1\} < 0. \tag{4.4}$$

In Equation (4.4), we know $D_1'(P_1) < 0$ and $D_1''(P_1) > 0$. As a result, the value in the first braces is always negative and the value in the second braces is positive. However, the first value is significantly greater than the second one. Consequently, $\frac{d^2\pi(P_1,t_3,P_2)}{dP_1^2}|_{(P_1=P_1^*)} < 0$ and the total profit is strictly concave at P_1^*.

Now, our objective is to determine the optimal time t_3^*. The optimal time t_3^* occurs when $\frac{d\pi(t_3,P_1,P_2)}{dt_3}$=0 is satisfied. From the two–market model of partial backlogging case with time dependent demand in previous chapter, we know the first derivative of $\pi(t_3, P_1, P_2)$ with respect to time t_3, i.e. $\frac{d\pi(P_1,t_3,P_2)}{dt_3}$ can be expressed as follows:

$$
\begin{aligned}
f(t_3) &= P_2 - \frac{P_2}{1+\delta(T-t_3)} - Ce^{\alpha t_3^\beta} + \frac{Ce^{\alpha t_2^\beta}}{1+\delta(T-t_3)} \\
&- he^{\alpha t_3^\beta}\int_0^{t_3} e^{-\alpha t^\beta}\,dt + \frac{he^{\alpha t_2^\beta}}{\beta+1}\{\frac{t_2(\beta+1)-\alpha t_2^{\beta+1}}{1+\delta(T-t_3)}\} - C_1\{e^{\alpha t_3^\beta}-1\} \\
&+ C_1\{\frac{e^{\alpha t_2^\beta}}{1+\delta(T-t_3)} - \frac{1}{1+\delta(T-t_3)}\} - C_4e^{-\alpha t_1^\beta}e^{\alpha t_3^\beta} \\
&+ \frac{C_4e^{-\alpha t_1^\beta}e^{\alpha t_2^\beta}}{1+\delta(T-t_3)} + \frac{(C_2+C_3\delta)(T-t_3)}{1+\delta(T-t_3)} = 0. \qquad (4.5)
\end{aligned}
$$

Using the same argument as we did in previous chapter for the two–market model of partial backlogging case with time dependent demand, we know that $f(t_3) = 0$ has a unique optimal solution such that $t_3^* \in (t_2, T)$ and $\frac{d^2\pi(t_3^*|P_1,P_2)}{dt_3^{*2}} < 0$, when the low–end market selling price is given.

Once the optimal value of t_3^* is obtained, the profit $\pi(t_3^*|P_1^*, P_2)$ can be found by substituting optimal t_3^*, P_1^* and the given P_2 in Equation (4.1).

Next, we study the condition under which the optimal selling price P_2 also exists uniquely. So the first derivative of $\pi(t_3, P_1, P_2)$ with respect to P_2 is

$$
\begin{aligned}
&\frac{d\pi(t_3, P_1, P_2)}{dP_2} \\
&= \frac{1}{T}[P_2D_2'(P_2)(t_3-t_2) + D_2(P_2)(t_3-t_2) \\
&+ P_2D_2'(P_2)\int_{t_3}^{T}\frac{1}{1+\delta(T-x)}\,dx + D_2(P_2)\int_{t_3}^{T}\frac{1}{1+\delta(T-x)}\,dx \\
&- CD_2'(P_2)\int_{t_2}^{t_3} e^{\alpha x^\beta}\,dx - D_2'(P_2)\int_{t_3}^{T}\frac{Ce^{\alpha t_2^\beta}}{1+\delta(T-x)}\,dx \\
&- hD_2'(P_2)\int_0^{t_2} e^{-\alpha t^\beta}[\int_{t_2}^{t_3} e^{\alpha x^\beta}\,dx]\,dt - hD_2'(P_2)\int_{t_2}^{t_3} e^{-\alpha t^\beta}[\int_{t}^{t_3} e^{\alpha x^\beta}\,dx]\,dt
\end{aligned}
$$

$$
\begin{aligned}
&- he^{\alpha t_2^\beta} D_2'(P_2) \int_0^{t_2} e^{-\alpha t^\beta} [\int_{t_3}^{T} \frac{1}{1+\delta(T-x)}\, dx]\, dt - C_1\{D_2'(P_2) \int_{t_2}^{t_3} e^{\alpha x^\beta}\, dx \\
&+ D_2'(P_2) \int_{t_3}^{T} \frac{e^{\alpha t_2^\beta}}{1+\delta(T-x)}\, dx - D_2'(P_2)(t_3 - t_2) - \int_{t_3}^{T} \frac{D_2'(P_2)}{1+\delta(T-x)}\, dx\} \\
&- C_4 e^{-\alpha t_1^\beta} D_2'(P_2) \int_{t_2}^{t_3} e^{\alpha x^\beta}\, dx - C_4 e^{-\alpha t_1^\beta} e^{\alpha t_2^\beta} D_2'(P_2) \int_{t_3}^{T} \frac{1}{1+\delta(T-x)}\, dx \\
&- (C_2 + C_3\delta) \int_{t_3}^{T} \frac{(T-x)}{1+\delta(T-x)}\, dx]. \qquad (4.6)
\end{aligned}
$$

In order to determine the optimal value of P_2^*, substitute $t_3 = t_3^*$ in the above equation. In other words, $\frac{d\pi(t_3,P_1,P_2)}{dP_2} = \frac{d\pi(P_2|t_3^*,P_1)}{dP_2}$. Let $\frac{d\pi(P_2|t_3^*,P_1)}{dP_2} = f(P_2|t_3^*, P_1) = 0$.

$$
\begin{aligned}
&f(P_2|t_3^*, P_1) \\
&= \frac{1}{T}[P_2 D_2'(P_2)(t_3^* - t_2) + D_2(P_2)(t_3^* - t_2) + \int_{t_3^*}^{T} \frac{P_2 D_2'(P_2)}{1+\delta(T-x)}\, dx \\
&+ \int_{t_3^*}^{T} \frac{D_2(P_2)}{1+\delta(T-x)}\, dx - C D_2'(P_2) \int_{t_2}^{t_3^*} e^{\alpha x^\beta}\, dx - \int_{t_3^*}^{T} \frac{D_2'(P_2) C e^{\alpha t_2^\beta}}{1+\delta(T-x)}\, dx \\
&- h D_2'(P_2) \int_0^{t_2} e^{-\alpha t^\beta} [\int_{t_2}^{t_3^*} e^{\alpha x^\beta}\, dx]\, dt - h e^{\alpha t_2^\beta} \int_0^{t_2} e^{-\alpha t^\beta} [\int_{t_3^*}^{T} \frac{D_2'(P_2)}{1+\delta(T-x)}\, dx]\, dt \\
&- h D_2'(P_2) \int_{t_2}^{t_3^*} e^{-\alpha t^\beta} [\int_{t}^{t_3^*} e^{\alpha x^\beta}\, dx]\, dt - C_1\{D_2'(P_2) \int_{t_2}^{t_3^*} e^{\alpha x^\beta}\, dx + \int_{t_3^*}^{T} \frac{D_2'(P_2) e^{\alpha t_2^\beta} dx}{1+\delta(T-x)} \\
&- D_2'(P_2)(t_3^* - t_2) - D_2'(P_2) \int_{t_3^*}^{T} \frac{1}{1+\delta(T-x)}\, dx\} - C_4 e^{-\alpha t_1^\beta} D_2'(P_2) \int_{t_2}^{t_3^*} e^{\alpha x^\beta}\, dx \\
&- C_4 e^{-\alpha t_1^\beta} e^{\alpha t_2^\beta} \int_{t_3^*}^{T} \frac{D_2'(P_2)}{1+\delta(T-x)}\, dx - \int_{t_3^*}^{T} \frac{(C_2 + C_3\delta) D_2'(P_2)(T-x)}{1+\delta(T-x)}\, dx] = 0. \qquad (4.7)
\end{aligned}
$$

According to Equation (4.7), $D_2(P_2) > 0$ and $D_2'(P_2) < 0$. Therefore, Equation (4.7) has some negative and positive terms. So using the iteration method one can find the value of P_2^* for the given optimal time t_3^*, which satisfies $f(P_2^*|t_3^*, P_1^*) = 0$.

Now the solutions to Equations (4.5) and (4.7) must satisfy the second order conditions for a local maximum. So the second derivative of the profit function with respect to P_2^* is

$$\frac{\partial^2\pi(P_2^*, t_3^*, P_1)}{\partial P_2^{*2}}$$
$$= \frac{1}{T}[D_2'(P_2^*)(t_3^* - t_2) + P_2^* D_2''(P_2^*)(t_3^* - t_2) + D_2'(P_2^*)(t_3^* - t_2)$$
$$+ D_2'(P_2^*)\int_{t_3^*}^{T} \frac{1}{1+\delta(T-x)}\, dx + P_2^* D_2''(P_2^*)\int_{t_3^*}^{T} \frac{1}{1+\delta(T-x)}\, dx$$
$$- C D_2''(P_2^*)\int_{t_2}^{t_3^*} e^{\alpha x^\beta}\, dx - D_2''(P_2^*)\int_{t_3^*}^{T} \frac{Ce^{\alpha t_2^\beta}}{1+\delta(T-x)}\, dx$$
$$- hD_2''(P_2^*)\int_{0}^{t_2} e^{-\alpha t^\beta}[\int_{t_2}^{t_3^*} e^{\alpha x^\beta}\, dx]\, dt - he^{\alpha t_2^\beta}\int_{0}^{t_2} e^{-\alpha t^\beta}[\int_{t_3^*}^{T} \frac{D_2''(P_2^*)}{1+\delta(T-x)}\, dx]\, dt$$
$$- hD_2''(P_2^*)\int_{t_2}^{t_3^*} e^{-\alpha t^\beta}[\int_{t}^{t_3^*} e^{\alpha x^\beta}\, dx]\, dt - C_1\{D_2''(P_2^*)\int_{t_2}^{t_3^*} e^{\alpha x^\beta}\, dx + \int_{t_3^*}^{T} \frac{D_2''(P_2^*)e^{\alpha t_2^\beta} dx}{1+\delta(T-x)}$$
$$- D_2''(P_2^*)(t_3^* - t_2) - D_2''(P_2^*)\int_{t_3^*}^{T} \frac{1}{1+\delta(T-x)}\, dx\} - C_4 e^{-\alpha t_1^\beta} D_2''(P_2^*)\int_{t_2}^{t_3^*} e^{\alpha x^\beta}\, dx$$
$$- C_4 e^{-\alpha t_1^\beta} e^{\alpha t_2^\beta}\int_{t_3^*}^{T} \frac{D_2''(P_2^*)}{1+\delta(T-x)}\, dx - \int_{t_3^*}^{T} \frac{(C_2 + C_3\delta)D_2''(P_2^*)(T-x)}{1+\delta(T-x)}\, dx] < 0. \tag{4.8}$$

In Equation (4.8), $D_2'(P_2)$ is negative and $D_2''(P_2)$ is also negative with negative sign. So in the above equation only $P_2 D_2''(P_2)(t_3^* - t_2)$ and $P_2 D_2''(P_2)\int_{t_3^*}^{T} \frac{1}{1+\delta(T-x)}\, dx$ are positive terms but these two values are significantly smaller than other terms.

The second derivative of total profit function respect to t_3^* is given by

$$\frac{\partial^2\pi(t_3^*, P_1^*, P_2^*)}{\partial t_3^{*2}}$$
$$= \frac{D_2(P_2^*)}{T}[\frac{-P_2^*\delta}{(1+\delta(T-t_3^*))^2} - C\alpha\beta t_3^{*\beta-1} e^{\alpha t_3^{*\beta}} + \frac{Ce^{\alpha t_2^\beta}\delta}{(1+\delta(T-t_3^*))^2}$$
$$- h\alpha\beta t_3^{*\beta-1} e^{\alpha t_3^{*\beta}}\int_0^{t_3^*} e^{-\alpha t^\beta}\, dt - h + \frac{he^{\alpha t_2^\beta}\delta}{\beta+1}\{\frac{t_2(\beta+1) - \alpha t_2^{\beta+1}}{1+\delta(T-t_3^*)^2}\}$$
$$- C_4\alpha\beta t_3^{*\beta-1} e^{-\alpha t_1^\beta} e^{\alpha t_3^{*\beta}} + \frac{C_4 e^{-\alpha t_1^\beta} e^{\alpha t_2^\beta}\delta}{1+\delta(T-t_3^*)^2} - C_1\alpha\beta t_3^{*\beta-1} e^{\alpha t_3^{*\beta}}$$
$$+ C_1\{\frac{e^{\alpha t_2^\beta}\delta}{1+\delta(T-t_3^*)^2} - \frac{\delta}{1+\delta(T-t_3^*)^2}\} - \frac{(C_2 + C_3\delta)}{(1+\delta(T-t_3^*))^2}] < 0. \tag{4.9}$$

Now, the derivative of profit function with respect to t_3^* and P_2^* is

$$\frac{\partial^2\pi(t_3^*, P_1^*, P_2^*)}{\partial t_3^*\partial P_2^*} = 0.$$

Now the determinant of the Hessain matrix at the stationary point (t_3^*, P_2^*) is

$$\det(\mathrm{H})= \begin{bmatrix} \frac{\partial^2\pi(t_3^*,P_1^*,P_2^*)}{\partial t_3^{*2}} & \frac{\partial^2\pi(t_3^*,P_1^*,P_2^*)}{\partial t_3^*\partial P_2^*} \\ \frac{\partial^2\pi(t_3^*,P_1^*,P_2^*)}{\partial P_2^*\partial t_3^*} & \frac{\partial^2\pi(t_3^*,P_1^*,P_2^*)}{\partial P_2^2 *} \end{bmatrix}.$$

From the Equation (4.8) and (4.9) we know $\frac{\partial^2\pi(P_2^*,t_3^*,P_1)}{\partial P_2^{*2}} < 0$ and $\frac{\partial^2\pi(t_3^*,P_1^*,P_2^*)}{\partial t_3^{*2}} < 0$. And $(\frac{\partial^2\pi(P_2^*,t_3^*,P_1)}{\partial P_2^{*2}} \times \frac{\partial^2\pi(t_3^*,P_1^*,P_2^*)}{\partial t_3^{*2}} - [\frac{\partial^2\pi(t_3^*,P_1^*,P_2^*)}{\partial P_2^*\partial t_3^*}]^2) > 0$. Thus, the Hessian is negative definite. This means that the profit function is strictly concave. In conclusion, we can say that the stationary points (t_3^*, P_2^*) is the global maximum solution to the problem of maximizing the total profit. Now substituting t_3^*, P_1^* and P_2^* in Equation (4.1), we can calculate the optimal total profit per unit time as follows:

$$\begin{aligned}
\pi(t_3^*, P_1^*, P_2^*) &= \frac{1}{T}[P_1^* D_1(P_1^*)\int_0^{t_1} dt + P_2^* D_2(P_2^*)\int_{t_2}^{t_3^*} dt + P_2^*\int_{t_3^*}^{T}\frac{D_2(P_2^*)}{1+\delta(T-x)}\,dx \\
&- C\{D_1(P_1^*)\int_0^{t_1} e^{\alpha x^\beta}\,dx + D_2(P_2^*)\int_{t_2}^{t_3^*} e^{\alpha x^\beta}\,dx + D_2(P_2^*)e^{\alpha t_2^\beta}\int_{t_3^*}^{T}\frac{1}{1+\delta(T-x)}\,dx\} \\
&- hD_1(P_1^*)\int_0^{t_1} e^{-\alpha t^\beta}[\int_t^{t_1} e^{\alpha x^\beta}\,dx]\,dt - hD_2(P_2^*)\int_0^{t_2} e^{-\alpha_1 t^{\beta_1}}[\int_{t_2}^{t_3^*} e^{\alpha x^\beta}\,dx]\,dt \\
&- he^{\alpha t_2^\beta}\int_0^{t_2} e^{-\alpha t^\beta}[\int_{t_3^*}^{T}\frac{D_2(P_2^*)}{1+\delta(T-x)}\,dx]\,dt - hD_2(P_2^*)\int_{t_2}^{t_3^*} e^{-\alpha t^\beta}[\int_t^{t_3^*} e^{\alpha x^\beta}\,dx]\,dt \\
&- C_1\{D_1(P_1^*)\int_0^{t_1} e^{\alpha x^\beta}\,dx + D_2(P_2^*)\int_{t_2}^{t_3^*} e^{\alpha x^\beta}\,dx + e^{\alpha t_2^\beta}D_2(P_2^*)\int_{t_3^*}^{T}\frac{1}{1+\delta(T-x)}\,dx \\
&- D_1(P_1^*)t_1 - D_2(P_2^*)(t_3^* - t_2) - D_2(P_2^*)\int_{t_3^*}^{T}\frac{1}{1+\delta(T-x)}\,dx\} \\
&- C_4 D_2(P_2^*)e^{-\alpha t_1^\beta}\int_{t_2}^{t_3^*} e^{\alpha x^\beta}\,dx - C_4 e^{-\alpha t_1^\beta}e^{\alpha t_2^\beta}D_2(P_2^*)\int_{t_3^*}^{T}\frac{1}{1+\delta(T-x)}\,dx \\
&- (C_2 + C_3\delta)D_2(P_2^*)\int_{t_3^*}^{T}\frac{(T-x)}{1+\delta(T-x)}\,dx]. \qquad (4.10)
\end{aligned}$$

And the optimal order quantity per cycle is

$$Q^* = D_1(P_1^*)\int_0^{t_1} e^{\alpha x^\beta}\,dx + D_2(P_2^*)\int_{t_2}^{t_3^*} e^{\alpha x^\beta}\,dx + e^{\alpha t_2^\beta}\int_{t_3^*}^{T}\frac{D_2(P_2^*)}{1+\delta(T-x)}\,dx. \qquad (4.11)$$

4.1.2 Complete Backlogging

The total profit $\pi(t_3, P_1, P_2)$ for complete backlogging can be calculated by substituting $\delta = 0$ in Equation (4.1)

$$\begin{aligned}
\pi(t_3, P_1, P_2) &= \frac{1}{T}[P_1 D_1(P_1) \int_0^{t_1} dt + P_2 D_2(P_2) \int_{t_2}^{t_3} dt + P_2 \int_{t_3}^{T} D_2(P_2)\, dx \\
&- C\{D_1(P_1) \int_0^{t_1} e^{\alpha x^\beta}\, dx + D_2(P_2) \int_{t_2}^{t_3} e^{\alpha x^\beta}\, dx + e^{\alpha t_2^\beta} \int_{t_3}^{T} D_2(P_2)\, dx\} \\
&- hD_1(P_1) \int_0^{t_1} e^{-\alpha t^\beta} [\int_t^{t_1} e^{\alpha x^\beta}\, dx]\, dt - hD_2(P_2) \int_0^{t_2} e^{-\alpha_1 t^{\beta_1}} [\int_{t_2}^{t_3} e^{\alpha x^\beta}\, dx]\, dt \\
&- he^{\alpha t_2^\beta} D_2(P_2) \int_0^{t_2} e^{-\alpha t^\beta} (T - t_3)\, dt - hD_2(P_2) \int_{t_2}^{t_3} e^{-\alpha t^\beta} [\int_t^{t_3} e^{\alpha x^\beta}\, dx]\, dt \\
&- C_1\{D_1(P_1) \int_0^{t_1} e^{\alpha x^\beta}\, dx + D_2(P_2) \int_{t_2}^{t_3} e^{\alpha x^\beta}\, dx + e^{\alpha t_2^\beta} D_2(P_2)(T - t_3) \\
&- D_1(P_1) t_1 - D_2(P_2)(t_3 - t_2) - D_2(P_2)(T - t_3) dx\} - C_4 D_2(P_2) e^{-\alpha t_1^\beta} \int_{t_2}^{t_3} e^{\alpha x^\beta}\, dx \\
&- C_4 e^{-\alpha t_1^\beta} e^{\alpha t_2^\beta} D_2(P_2)(T - t_3) - C_2 D_2(P_2) \int_{t_3}^{T} (T - x)\, dx].
\end{aligned} \tag{4.12}$$

The optimal value of P_1 will be as same as the previous partial backlogging case because shortage occurs only at the low–end market. And the optimal value of t_3 and P_2 can be obtained by substituting $\delta = 0$ in the proper equations of the previous partial backlogging model. Then, the optimal total profit can be calculated by substituting optimal t_3^*, P_1^* and P_1^* in Equation (4.12).

And the optimal order quantity per cycle is

$$Q^* = D_1(P_1^*) \int_0^{t_1} e^{\alpha x^\beta}\, dx + D_2(P_2^*) \int_{t_2}^{t_3^*} e^{\alpha x^\beta}\, dx + e^{\alpha t_2^\beta} \int_{t_3^*}^{T} D_2(P_2^*)\, dx. \tag{4.13}$$

4.2 One–Market Model with Discounting

The inventory profile of one–market model with discounting follows the pattern which is shown in Figure 3.3. Before discounting demand is $D_1(P_1)$ and after discounting demand is $D_d(P_d)$. It is assumed, before discounting the selling price in this one–market model is as same as the optimally determined selling price P_1^* of the two–market model. And the discounted selling price P_d is determined such that the total profit of this one–market with discounting model can be as same as the total profit of a two–market model. The optimal time at which inventory level goes to zero, t_3^*, the optimal order quantity, Q^*, and the optimal backlogged quantity, B^* are also determined.

4.2.1 Partial Backlogging

The total profit for partial backlogging can be expressed as:

$$\begin{aligned}
\pi(t_3, P_1, P_d) &= \frac{1}{T}[P_1 D_1(P_1)\int_0^{t_1} dx + P_d D_d(P_d)\int_{t_1}^{t_3} dx + P_d\int_{t_3}^{T}\frac{D_d(P_d)}{1+\delta(T-x)}\,dx \\
&- C\{D_1(P_1)\int_0^{t_1} e^{\alpha x^\beta}\,dx + D_d(P_d)\int_{t_1}^{t_3} e^{\alpha x^\beta}\,dx + D_d(P_d)\int_{t_3}^{T}\frac{1}{1+\delta(T-x)}\,dx\} \\
&- h\{D_1(P_1)\int_0^{t_1} e^{-\alpha t^\beta}[\int_t^{t_1} e^{\alpha x^\beta}\,dx]\,dt\} - h\{D_d(P_d)\int_0^{t_1} e^{-\alpha t^\beta}[\int_{t_1}^{t_3} e^{\alpha x^\beta}\,dx]\,dt\} \\
&- h\{D_d(P_d)\int_{t_1}^{t_3} e^{-\alpha t^\beta}[\int_t^{t_3} e^{\alpha x^\beta}\,dx]\,dt\} - (C_2 + C_3\delta)D_d(P_d)\int_{t_3}^{T}\frac{(T-x)}{1+\delta(T-x)}\,dx \\
&- C_1\{D_1(P_1)\int_0^{t_1} e^{\alpha x^\beta}\,dx + D_d(P_d)\int_{t_1}^{t_3} e^{\alpha x^\beta}\,dx - D_1(P_1)t_1 - D_d(P_d)(t_3 - t_1)\}]. \quad (4.14)
\end{aligned}$$

In order to determine optimal time t_3^* at which inventory level becomes zero, we take the first derivative of the total profit function with respect to t_3, which is:

$$\begin{aligned}
\frac{d\pi(t_3, P_1, P_d)}{dt_3} &= \frac{D_d(P_d)}{T}[P_d - \frac{P_d}{1+\delta(T-t_3)} - Ce^{\alpha t_3^\beta} + \frac{C}{1+\delta(T-t_3)} \\
-he^{\alpha t_3^\beta}\int_0^{t_3} e^{-\alpha t^\beta}\,dt &- C_1\{e^{\alpha t_3^\beta} - 1\} + \frac{(C_2 + C_3\delta)(T-t_3)}{1+\delta(T-t_3)}] = 0. \quad (4.15)
\end{aligned}$$

In Equation (4.15), t_3 and P_d are two unknown variables. So for a given P_d, solve Equation (4.15) and determine optimal value of t_3^*. Consequently it has been shown in one–market with discounting model for time dependent demand that $\frac{d^2\pi(t_3^*, P_1, P_d)}{dt_3^{*2}} < 0$ for a given P_d ($P_d > C$). Then, by solving Equations (4.15) and (4.14) simultaneously, one can determine the discounted selling price P_d, which can result in the same profit as in the two–market model.

4.2.2 Complete Backlogging

Total profit per for the complete backlogging can be calculated by substituting $\delta = 0$ in Equation (4.14)

$$\pi(t_3, P_1, P_d) = \frac{1}{T}[P_1 D_1(P_1) \int_0^{t_1} dx + P_d D_d(P_d) \int_{t_1}^{t_3} dx + P_d \int_{t_3}^{T} D_d(P_d)\, dx$$
$$- C\{D_1(P_1) \int_0^{t_1} e^{\alpha x^\beta}\, dx + D_d(P_d) \int_{t_1}^{t_3} e^{\alpha x^\beta}\, dx + D_d(P_d) \int_{t_3}^{T} dx\}$$
$$- h\{D_1(P_1) \int_0^{t_1} e^{-\alpha t^\beta} [\int_t^{t_1} e^{\alpha x^\beta}\, dx]\, dt\} - h\{D_d(P_d) \int_0^{t_1} e^{-\alpha t^\beta} [\int_{t_1}^{t_3} e^{\alpha x^\beta}\, dx]\, dt\}$$
$$- h\{D_d(P_d) \int_{t_1}^{t_3} e^{-\alpha t^\beta} [\int_t^{t_3} e^{\alpha x^\beta}\, dx]\, dt\} - C_2 D_d(P_d) \int_{t_3}^{T} (T - x)\, dx$$
$$- C_1\{D_1(P_1) \int_0^{t_1} e^{\alpha x^\beta}\, dx + D_d(P_d) \int_{t_1}^{t_3} e^{\alpha x^\beta}\, dx - D_1(P_1) t_1 - D_d(P_d)(t_3 - t_1)\}]. \quad (4.16)$$

The objective is to determine the optimal value of t_3^* and a discounted selling price P_d. The optimal value of t_3^* can be obtained from the partial backlogging case by substituting $\delta = 0$ in Equation (4.15). After substututing optimal value of t_3 in Equation (4.16), one can determine a discounted selling price P_d using an iterative method from Equation (4.16), which can result in the same profit as in the two–market model.

4.3 One–Market Model

The inventory profile follows the pattern which is shown in Figure 3.4. Let $D_o(P_o)$ be the demand rate for the one–market model where P_o is the selling price, which is not optimally determined. P_o is determined such that the total profit obtained from this model is as same as that of the two–market model. The optimal time at which inventory level becomes zero, t_3^*, the optimal order quantity, Q^*, and the optimal backlogged quantity, B^* are also determined.

4.3.1 Partial Backlogging

The total profit for partial backlogging can be expressed as:

$$\pi(t_3, P_o) = \frac{1}{T}[P_o \int_0^{t_3} D_o(P_o)\,dx + P_o \int_{t_3}^{T} \frac{D_o(P_o)}{1+\delta(T-x)}\,dx - CD_o(P_o) \int_0^{t_3} e^{\alpha x^{\beta}}\,dx$$
$$- CD_o(P_o) \int_{t_3}^{T} \frac{1}{1+\delta(T-x)}\,dx - hD_o(P_o) \int_0^{t_3} e^{-\alpha t^{\beta}} [\int_t^{t_3} e^{\alpha x^{\beta}}\,dx]\,dt$$
$$- C_1 D_o(P_o)\{\int_0^{t_3} e^{\alpha x^{\beta}}\,dx - t_3\} - (C_2 + C_3\delta) D_o(P_o) \int_{t_3}^{T} \frac{(T-x)}{1+\delta(T-x)}\,dx]. \tag{4.17}$$

The objective is to determine the optimal value of t_3^* and a selling price P_o. Optimal value of t_3^* can be obtained by taking the first derivative of the total profit function with respect to t_3, which can be expressed as:

$$\frac{d\pi(t_3, P_o)}{dt_3} = \frac{D_o(P_o)}{T}[P_o - \frac{P_o}{1+\delta(T-t_3)} - Ce^{\alpha t_3^{\beta}} + \frac{C}{1+\delta(T-t_3)}$$
$$-he^{\alpha t_3^{\beta}} \int_0^{t_3} e^{-\alpha t^{\beta}}\,dt - C_1\{e^{\alpha t_3^{\beta}} - 1\} + \frac{(C_2 + C_3\delta)(T-t_3)}{1+\delta(T-t_3)}] = 0. \tag{4.18}$$

In Equation (4.18), t_3 and P_o are two unknown variables. So for a given P_o, Equation (4.18) is solved to determine optimal value of t_3^*. Consequently, it is shown in one–market model of time dependent demand that $\frac{d^2\pi(t_3^*, P_o)}{dt_3^{*2}} < 0$ for a given P_o ($P_o > C$). Thus, by solving Equations (4.17) and (4.18) simultaneously, one can determine a selling price P_o, which can result in the same profit as in the two–market model.

4.3.2 Complete Backlogging

Substituting $\delta = 0$ in Equation (4.17), we obtain the total profit of the complete backlogging case as follows:

$$\pi(t_3, P_o) = \frac{1}{T}[P_o \int_0^{t_3} D_o(P_o)\,dx + P_o \int_{t_3}^{T} D_o(P_o)\,dx - CD_o(P_o) \int_0^{t_3} e^{\alpha x^{\beta}}\,dx$$
$$- CD_o(P_o) \int_{t_3}^{T} dx - hD_o(P_o) \int_0^{t_3} e^{-\alpha t^{\beta}} [\int_t^{t_3} e^{\alpha x^{\beta}}\,dx]\,dt$$
$$- C_1 D_o(P_o)\{\int_0^{t_3} e^{\alpha x^{\beta}}\,dx - t_3\} - C_2 D_o(P_o) \int_{t_3}^{T} (T-x)\,dx]. \tag{4.19}$$

The objective is to determine the optimal value of t_3^* and a selling price P_o. The optimal value of t_3^* can be obtained from the partial backlogging case by substituting $\delta = 0$ in Equation (4.18). After substituting this optimal value of t_3 in Equation (4.19), one can determine a selling price P_o using an iterative method from Equation (4.19), which can result in a same profit as in the two–market model.

Chapter 5

Examples and Sensitivity Analysis for Time–Dependent Demand

In this chapter, numerical examples are presented for the models developed in chapter 3. To illustrate the models numerically, the following parameter values are considered: holding cost $h = \$2$ per unit per unit time, deterioration cost $C_1 = \$1$ per unit, backorder cost $C_2 = \$32$ per unit, opportunity cost due to lost sale $C_3 = \$35$ per unit, transportation cost $C_4 = \$0.5$ per unit, time proportional constant for backlogging $\delta = 0.1$, purchasing cost $C = \$30$ per unit, scale parameter $\alpha = 0.3$, shape parameter $\beta = 2$, and cycle time $T = 1$.

5.1 Two–Market model:

The high–end market demand $D_1(t)$ is expressed as $D_1(t) = a_1 + b_1 t$, where $a_1 = 700$ and $b_1 = 10$. The low–end market demand $D_2(t)$ is expressed as $D_2(t) = a_2 + b_2 t$, where $a_2 = 900$ and $b_2 = 20$. At time $t_1 = 0.41$, the leftover inventory is transferred from the high–end market to the low–end market and the transportation time is $t_2 - t_1$=0.011. The high–end market selling price is $P_1 = \$50$ per unit and the low–end market selling price is $P_2 = \$45$ per unit.

5.1.1 Partial Backlogging

First, the optimal time t_3^* is determined iteratively by using Equation (3.28). Initially, for example, substitute $t_3 = 0.86$ in Equation (3.28):

$$\begin{aligned} f(0.86) &= 0.6213 - 6.3365 - 0.982 - 0.1007 - 0.3948 + 4.9014 \\ &= -2.294. \end{aligned}$$

Then, the value of t_3 is decreased until it satisfies the condition $f(t_3) = 0$. Figure 5.1 shows, $t_3^* = 0.8201$ satisfies that condition. Figure 5.2 also shows that the total profit is a concave function on t_3 and the maximum total profit is at $t_3 = 0.8201$.

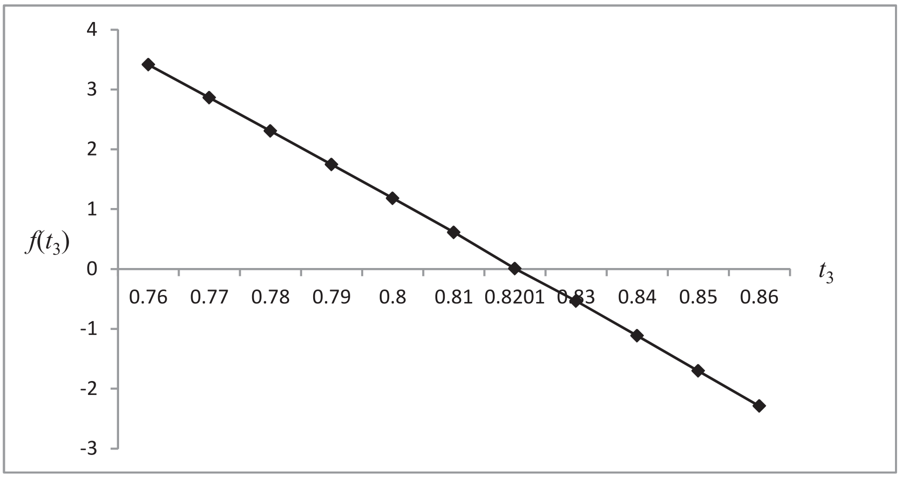

Figure 5.1: The optimal value at which inventory becomes zero with time–dependent demand and partial backlogging

Second, the optimal total profit per year, optimal order quantity, and optimal backlogged quantity are calculated with $t_3^* = 0.8201$. The optimal profit per year is equal to the optimal sales revenue minus the optimal total cost. Hence, the optimal total cost is calculated using Equation (3.24) as follows:

$$\Gamma_c^*(0.8201) = \frac{1}{T}[P_c + H_c + U_c + M_c + B_c + L_c] = \$27946 \text{ per unit time.}$$

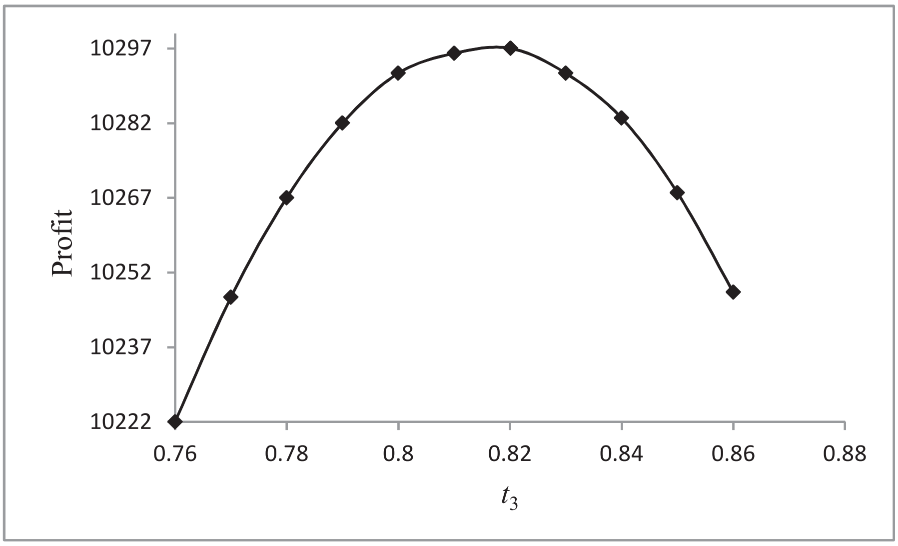

Figure 5.2: The total profit with respect to t_3 for time–dependent demand and partial backlogging

The optimal sales revenue is determined by using Equation (3.25) as follows:

$$SR^* = \frac{1}{T}[P_1 \int_0^{t_1} D_1(x)\,dx + P_2 \int_{t_2}^{t_3^*} D_2(x)\,dx + P_2 \int_{t_3^*}^{T} \frac{D_2(x)}{1+\delta(T-x)}\,dx]$$

$$= 14040 + 16836 + 7367$$

$$= \$38243 \ \text{ per unit time.}$$

Therefore, the optimal total profit,

$$\pi^*\ (0.8201) = [\text{The optimal sales revenue - The optimal total cost}]$$

$$= 38243 - 27946$$

$$= \$10297 \text{ per unit time.}$$

The optimal order quantity Q^* is determined by using Equation (3.32) as given blow:

$$Q^* = \int_0^{t_1} D_1(x) e^{\alpha x^\beta}\,dx + \int_{t_2}^{t_3^*} D_2(x) e^{\alpha x^\beta}\,dx + e^{\alpha t_2^\beta} \int_{t_3^*}^{T} \frac{D_2(x)}{1+\delta(T-x)}\,dx$$

$$= 285.32 + 418.20 + 172.18$$

$$= 875.70 \text{ units,}$$

and the optimal backlogged quantity is

$$B^* = \int_{t_3^*}^{T} \frac{D_2(x)}{1+\delta(T-x)}\,dx$$
$$= 163.71 \text{ units.}$$

Next, in order to study the sensitivity to change in the input parameters C, h, α, β, C_1, C_2, C_3, C_4, and δ on the optimal time in which the inventory becomes zero, t_3^*, optimal order quantity, Q^*, optimal backlogged quantity, B^*, optimal total cost, Γ_c^*, and optimal total profit, π^*, the sensitivity analysis of the two–market model for partial backlogging is studied by changing one parameter at a time while keeping the remaining parameters at their original value.

Table 5.1: Sensitivity analysis for the two–market model with time–dependent demand and partial backlogging

Parameters	Changing values	t_3^*	Q^*	B^*	Γ_c^*	π^*
Panel 1						
	15	0.8369	885.07	112.41	14801	23477
C	30	0.8201	875.70	163.71	27946	10297
	45	0.7761	869.23	203.22	40982	-2775
Panel 2						
	1	0.8294	877.14	155.34	27566	10683
h	2	0.8201	875.70	163.71	27946	10297
	3	0.8130	874.60	170.10	28317	9920.8
Panel 3						
	0.15	0.8394	853.01	104.70	27002	11280
α	0.30	0.8201	875.70	163.71	27946	10297
	0.45	0.8034	894.12	207.78	28781	9421.4
Panel 4						
	1	0.8122	925.03	170.82	29527	8710.8
β	2	0.8201	875.70	163.71	27946	10297
	3	0.8286	853.76	156.06	27205	11044
Panel 5						
	0.5	0.823	876.14	161.10	27892	10354
C_1	1.0	0.8201	875.70	163.71	27946	10297

Table 5.1: Sensitivity analysis (cont.)

Parameters	Changing values	t_3^*	Q^*	B^*	Γ_c^*	π^*
	1.5	0.8176	875.30	165.96	28001	10241
Panel 6						
	16	0.7498	865.69	226.73	27539	10644
C_2	32	0.8201	875.70	163.71	27946	10297
	48	0.8409	882.03	128.34	28163	10106
Panel 7						
	17.5	0.8151	874.92	168.21	27915	10325
C_3	35.0	0.8201	875.70	163.71	27946	10297
	52.5	0.8256	881.06	153.54	28063	9949.1
Panel 8						
	0.375	0.8285	877.00	156.15	27885	10364
C_4	0.500	0.8201	875.70	163.7171	27946	10297
	0.625	0.8142	874.78	169.02	28012	10227
Panel 9						
	0.05	0.8140	875.57	169.98	27942	10332
δ	0.10	0.8201	875.70	163.71	27946	10297
	0.15	0.8270	876.06	156.83	27953	10265

The following characteristics are observed from Table 5.1

1. The total cost increases and total profit decreases with an increase in the value of all cost parameters $(C, h, C_1, C_2, C_3, C_4)$. In order to increase the profit, these costs should be reduced.

2. When the value of C increases, the total cost and B^* increase while t_3^*, Q^*, and π^* decrease. The obtained results show that the total profit and total cost are highly sensitive to change in the value of C. When $C = 45$, the profit is negative, because purchase cost per unit should be less than the selling price per unit.

3. If holding cost increases, one tries to hold less inventory, hence Q^* decreases. This decreases t_3^* and increases B^*. Consequently, the total cost increases and the profit

decreases.

4. Deterioration cost increases with an increase in the value of deterioration parameter α. As a result, the total cost increases and the total profit decreases with an increase in the value of α.

5. Q^* and B^* decrease as β increases. Hence, the total cost decreases and the total profit increases.

6. When deterioration cost C_1 increases, one will hold less inventory. Therefore, t_3^* and Q^* decrease while B^* increases with an increase in C_1.

7. When backorder and opportunity costs increase, more amount of inventory will be hold to reduce the costs. As a result, t_3^* and Q^* increase and B^* decreases with an increase in C_2 and C_3.

8. The value of δ is the proportion of customers who do not want to accept their demand to be backlogged. Increasing the value of δ will increase the total cost and decrease the optimal profit, hence, maximum profit occurs at $\delta = 0$ and minimum profit occurs at $\delta = \infty$.

5.1.2 Complete Backlogging

Our objective is to determine the optimal time t_3 that maximizes the total profit. Let $t_3 = 0.85$ and then

$$\begin{aligned} f(0.85) &= -36.5161 + 30 - 1.853 - 0.089 - 0.3317 + 6.02 \\ &= -2.7698. \end{aligned}$$

Now, the value of t_3 is decreased, until it satisfies the condition $f(t_3^*) = 0$. Figure 5.3 shows that $t_3^* = 0.8$ satisfies $f(t_3^*) = 0$ and Figure 5.4 shows the profit function is a concave function on t_3 and the maximum total profit is at $t_3 = 0.8$.

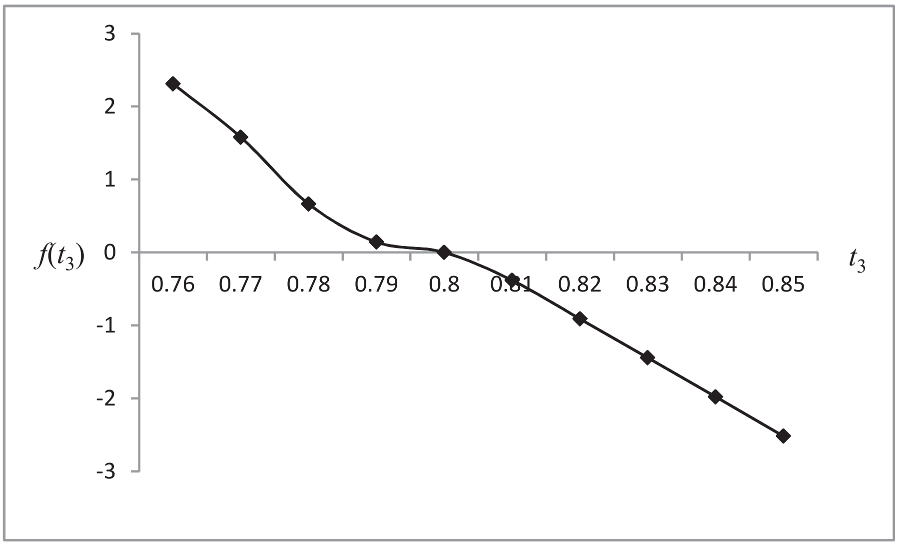

Figure 5.3: The optimal time at which inventory becomes zero with time–dependent demand and complete backlogging

Once t_3^* is determined, the optimal total cost, Γ_c^*, optimal sales revenue, SR^*, optimal total profit, π^*, optimal order quantity, Q^*, and optimal backlogged quantity, B^*, are calculated, respectively. Hence, the optimal total cost per unit time

$$\Gamma_c^*\ (0.80) = \frac{1}{T}[P_c + H_c + U_c + M_c + B_c + L_c] = \$27939 \text{ per unit time.}$$

The optimal sales revenue:

$$\begin{aligned} SR^* =& \frac{1}{T}[P_1 \int_0^{t_1} D_1(x)\,dx + P_2 \int_{t_2}^{t_3^*} D_2(x)\,dx + P_2 \int_{t_3^*}^{T} D_2(x)\,dx] \\ &= 14040 + 16007 + 8263 \\ &= \$38310 \ \text{ per unit time.} \end{aligned}$$

Hence, the optimal total profit is determined as follows:

$$\begin{aligned} \pi^*\ (0.8) &= \frac{1}{T}[\text{The total sales revenue – The total cost}] \\ &= 38310 - 27939 \\ &= \$10371 \text{ per unit time.} \end{aligned}$$

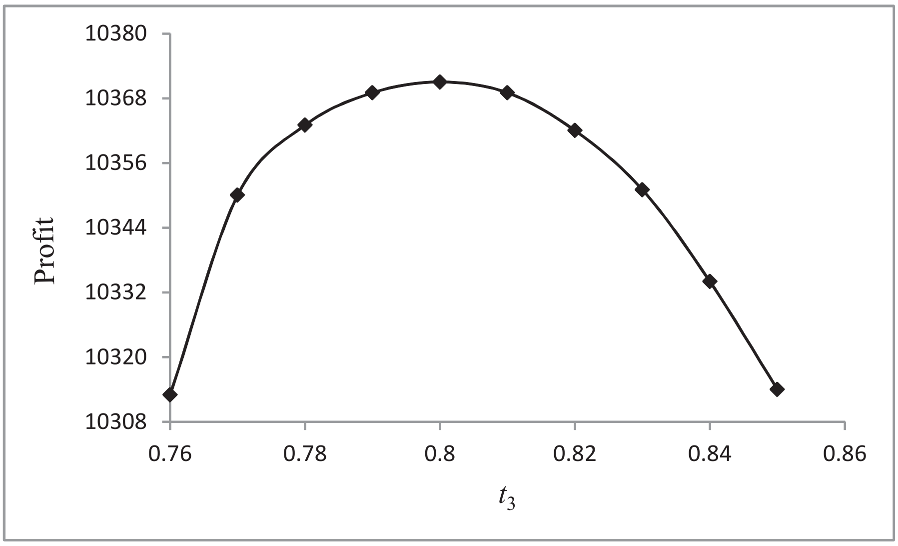

Figure 5.4: The total profit with respect to t_3 for time–dependent demand and complete backlogging

The optimal order quantity is given by

$$\begin{aligned} Q^* &= \int_0^{t_1} D_1(x)e^{\alpha x^\beta}\,dx + \int_{t_2}^{t_3^*} D_2(x)e^{\alpha x^\beta}\,dx + e^{\alpha t_2^\beta}\int_{t_3^*}^{T} D_2(x)\,dx \\ &= 285.29 + 396.18 + 193.09 \\ &= 874.56 \text{ units,} \end{aligned}$$

and the optimal backlogged quantity is given by

$$\begin{aligned} B^* &= \int_{t_3^*}^{T} D_2(x)\,dx \\ &= 183.60 \text{ units.} \end{aligned}$$

If we compare complete and partial backlogging, we can see that total profit of the complete backlogging is usually greater than that of the partial backlogging. Because, in partial backlogging, some percentages of demand are lost and opportunity cost is added due to lost sales. However, in complete backlogging, all demands are backlogged with no lost sales. For complete backlogging case, the sensitivity of the input parameters are studied and the results are given in Table 5.2

Table 5.2: Sensitivity analysis for the two–market model with time–dependent demand and complete backlogging

Parameters	Changing values	t_3^*	Q^*	B^*	Γ_c^*	π^*
Panel 1						
	15	0.8411	883.26	127.59	14778	23532
C	30	0.8000	874.56	183.60	27939	10371
	45	0.7586	869.01	226.53	41021	-2711.2
Panel 2						
	1	0.8087	875.69	175.63	27567	10742
h	2	0.8000	874.56	183.60	27939	10371
	3	0.7991	874.44	184.42	28304	10005
Panel 3						
	0.15	0.8337	851.02	135.02	26986	11324
α	0.30	0.8000	874.56	183.60	27939	10371
	0.45	0.7395	891.54	238.98	28795	9514.8
Panel 4						
	1	0.7912	919.90	189.79	29390	8732.1
β	2	0.8000	874.56	183.60	27939	10371
	3	0.8098	852.64	174.62	27192	11117
Panel 5						
	0.5	0.8037	875.04	180.03	27884	10425
C_1	1.0	0.8000	874.56	183.60	27939	10371
	1.5	0.7994	874.48	184.14	27993	10316
Panel 6						
	16	0.7245	866.08	252.70	27513	10796
C_2	32	0.8000	874.56	163.60	27939	10371
	48	0.8305	878.69	155.65	28170	10139

5.2 One–Market Model with Discounting

All parameters are the same as in the two–market model. The only difference is that the selling price is discounted at time t_1 without transporting the items to the low–end market. From time 0 to t_1, demand, $D_1(t) = 700 + 10t$, and selling price, $P_1 = 50$, are the same as in the high–end market of the two–market model. In this model, after time t_1, demand is $D_d(t) = a_d + b_d t$, where $a_d = 905$ and $b_d = 20$, and discounted selling price is P_d, which will be determined by obtaining the same profit as in the two–market model.

5.2.1 Partial Backlogging

In order to determine the discounted selling price P_d, we apply the following algorithm:

Algorithm 1

1. Start with $\hat{P}_d = P_d$, where $C < \hat{P}_d < P_1$.

2. Determine the optimal time t_3^* by substituting $\hat{P}_d$ in Equation (3.45). Using this t_3^*, calculate the total profit $\pi(t_3^*)$ of the one–market model with discounting from Equation (3.44).

3. Calculate the total profit $\pi(t_3^*)$ of the two–market model from Equation (3.26). If the profit from Equation (3.44) is equal to the profit from Equation (3.26), go to step 5; otherwise go to step 4.

4. Initialize another $\hat{P}_d$ and go to step 2.

5. Stop and take this P_d as the discounted selling price of the one–market model with discounting.

From Equation (3.26), we get the optimal profit of the two–market model as \$10297 per unit time. In order to determine t_3^*, we start with $\hat{P}_d = 43.800$ in Equation (3.45) and we obtain $t_3^* = 0.7767$ as follows:

$$\begin{aligned} f(0.7767) &= 44 - 43.0394 - 35.9534 + 29.3450 - 1.7496 - 2.0967 + 2 + 7.7504 \\ &= 0.0041 \approx 0. \end{aligned}$$

Then, substituting $t_3 = 0.7767$ and $\hat{P}_d = 43.800$ in Equation (3.44), we obtain total profit of the one–market model with discounting $\pi(t_3) = \$10145 < \$10297$. Now, we increase the value of $\hat{P}_d$ as shown in Table 5.3. For $P_d = 44.052$, we obtain that profit from the one–market model with discounting is the same as the profit from the two–market model. Hence, optimal time $t_3^* = 0.7773$, the optimal total profit $\pi^* = \$10297$ per unit time, the optimal order quantity $Q^* = 880.33$ units, and the optimal backlogged quantity $B^* = 203.35$ units.

Table 5.3: Discounted selling price for time–dependent demand with partial backlogging

No	P_d	t_3^*	Total profit
1	43.800	0.7767	10145
2	43.900	0.7770	10199
3	44.000	0.7771	10254
4	44.030	0.7772	10279
5	44.052	0.7773	10297

5.2.2 Complete Backlogging

In Section 5.1.2, we obtain the optimal profit of \$10371 per unit time for the two–market model with complete backlogging. Now, we use the iterative method to find P_d for complete backlogging. The summary of the iteration is given in Table 5.6. The optimal solution at which inventory becomes zero does not depend on selling price for complete backlogging. As a result, the optimal time $t_3^* = 0.7585$ remains the same with changing the discounted selling price. We also calculate the optimal order quantity $Q^* = 878.05$ units, and the optimal backlogged quantity $B^* = 222.80$ units.

Next, in order to investigate how the discounted price in the one–market model behaves with respect to prices in high–end and low–end markets in the two–market model,

Table 5.4: Discounted selling price for time–dependent demand with completel backlogging

No	t_3	P_d	Total profit
1	0.7585	43.30	9992.0
2	0.7585	43.50	10157
3	0.7585	43.70	10212
4	0.7585	43.90	10304
5	0.7585	44.04	10371

we determine the total profit of the two–market model for different selling prices in both high–end and low–end markets, and then we also determine the corresponding discounted selling price in the one–market model. Figures 5.5 and 5.6 show that when the profit in the one–market model is the same as that of the two–market model, discounted selling price is less than that of both high–end market and low–end market. One way to explain this behavior is that the total cost of the two–market model is greater than that of the one–market model with discounting, because of transportation. This helps to offer a discounted selling price, which is less than that of the high–end and low-end markets in the two–market model.

Finally, there is a fundamental problem when dealing with deteriorating items. The longer time one keeps the inventory, the more products deteriorate. Consequently, discount pricing strategy is used to increase the demand. However, one has to decide when to discount the price and by how much. Therefore, we investigate the relationship between the discounted selling price and the time to introduce the discount. In Figure 5.7, at each and every points profit is the same. This figure shows that if one offers discount early, the items can be sold at a higher discounted price. As time passes the value of

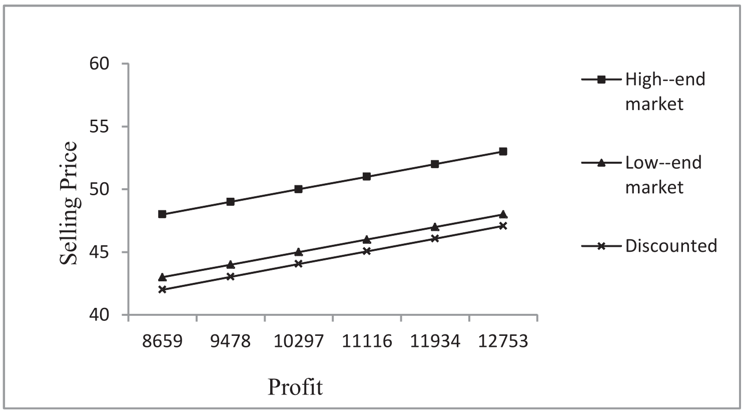

Figure 5.5: Relationships among high–end, low–end, and discounted selling prices for partial backlogging

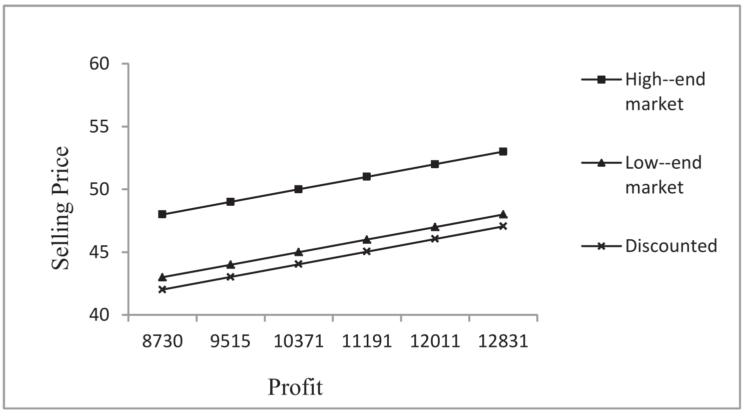

Figure 5.6: Relationships among high–end, low–end, and discounted selling prices for complete backlogging

the items drop very sharply because of deterioration. This figure indicates that there is a critical time that makes the difference in the discounted price. After that critical time discounted selling price does not change much and becomes constant. This indicates that each item has its lowest selling price to make the business profitable. Hence, one is not able to offer discount beyond that selling price.

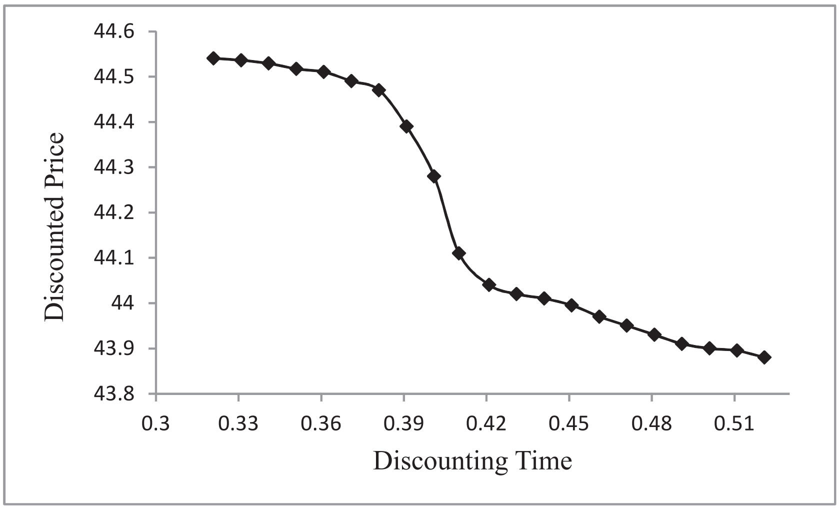

Figure 5.7: Relationship between discounting time and discounted selling price for time–dependent demand

5.3 One–Market Model

Demand for the one–market without discounting is assumed to be $D_o(t) = a_o + b_o t$, where $a_o = 800$, and $b_o = 20$ and the rest of the input parameters remain same as in the two–market model. The objective is to determine the selling price which leads to the same profit as in the two–market model.

5.3.1 Partial Backlogging

The objective is to determine the selling price, P_o, that can result in the same profit as in the two–market model. To determine P_o the algorithm 1 is applied, which is presented in Section 5.2.1. Table 5.5 shows the summary of the iterations. The selling price P_o = \$46.77, the optimal time $t_3^* = 0.77811$, the optimal order quantity $Q^* = 846.26$ units, and the optimal backlogged quantity $B^* = 179.45$ units.

Table 5.5: Selling price for the one–market model with time–dependent demand and partial backlogging

No	P_o	t_3^*	Total profit
1	46.40	0.77803	9997.6
2	46.50	0.77805	10078
3	46.60	0.77807	10159
4	46.70	0.77809	10240
5	46.77	0.77811	10297

5.3.2 Complete Backlogging

To determine P_o, an iterative method is applied. Table 5.6 shows that, $P_o = \$46.751$ per unit that can generate the profit of \$10371, which is the profit in the two–market model with complete backlogging. Hence, the optimal time $t_3^* = 0.7612$, the optimal order quantity $Q^* = 845.66$ units, and the optimal order quantity $B^* = 195.24$ units.

Table 5.6: Selling price for the one–market model with time–dependent demand and complete backlogging

No	t_3	P_o	Total profit
1	0.7612	46.500	10169
2	0.7612	46.600	10250
3	0.7612	46.700	10331
4	0.7612	46.720	10356
5	0.7612	46.751	10371

Now, in order to investigate the behaviour of the selling price of one–market model with respect to high–end and low–end market selling prices in the two–market model, we determine the total profit of the two–market model for different selling prices in both high–end and low–end markets. Then, we determine corresponding selling price of one–market model. Figures 5.8 and 5.9 show that the selling price of one–market model is less than high–end market selling price but greater than low–end market selling price when the profit in the one–market model is the same as that of two–market model. For one–market model, the same selling price is continued through the whole cycle time. This hepls to offer a selling price, which is greater than the low–end market selling price but less than the high–end market selling price in the two–market model.

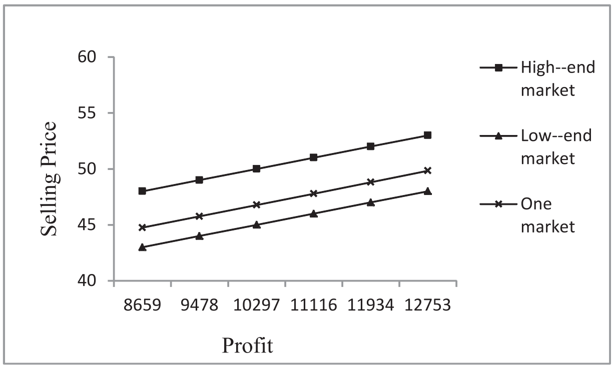

Figure 5.8: Relationships among high–end, low–end, and one–market model selling prices for partial backlogging

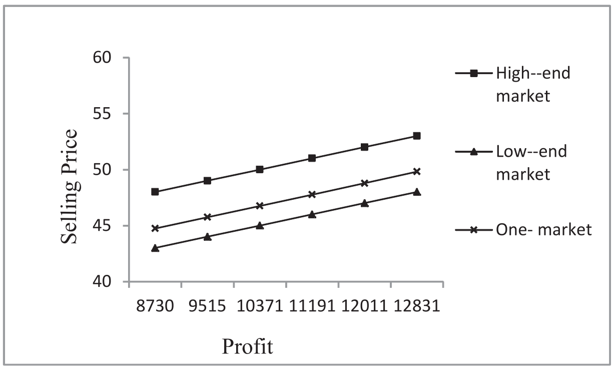

Figure 5.9: Relationships among high–end, low–end, and one–market model selling prices for complete backlogging

Chapter 6

Examples and Sensitivity Analysis for Price–Dependent Demand

In this chapter, numerical examples are presented for the models which are developed in Chapter 4. To illustrate the models numerically, the following parameter values are considered: holding cost $h = \$3$ per unit per unit time, cycle time $T = 0.25$, and other parameters are assumed same as the previous time–dependent demand model.

6.1 Two–Market Model

The high–end market demand $D_1(P_1) = a_1 P_1^{-b_1}$, where $a_1 = 7 \times 10^7$ and $b_1 = 3.2$. The low–end market demand $D_2(P_2) = a_2 P_2^{-b_2}$, where $a_2 = 8 \times 10^9$ and $b_2 = 3.9$. At time $t_1 = 0.1$, the leftover inventory is transferred from the high–end market to low–end market and the transportation time $t_2 - t_1 = 0.011$.

6.1.1 Partial Backlogging

In the total profit function t_3, P_1, and P_2 are three unknown variables. At first, the optimal selling price of the high–end market P_1^* is determined by the iterative method from Equation (4.3). Initially, substitute $P_1 = 65$ in Equation (4.3):

$$\begin{aligned} f(65) =& 11.0605 + 16.3519 + 0.1099 + 2.1988 - 2.1966 - 40.3936 \\ =& -12.8691. \end{aligned}$$

Then, the value of P_1 is decreased until it satisfies the condition $f(P_1) = 0$. Figure 6.1 shows, $P_1^* = 44.632$ satisfies the optimal condition.

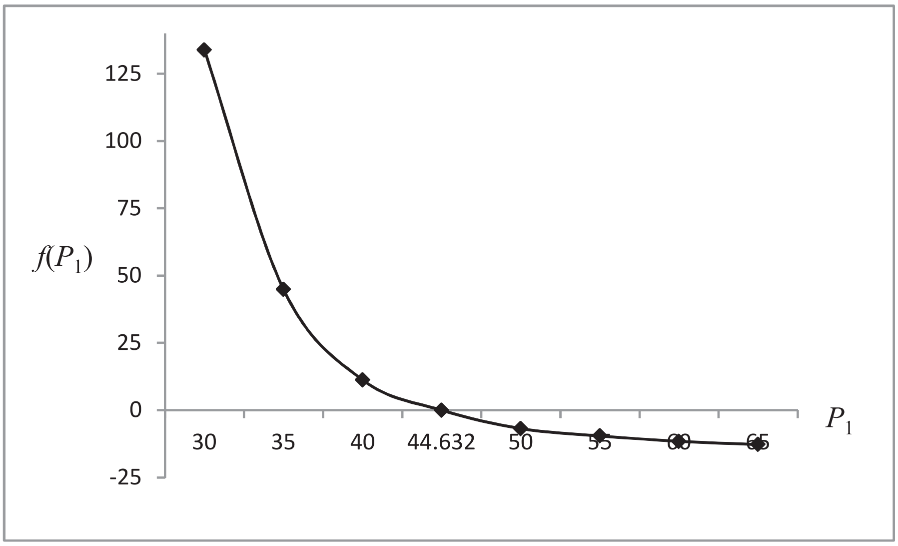

Figure 6.1: The optimal value of P_1 for partial backlogging

Now, the optimal values of t_3^* and P_2^* are determined by solving Equations (4.5) and (4.7) simutaneously. To solve these two equations, the following algorithm is applied.

Algorithm 2

1. Start with P_2 value, where, $P_2 > C$.

2. For the current P_2, solve $f(t_3) = 0$ from Equation (4.5) and denote the optimal solution by $\hat{t_3}$.

3. Let t_3=$\hat{t_3}$ solve $f(P_2) = 0$ from Equation (4.7). Let the solution to (4.7) be the current P_2. The objective function value of $\pi(t_3, P_1, P_2)$ should change in step

2. Similarly, when $f(t_3)=0$ is solved for current P_2, $\pi(t_3,P_1,P_2)$ should change. Repeat steps 2 and 3 until no change is seen in the objective function value of $\pi(t_3,P_1,P_2)$.

Applying Algorithm 2, after some iterations we obtain the optimal time $t_3^*=0.215$ and the optimal selling price of the low–end market $P_2^*=40.26$.

Figure 6.2 indicates the total profit per unit time $\pi(P_2|t_3^*,P_1^*)$ is strictly concave in P_2 and Figure 6.3 shows the total profit per unit time $\pi(t_3|P_1^*,P_2^*)$ is strictly concave function in t_3. Consequently, we are sure that the local maximum solution $t_3^*=0.215$ and $P_2^*=40.26$ are the global maximum solution for maximizing total profit.

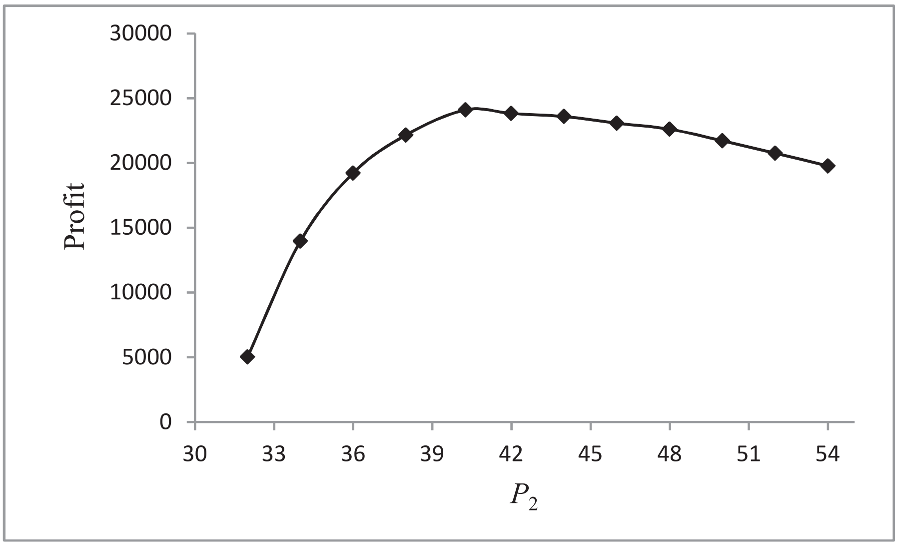

Figure 6.2: Graphical representation of $\pi(P_2|t_3^*,P_1^*)$

Therefore, the optimal total profit per year, optimal order quantity, and optimal backlogged quantity are calculated as $t_3^*=0.215$, $P_1^*=44.632$, and $P_2^*=40.26$. We obtain the optimal total cost $\Gamma_c^*=\$70322$ per year and the optimal sales revenue $SR^*=\$94410$ per year.

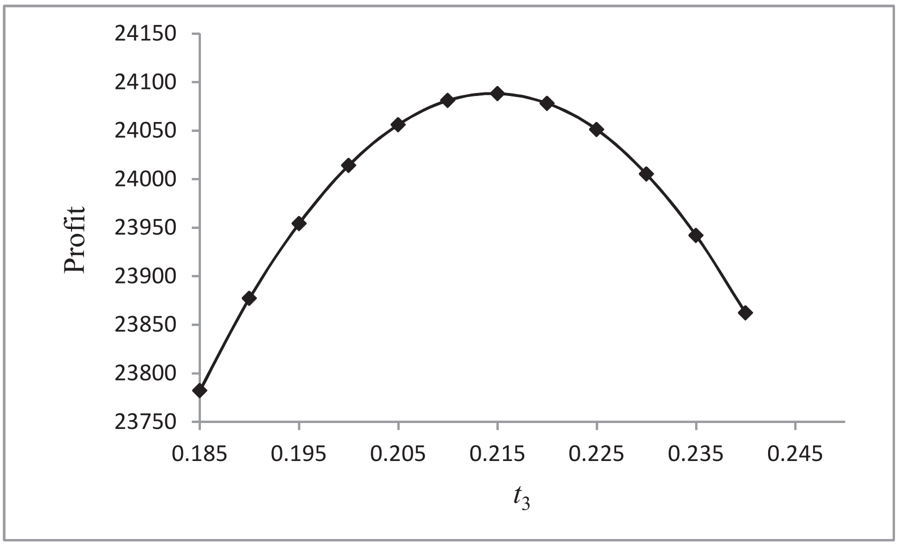

Figure 6.3: Graphical representation of $\pi(t_3|P_1^*, P_2^*)$

$$\begin{aligned}\text{Therefore, the optimal total profit, } \pi(t_3^*, P_1^*, P_2^*) &= (SR^* - \Gamma_c^*)\\ &= (\$94410 - \$70322)\\ &= \$24088\text{per unit time.}\end{aligned}$$

The optimal order quantity Q^* is:

$$\begin{aligned}Q^* =& D_1(P_1^*)\int_0^{t_1} e^{\alpha x^\beta}\,dx + D_2(P_2^*)\int_{t_2}^{t_3^*} e^{\alpha x^\beta}\,dx + e^{\alpha t_2^\beta}\int_{t_3^*}^{T}\frac{D_2(P_2^*)}{1+\delta(T-x)}\,dx\\ =& \,563.38 \text{ units,}\end{aligned}$$

and the optimal backlogged quantity is:

$$B^* = \int_{t_3^*}^{T}\frac{D_2(P_2^*)}{1+\delta(T-x)}\,dx \;= 130.53 \text{ units.}$$

Sensitivity analysis of the two–market model for the partial backlogging case is preformed to study the effect of the input parameters on the optimal time, (t_3^*), the optimal selling price of the high–end market, (P_1^*), the optimal selling price of the low–end market, (P_2^*), the optimal total cost, (Γ_c^*), and the optimal total profit, (π^*). The sensitivity analysis results are presented in Table (6.1)

Table 6.1: Sensitivity analysis for the two–market model with price–dependent demand and partial backlogging

Parameters	Changing values	t_3^*	P_1^*	P_2^*	Γ_c^*	π^*
Panel 1						
	15	0.2198	27.602	24.133	308360	148720
C	30	0.2150	44.632	40.260	70322	24088
	45	0.2045	65.020	61.218	24416	7903
Panel 2						
	1.5	0.2191	44.600	39.976	71923	24441
h	3.0	0.2150	44.632	40.260	70322	24088
	4.5	0.2095	44.671	40.341	70012	22979
Panel 3						
	0.15	0.2176	44.350	39.877	70932	24331
α	0.30	0.2150	44.632	40.260	70322	24088
	0.45	0.2045	44.710	41.869	65451	23845
Panel 4						
	1	0.1747	44.78	42.709	68628	21440
β	2	0.2150	44.632	40.260	70322	24088
	3	0.2161	44.498	39.826	73542	24461
Panel 5						
	0.5	0.2164	44.596	40.133	71054	24096
C_1	1.0	0.215	44.632	40.260	70322	24088
	1.5	0.2132	44.658	40.350	69724	24077
Panel 6						
	16	0.1989	44.632	40.018	72790	24286
C_2	32	0.2150	44.632	40.260	70322	24088
	48	0.2203	44.632	40.412	65666	23979
Panel 7						
	17.5	0.1943	44.632	39.696	72931	24158
C_3	35.0	0.2150	44.632	40.260	70322	24088
	52.5	0.2199	44.632	40.834	67506	23949
Panel 8						
	0.375	0.2182	44.611	40.178	70512	24290
C_4	0.500	0.2150	44.632	40.260	70322	24088
	0.625	0.2105	44.689	40.470	69317	23934
Panel 9						

Table 6.1: Sensitivity analysis (cont.)

Parameters	Changing values	t_3^*	P_1^*	P_2^*	Γ_c^*	π^*
	0.05	0.2141	44.632	40.030	70933	24105
δ	0.10	0.2150	44.632	40.260	70322	24088
	0.15	0.2167	44.632	40.422	69721	24072

The following characteristics are observed from Table 6.1

1. Selling prices for both markets increase with the value of cost parameters (C, h, C_1). Consequently, demand decreases due to high selling price, which leads to decrease in the total profit as well as the total cost.

2. The obtained results show that, $P_1^*, P_2^*, \Gamma_c^*, \pi^*$ are highly sensitive to change in the value of C. The selling prices highly depend on the purchase cost. For a lower purchase cost, one can offer a lower selling price. And this lower selling price can generate more selling rate and more total profit.

3. The total profit and the total cost decrease, however, selling prices of the high–end and the low–end markets increase with the value of α.

4. P_1^* and P_2^* decrease as β increases. Hence, t_3^*, Γ_c^*, and π^* increase in β.

5. The high–end market selling price does not change with the backorder cost C_2, the opportunity cost C_3 or the backlogging parameter δ, because shortage does not occur at the high–end market. As a result, if C_2, C_3 or δ increases, only the low–end market selling price increases.

6.1.2 Complete Backlogging

For complete backlogging case, the optimal selling price of the high–end market will be as same as in partial backlogging because backlogged occurs only at the low–end market.

In order to determine the optimal time t_3^*, we initialize $t_3 = 0.24$:

$$f(t_3 = 0.24) = -0.4138 - 0.7283 - 0.0069 - 0.0276 + 0.32 \\ - 0.8566.$$

Now, the value of t_3 is decreased until it satisfies the condition $f(t_3^*) = 0$. Figure 6.4 shows that $t_3 = 0.2184$ satisfies that condition. Then, using this optimal value of t_3^*, the optimal value of P_2 is determined. To determine the optimal value of P_2, we initialize $P_2 = 50$. However, $f(P_2 = 50) = -130.25340$, which is not equal to zero.

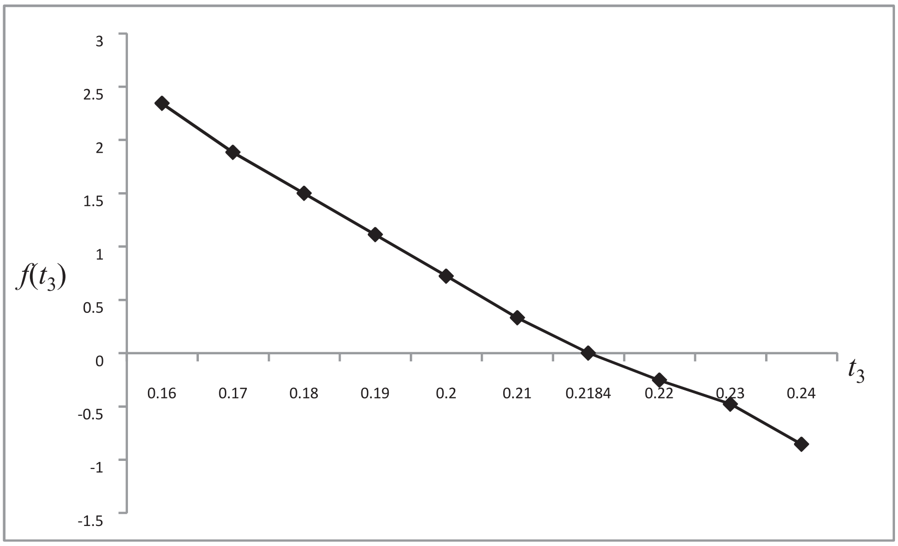

Figure 6.4: The optimal time at which inventory becomes zero with price–dependent demand and complete backlogging

Now the value of P_2 is decreased until it satisfies the condition $f(P_2^*) = 0$. Figure (6.5) shows that $P_2 = 40.15$ satisfies the optimal condition.

Once t_3^*, P_1^*, and P_2^* are determined, the optimal total cost, Γ_c^*, optimal sales revenue, SR^*, the optimal total profit, π^*, the optimal order quantity, Q^*, and the optimal backlogged quantity B^* are calculated, respectively. The optimal total cost $\Gamma_c^* = \$81701$ per unit time and the optimal sales revenue $SR^* = \$106710$ per unit time.

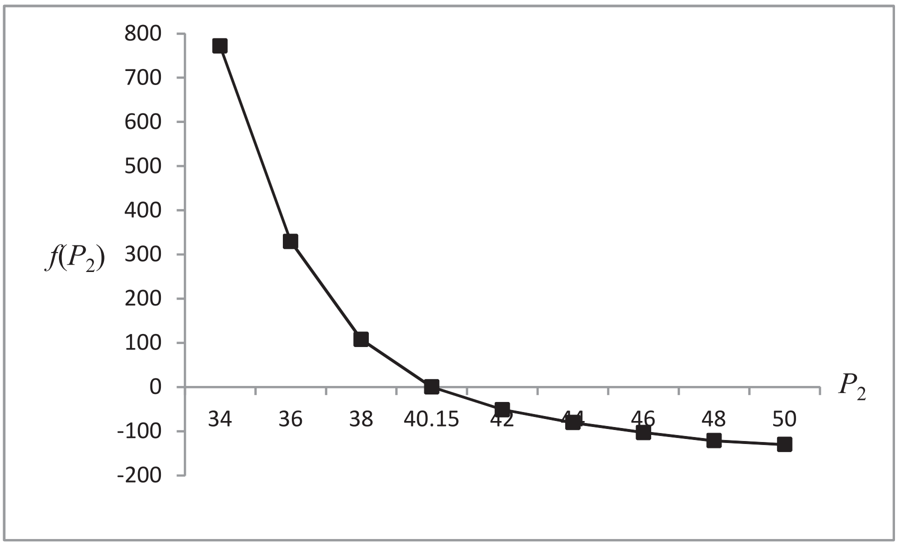

Figure 6.5: The optimal value of P_2 for complete backlogging

$$\text{Therefore, the optimal total profit,}\pi(t_3^*, P_1^*, P_2^*) = (SR^* - \Gamma_c^*)$$
$$= (\$106710 - \$81701)$$
$$= \$25009 \text{ per unit time.}$$

The optimal order quantity is given by

$$Q^* = D_1(P_1^*) \int_0^{t_1} e^{\alpha x^\beta}\, dx + D_2(P_2^*) \int_{t_2}^{t_3^*} e^{\alpha x^\beta}\, dx + e^{\alpha t_2^\beta} \int_{t_3^*}^{T} D_2(P_2^*)\, dx$$
$$= 664.95 \text{ units,}$$

and the optimal baclogged quantity is given by

$$B^* = \int_{t_3^*}^{T} D_2(P_2^*)\, dx \ = 140.73 \text{ units.}$$

Some sensitivity analysis of this model is performed by changing the paratemer values. The results are presented in Table 6.2

Table 6.2: Sensitivity analysis for the two–market model with price–dependent demand and complete backlogging

Parameters	Changing values	t_3^*	P_1^*	P_2^*	Γ_c^*	π^*
Panel 1						
	15	0.2207	27.602	25.065	258310	153240
C	30	0.2184	44.632	40.150	81701	25009
	45	0.2012	65.020	61.930	23243	8126.9
Panel 2						
	1.5	0.2201	44.600	40.033	83913	25294
h	3.0	0.2184	44.632	40.150	81701	25009
	4.5	0.2139	44.671	40.219	80841	24360
Panel 3						
	0.15	0.2206	44.350	39.823	82913	25919
α	0.30	0.2184	44.632	40.150	81701	25009
	0.45	0.2031	44.710	41.682	71590	24791
Panel 4						
	1	0.1767	44.780	42.520	67981	22162
β	2	0.2184	44.632	40.150	81701	25009
	3	0.2197	44.498	39.910	83045	25466
Panel 5						
	0.5	0.2189	44.596	40.070	82293	25012
C_1	1.0	0.2184	44.632	40.150	81701	25009
	1.5	0.2146	44.658	40.191	81442	24965
Panel 6						
	16	0.1821	44.632	40.080	82358	25202
C_2	32	0.2184	44.632	40.150	81701	25009
	48	0.2205	44.632	40.345	80283	24812

Table 6.2 shows the same characteristics as Table 6.1. In other words, the total profit and the total cost decrease but the selling price increases with cost parameters.

6.2 One–Market Model with Discounting

From time 0 to t_1, demand, $D_1(P_1) = a_1 P_1^{-b_1}$, where $a_1 = 7 \times 10^7$ and $b_1 = 3.2$, are the same as in the high–end market of two–market model and P_1 is the same as the optimally determined selling price of the high–end market of the two–market model. After time t_1, demand $D_d(P_d) = a_d P_d^{-b_d}$, where $a_d = 7 \times 10^9$, $b_d = 3.9$, and P_d is the discounted selling price, which will be detemined to obtain the same profit as in the two–market model. All parameters are the same as in the two–market model. The only difference is at time t_1, a discount on selling price is offered instead of transporting the items from the high–end market to the low–end market.

6.2.1 Partial Backlogging

In order to determine the discounted selling peice P_d, Algorithm 1 is applied. Table 6.3 shows, for P_d = \$40.87, we obtain that the profit from the one–market model with discounting is the same as the profit from the two–market model. Hence, the optimal time, $t_3^* = 0.21802$, the total order quantity, $Q^* = 595.66$ units, and the total backlogged quantity, $B^* = 118.40$ units.

Table 6.3: Discounted selling price for price–dependent demand with partial backlogging

No	t_3	P_d	Total profit
1	0.21750	38.00	22910
2	0.21770	38.50	23331
3	0.21782	39.00	23600
4	0.21795	39.50	23805
5	0.21802	40.87	24088

6.2.2 Complete Backlogging

The optimal total profit is \$25009 per unit time for the two–market model with complete backlogging. In order to determine the discounted selling price P_d, we apply an iterative method. The summery of the iteration is given in Tabel 6.4. The discounted selling price, $P_d = \$40.55$, the optimal time, $t_3^* = 0.2035$, the total order quantity, $Q = 695.84$ units, and the total backlogged quantity, $B = 202.48$ units. Now, we investigate the relationship

Table 6.4: Discounted selling price for price–dependent demand with complete backlogging

No	t_3	P_d	Total profit
1	0.2035	38.85	24344
2	0.2035	39.00	24469
3	0.2035	39.45	24587
4	0.2035	39.85	24662
5	0.2035	40.55	25009

between the discounting time and discounted selling price for price–dependent demand. In Figure 6.6, at each point the profit is the same. This Figure shows that, if one offers discount early, items can be sold at a lower discounted price. If one delays the discount, the items are sold at a higher price for a longer period, and in price-dependent demand that has a reverse effect on profit because of decrease in demand rate. As a result, the discounted price increases very sharply with an increase in discounting time. This figure indicates that there is a critical time that makes the difference in the discounted selling price. After that time, the discounted selling price does not change much and becomes constant. This indicates that no one is interested in buying the old items beyond a certain discounted selling price.

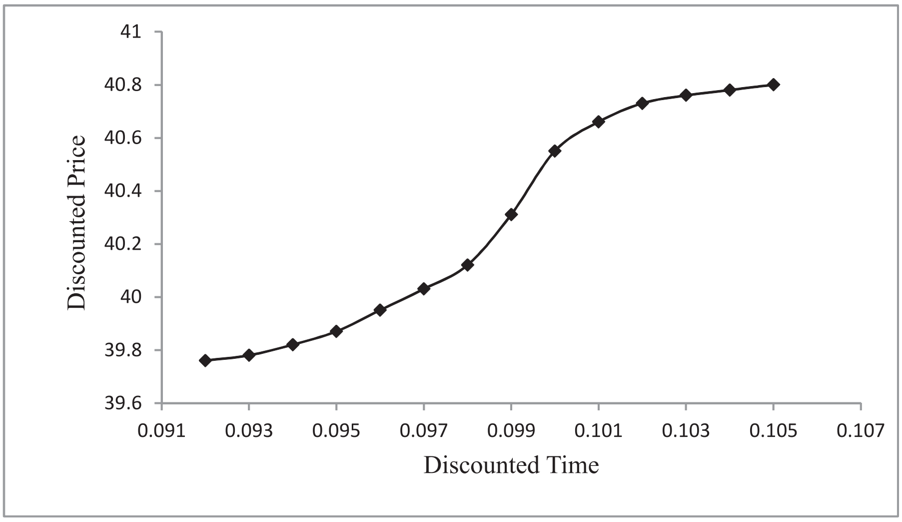

Figure 6.6: Relationship between discounting time and discounted selling price for price–dependent demand

6.3 One–Market Model

Demand for the one–market model is $D_o(P_o) = a_o P_o^{-b}$, where $a_o = 5 \times 10^9$ and $b_o = 3.9$ and the rest of the input parameters remain as same as in the two–market model. The aim is to determine the selling price P_o that results in the same profit as in the two–market model.

6.3.1 Partial Backlogging

In order to determine P_o, we apply Algorithm 1 which is presented in Section 5.2.1. Table 6.5 shows the summary of the iterations. From Section 6.1.1, we obtain the optimal selling price of high–end market $P_1^* = \$44.632$ and the optimal selling price of low–end market $P_2^* = \$40.26$, which results in the optimal total profit $\pi^* = \$24088$ per unit time. Table 6.5 shows, for $P_o = \$42.87$, we obtain the same total profit, which implies the selling price of one–market model is less than the high–end market selling price but greater than the low–end market selling price in the two–market model. In addition, the optimal order quantity, $Q^* = 507.93$ units, and the optimal backlogged quantity, $B^* = 101.17$ units.

Table 6.5: Selling price for the one–market model with price–dependent demand and partial backlogging

No	P_o	t_3^*	Total profit
1	40.00	0.2204	22814
2	40.50	0.2207	23013
3	41.00	0.2209	23151
4	42.00	0.2210	23698
5	42.87	0.2211	24088

6.3.2 Complete Backlogging

For complete backlogging, Table 6.6 shows that, for P_o = \$42.23, we obtain the same profit as in the two–market model. The optimal order quantity, $Q^* = 602.74$ units, and the optimal backlogged quantity, $B^* = 98.56$ units.

Table 6.6: Selling price for the one–market model with price–dependent demand and complete backlogging

No	P_o	t_3^*	Total profit
1	41.40	0.2157	23901
2	41.60	0.2157	24361
3	41.80	0.2157	24579
4	42.00	0.2157	24825
5	42.23	0.2157	25009

Chapter 7

Conclusion

The contribution of this thesis is developing EOQ models considering a two–market model, price–dependent demand, time–dependent demand, complete backlogging, and partial backlogging. In this thesis, three different EOQ models are developed for both time–dependent demand and price–dependent demand. The first EOQ model is developed for two different markets (high–end market and low–end market). In this model the items are first sold at the high–end market at a higher price for a given period of time and then the leftover inventory is transferred to a low–end market where the items are sold at a lower price. At the low–end market, when the inventory becomes zero, the demand is backlogged either partially or completely. The second EOQ model is developed to generate the same amount of profit that is obtained from the first EOQ model, but the items are sold only at the high–end market and the price is discounted after a certain time. When the inventory becomes zero, the demand is backlogged either partially or completely. The third EOQ model is developed to generate the same amount of profit that is obtained from the first EOQ model; however, the items are sold only at the high–end market without any price discount. When the inventory becomes zero, the demand is backlogged either partially or completely.

It has been proved that there exists an optimal time at which inventory becomes zero, and the total profit is a concave function in all scenarios investigated. It has been shown that the discounted price in the second EOQ model is lower than the price at the high-end market and the price at the low-end market. Furthermore, the relationship between discounted price and time of discounting has been investigated, and it is shown that there is a critical time that makes a difference in the discounted price.

This research can be extended by considering stock–dependent demand and also stochastic demand.

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