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# Film flow with temperature dependent fluid properties over heated inclined surfaces

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# FILM FLOW WITH TEMPERATURE DEPENDENT FLUID PROPERTIES OVER HEATED INCLINED SURFACES

by

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## Abstract

The gravity driven film flow down a heated inclined ramp and how it is affected by temperature dependent fluid properties is examined. The five temperature dependent fluid properties examined were: surface tension, mass density, dynamic viscosity, thermal conductivity and specific heat capacity.

The investigation utilized a theoretical model based on the conservation of mass, momentum and energy, including the physically appropriate Newton's Law of Cooling to incorporate temperature changes on the surface of the film. A two-scale model of this system was also considered and a Benney equation was derived. A depth-integrated model was also considered and modified Integral Boundary Layer (IBL) equations were generated. A linear stability analysis was carried out in all cases.

Numerical simulations were carried out on the nonlinear modified IBL equations and their agreement to the linear approximations was good. The nonlinear analysis was also used to determine the evolution of the unstable flow.

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## CHAPTER 1 - INTRODUCTION

The purpose of this thesis will be to examine how the flow of a thin fluid film down a heated inclined plane is affected by temperature dependent fluid properties. When a gravity-driven fluid film is heated, the variation in its surface tension with temperature can combine with the inertial effects of the flow to generate interfacial instability. Other fluid properties, like mass density, dynamic viscosity, thermal conductivity and specific heat capacity can also be significantly affected by changes in temperature. While the effect of variable surface tension on interfacial instability has been thoroughly investigated, very little has been done in studying the effect of temperature dependence in the other fluid properties. This thesis will examine the combined effect of temperature variation in all the fluid properties.

The investigation will be carried out by implementing a theoretical model based on the conservation of mass, momentum and energy. This model will exploit the assumed shallowness of the fluid layer, and will incorporate the physically appropriate temperature dependence of the fluid properties. The thesis will report on linear and nonlinear stability analyses of the model. The linear analysis provides analytic results and predicts the critical conditions under which the equilibrium flow becomes unstable. The nonlinear analysis involves numerical approximations of the governing equations, and will reveal if nonlinear effects will significantly alter the predictions of the linear theory. The nonlinear analysis will also be used to determine the evolution of the unstable flow.

### A. History

The flows of thin fluid films exist in various aspects of daily life. In engineering applications we see their usage in distillation units, condensers and heat exchangers. In geophysical events we see thin fluid films in the forms of gravity currents, mud, granular and debris flow, snow avalanches and lava flows. In biological systems we find thin fluid films lining the airways in the lungs and thin tear films covering the eye. These are just a few of the many examples of daily occurrences of thin fluid films.

Although these occurrences seem very different with little in common with each other, they can all be modeled using the same mathematical principles. Much of the mathematics for the flow of thin fluid films is based on the experiments of Tower (1884) and the theoretical work of Reynolds (1886).

The idea behind the modeling of thin fluid film flows is to track the interfacial position while simultaneously solving the full governing fluid equations, along with any other relevant equations (i.e. electrostatic forces, temperature or chemical concentrations) required to do so. This is often quite difficult or impossible due to the complexity of the equations involved. However, certain simplifications can be made by assuming the film flow can be classified as a thin fluid film.

Thin fluid films are modeled using the Navier-Stokes equations (plus any other relevant equations, as mentioned earlier). But in a thin fluid film we recognize that there is a large disparity between the vertical and lateral length scales; the lateral length scale is much larger than the vertical length scale. This gives rise to small aspect ratios that provide the small parameters necessary for perturbation expansions. In other words, we can simplify the model by filtering out the explicit dependence on the depth coordinate.

The particular case of an isothermal film falling down a planar substrate was first observed in the experiments of Kapitza and Kapitza (1949). Yih (1955) carried out a linear stability analysis of the flow by employing the Orr-Sommerfeld equation. Using a numerical method he calculated the critical conditions for instability, however his results were imprecise. Benjamin (1957) performed a similar calculation, but used analytic results rather than numerical ones. Benjamin's values were more in accordance with experiments done by Binnie (1957). Yih (1963) redid his analysis using a perturbation expansion. This approach was much simpler than his earlier approach and provided results in agreement with Benjamin's. Both Benjamin's (1957) and Yih's (1963) calculations predicted a critical Reynolds number of $(5/6)\cot\beta$, where $\beta$ is the angle of inclination. This result was verified by the experiments of Liu et al. (1993).

Benney (1966) applied a long wave expansion (the lateral length scale is much larger than the vertical length scale) leading to a single nonlinear evolution equation for the interfacial position for the free surface. Although this approach was accurate for determining critical conditions, it failed to correctly describe nonlinear waves far from criticality.

Shkadov (1967) attempted an integral boundary layer (IBL) approximation, which combined the boundary layer approximation of the Navier-Stokes equations assuming a self-similar parabolic velocity profile and long waves on the interface with the Karman-Pohlhausen averaging method in boundary layer theory. This procedure resulted in two equations in two unknowns (the unknowns being the flow thickness and flow rate). While this procedure was successful in describing nonlinear waves far from criticality, there were a few problems, one being an erroneous prediction of the critical Reynolds number as $\cot\beta$, instead of the correct value of $(5/6)\cot\beta$.

Ruyer-Quil and Manneville (2000) used a modified IBL approach, where they combined a gradient expansion with a weighted residual technique using polynomial test functions. This approach lead to a similar two-equation result like Shkadov's, but predicted the correct instability threshold. Trevelyan et al. (2007) extended this approach for the basic non-isothermal problem (temperature variation in surface tension only).

In any study of thin fluid film flow with non-uniform surface tension, the Marangoni effect must be considered. Because of the Marangoni effect, surface segments with high surface tension will tend to pull more strongly on the surrounding liquid than segments with lower surface tension. As a result, fluid will flow from areas of lower surface tension to areas of higher surface tension. Surface tension gradients can be created or enhanced by changes in temperature, since surface tension is a function of temperature. So when studying the effects of thin fluid flow down a heated inclined plane, not only must we consider the typical long-wave instability resulting from isothermal flows, as was studied by Kapitza and Kapitza (1949); but in a non-isothermal flow we must also consider instability due to the Marangoni effect. These competing factors and how they interact with each other will need to be considered when studying the instability of a thin fluid flow down a heated inclined plane.

There has been some research done in recent years for the problem of a thin fluid flow down a heated inclined plane. However, this research only considered the surface tension effects (Marangoni effect) arising from the heated plane. Kalliadasis et al. (2003) used the IBL approach. They adopted a linear test function for the temperature combined with a weighted residuals approach for the energy equation and obtained a three-equations model for flow height, flow rate, and temperature (or energy). However, this method suffers the same problems as does the Shkadov method. It does not accurately predict the onset of flow instability and the critical Reynolds number prediction is erroneous, on the order of 20%.

Later, both Ruyer-Quil et al. (2005) and Scheid et al. (2005), building on the work of Kalliadasis et al. (2003) for thin fluid flow down a heated inclined plane and the work of Ruyer-Quil and Manneville (2000) for thin fluid flow down an inclined plane (isothermal case), used a procedure based on a high-order weighted residuals approach combined with a Galerkin projection with polynomial test functions for both velocity and temperature fields. This modified IBL approach suffers none of the shortcomings of the approach used by Kalliadasis et al. (2003).

In 2007, Trevelyan et al. (2007) employed further refinements to the modified IBL approach. Some of these refinements included an energy equation based on a high-order Galerkin projection in terms of polynomial test functions which satisfy all boundary conditions and a numerical solution to the full energy equation; to name a few. Trevelyan et al. (2007) also considered the case where the heat flux is prescribed at the bottom instead of the temperature.

Work has also been done for this problem considering other temperature dependent fluid properties, aside from surface tension. Goussis and Kelly (1985) examined the role of temperature variation in the viscosity only. They performed a linear stability analysis on the Navier-Stokes equations and found that heating a film whose viscosity decreases with temperature has the effect of destabilizing the flow. Their work assumes a prescribed constant temperature at the surface of the fluid (no temperature changes between the surface and the environment). As a result, the Marangoni effect does not play a role. In order to capture the Marangoni effect with the model, we must apply the physically appropriate Newton's Law of Cooling at the liquid-air interface. Hwang and Weng (1988) considered the same problem as Goussis and Kelly (1985), and made the same assumptions but set up a Benney equation and performed a linear and weakly-nonlinear stability analysis on it.

This thesis will extend the basic non-isothermal problem with temperature dependent surface tension, to also include temperature variation in mass density, dynamic viscosity, thermal conductivity and specific heat.

In the non-isothermal case mass density, dynamic viscosity, thermal conductivity, specific heat and surface tension are all affected by temperature. Our equations, based on the conservation of mass, momentum and energy, will need to capture these effects. This is unlike the previous cases cited above since we are examining the simultaneous effects of five different temperature dependent fluid properties. We also implement the physically realistic Newton's Law of Cooling at the surface of the fluid film.

With this system, we will take two approximation approaches. One will be based on the modified IBL approach, used by Ruyer-Quil and Manneville (2000) in the isothermal problem and later extended by Trevelyan et al. (2007) for the basic non-isothermal problem. The other will be based on the long wave expansions pioneered by Benney (1966). We will also solve the full equations analytically, without the use of these approximation techniques, for special cases. We will then compare these results with each other and with numerical simulations to test the accuracy of our model.

## CHAPTER 2 – Governing Equations

### A. Conservation Equations for the General Case with Variable Fluid Properties

We model our fluid flow down a heated inclined plane based on the conservation of mass, momentum and energy. Consider the diagram below:

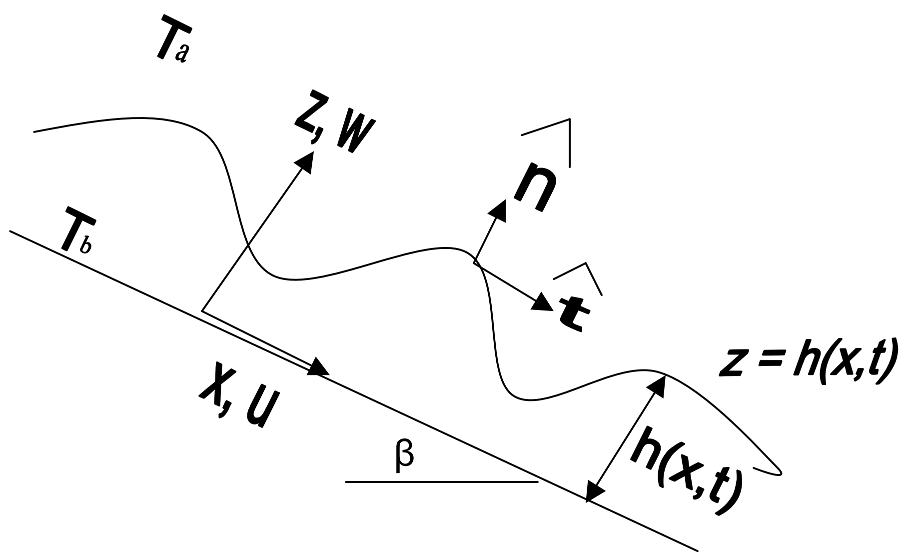


Figure 1.1: The Flow Configuration

We assume that the inclined surface along which the fluid is flowing is even and impermeable. We also assume that the flow is two-dimensional. So the flow is dependent on x and z but not on y. In other words the flow moves in a downhill direction and the height may change, but there is no change in breadth (across the flow). $T_a$ refers to the temperature of the ambient medium and $T_b$ ($>T_a$) refers to the temperature of the ramp. The velocity components in the x and z directions are denoted by u and w respectively.

Applying these assumptions yields the general two dimensional Navier-Stokes equations (equations of motion), where $\frac{D}{Dt} = \partial_t + u\partial_x + w\partial_z$:

$$\frac{D(\rho u)}{Dt} = -p_x + \rho g \sin\beta + [\mu u_x]_x + [\mu u_z]_z + \mu_x u_x + \mu_z w_x \qquad \text{x-Momentum Equation}$$

$$\frac{D(\rho w)}{Dt} = -p_z - \rho g \cos\beta + [\mu w_x]_x + [\mu w_z]_z + \mu_x u_z + \mu_z w_z \qquad \text{z-Momentum Equation}$$

$$\frac{D\rho}{Dt} + \rho(u_x + w_z) = 0 \qquad \text{Continuity Equation}$$

$$\frac{D}{Dt}(c_p \rho T) = [KT_x]_x + [KT_z]_z \qquad \text{Temperature Equation}$$

where p=pressure, ρ=mass density, μ=dynamic viscosity, g=acceleration due to gravity, $c_p$=specific heat, T=Temperature, K=thermal conductivity. These equations are general in the sense that they apply to fluid flows with variable fluid properties.

**B. The Form of the Temperature Variation of the Fluid Properties**

Since we are testing the effects of temperature dependent fluid properties, we allow the properties of the fluid (mass density, dynamic viscosity, thermal conductivity, specific heat capacity and σ=surface tension) to vary with temperature as follows:

$$\rho = \rho_0[1-\hat{\alpha}(T-T_a)]$$
$$\sigma = \sigma_0 - \gamma(T-T_a)$$
$$K = K_0 + \hat{\Lambda}(T-T_a)$$
$$\mu = \mu_0 - \hat{\lambda}(T-T_a)$$
$$c_p = c_{p_0} + \hat{S}(T-T_a)$$

where $\hat{\alpha}, \gamma, \hat{\Lambda}, \hat{\lambda}$ and $\hat{S}$ are positive parameters measuring the rate of change with respect to the temperature. Also note that the expression for μ is a simplification of the commonly assumed Arrhenius-type exponential relation $\mu = \mu_0 e^{-\frac{\hat{\lambda}}{\mu_0}(T-T_a)}$, used, for example, to describe the viscosity of lava flows and ice sheets (Craster and Matar (2009)). The linear formulation was initially employed and justified by Reisfeld and Bankoff (1990).

**C. The Boussinesq Approximation**

Before we move on to the boundary conditions (conditions at the interfaces), we make a simplification by applying the Boussinesq approximation. This approximation assumes mass density to be constant ($\rho=\rho_0$) except where it appears in a gravitational term. Thus our first four equations become:

$$\frac{Du}{Dt} = \frac{-p_x}{\rho_0} + [1-\alpha(T-T_a)]g\sin\beta + [\nu u_x]_x + [\nu u_z]_z + \nu_x u_x + \nu_z w_x \quad \text{x-Momentum Equation}$$

$$\frac{Dw}{Dt} = \frac{-p_z}{\rho_0} - [1-\alpha(T-T_a)]g\cos\beta + [\nu w_x]_x + [\nu w_z]_z + \nu_x u_z + \nu_z w_z \quad \text{z-Momentum Equation}$$

$$u_x + w_z = 0 \quad \text{Continuity Equation}$$

$$\frac{D}{Dt}(c_p \rho_0 T) = [KT_x]_x + [KT_z]_z \quad \text{Temperature Equation}$$

where $\nu = \frac{\mu}{\rho_0} = \frac{\mu_0}{\rho_0} - \frac{\hat{\lambda}}{\rho_0}(T-T_a)$, is defined as the kinematic viscosity.

## D. Interface Conditions

At the bottom of the fluid flow, where z=0, there is no slip and no penetration (earlier we had assumed that the surface along which the fluid is flowing is even and impermeable) so

$u = w = 0$
$T = T_b$

At the free surface, where z=h(x,t), we assume that the ambient atmosphere does not exert a force on the fluid film. Thus, the force balance vector is given as follows:

$$(-p\vec{\vec{I}} + \vec{\vec{\tau}})\hat{n} = -\kappa\sigma\hat{n} - \frac{\partial}{\partial s}\sigma\hat{t}$$

where $\vec{\vec{I}}$ is the 2x2 Identity matrix, $\hat{n} = \frac{1}{\sqrt{h_x^2+1}}(-h_x, 1)$ is the outward pointing unit normal vector to the surface, $\hat{t} = \frac{1}{\sqrt{h_x^2+1}}(1, h_x)$ is the unit tangent vector to the surface, $\frac{\partial}{\partial s}$ is the derivative with respect to arc length, $\frac{\partial}{\partial s}\sigma = \vec{\nabla}\sigma \bullet \hat{t}$, with $\nabla = (\partial_x, \partial_z)$, κ = curvature = $-\vec{\nabla} \bullet \hat{n} = \frac{h_{xx}}{[1+(h_x)^2]^{3/2}}$, and $\vec{\vec{\tau}} = \mu\begin{bmatrix} 2u_x & u_z + w_x \\ u_z + w_x & 2w_z \end{bmatrix}$ is the deviatoric stress tensor.

To obtain the **normal component** we use the dot product of the force balance vector with $\hat{n}$. This gives us the following equation:

$$\left[(-p\vec{\vec{I}} + \vec{\vec{\tau}})\hat{n}\right] \bullet \hat{n} = \frac{\sigma h_{xx}}{[1+(h_x)^2]^{3/2}}$$

For the **tangential component** we use the dot product of the force balance vector with $\hat{t}$. This gives us the following equation:

$$\left[(-p\vec{\vec{I}} + \vec{\vec{\tau}})\hat{n}\right] \bullet \hat{t} = -\vec{\nabla}\sigma \bullet \hat{t} = \frac{-\gamma(T_x + h_x T_z)}{[1+(h_x)^2]^{1/2}}$$

We assume there are no evaporation effects, so the mass of fluid at the surface is conserved (no fluid is gained or lost). This yields the following equation:
$w = h_t + u h_x$ at z = h(x,t)

This is also known as the **kinematic condition** and can be written as $\frac{D}{Dt}(z-h) = 0$.

### E. Newton's Law of Cooling

Since our fluid is heated we also need to consider Newton's Law of Cooling. The heat flux across the surface will be changing at a rate that is proportional to the difference between the surface temperature and the ambient temperature. This gives us the following energy condition at the surface (z=h(x,t)):

$K\vec{\nabla}T \bullet \hat{n} = -\alpha_g (T - T_a)$,  where $\alpha_g$ = heat transfer coefficient

This set of equations is a model for our system. However, before we begin using this model, we will scale the variables to create non-dimensional equations.

### F. Scaling

Finally, we will make our equations non-dimensional by scaling them. To do this we consider the uniform and steady isothermal flow for the problem. Under these conditions the x and t derivatives, as well as w are all zero. Our x-momentum equation then reduces to:

$\frac{\mu_0}{\rho_0} u_{zz} + g\sin\beta = 0$, along with the boundary conditions u=0 at z=0 and $u_z$=0 at z=h(x,t), which arises from the tangential component of force at the surface.

Solving this problem yields the following equation:

$$u(z) = \frac{g\rho_0 \sin\beta}{2\mu_0}(2hz - z^2)$$

We then "depth average" u(z), and obtain our velocity scale U:

$$U \equiv \frac{1}{h}\int_0^h u(z)dz = \frac{\rho_0 g\sin\beta}{3\mu_0} h^2$$

Solving for h, we obtain the Nusselt thickness, H:

$$H = \left(\frac{3\mu_0 U}{\rho_0 g\sin\beta}\right)^{1/2}$$

which will serve as our length scale.

We now scale our equations of motion, using the following transformation (where the *s denote the non-dimensional quantities):

$x=Hx^*$, $z=Hz^*$, $h=Hh^*$, $u=Uu^*$, $w=Uw^*$, $t=(H/U)t^*$, $p=\rho_0 U^2 p^*$, $T-T_a=\Delta T T^*$, where $\Delta T=T_b-T_a$

As a result, the following non-dimensional numbers will be introduced:

$S = \frac{\hat{S}\Delta T}{c_{p_0}}$, scaled specific heat gradient

$\Lambda = \frac{\hat{\Lambda}\Delta T}{K_0}$, scaled thermal conductivity gradient

$\alpha = \hat{\alpha}\Delta T$ , scaled mass density gradient

$\lambda = \frac{\hat{\lambda}}{\mu_0}\Delta T$ , scaled dynamic viscosity gradient

$\Delta T_r = \frac{\Delta T}{T_a} = \frac{T_b - T_a}{T_a}$ , relative temperature difference

$Re = \frac{\rho_0 U H}{\mu_0}$, the Reynolds number (inertial forces/viscous forces)

$We = \frac{\sigma_0}{\rho_0 U^2 H}$, the Weber number (surface tension parameter)

$Pr = \frac{\mu_0 c_{p_0}}{K_0}$, the Prandtl number

$Ma = \frac{\gamma \Delta T}{\rho_0 U^2 H}$, the Marangoni number (scaled surface tension gradient)

$Bi = \frac{\alpha_g H}{K_0}$, the Biot number (scaled heat transfer coefficient)

## G. The Full Equations of Motion With Temperature Dependent Properties

The non-dimensional governing equations can be expressed as (dropping the *s for notational convenience):

$$u_x + w_z = 0 \tag{1}$$

$$\begin{aligned} Re(u_t + uu_x + wu_z) = -Rep_x + 3(1-\alpha T) + [(1-\lambda T)u_x]_x + [(1-\lambda T)u_z]_z + (1-\lambda T)_x u_x \\ + (1-\lambda T)_z w_x \end{aligned} \tag{2}$$

$$\begin{aligned} Re(w_t + uw_x + ww_z) = -Rep_z - 3\cot\beta(1-\alpha T) + [(1-\lambda T)w_x]_x + [(1-\lambda T)w_z]_z \\ + (1-\lambda T)_x u_z + (1-\lambda T)_z w_z \end{aligned} \tag{3}$$

$$PrRe \cdot \frac{D}{Dt}[(1 + S/\Delta T_r)T + ST^2] = [(1+\Lambda T)T_x]_x + [(1+\Lambda T)T_z]_z \tag{4}$$

Boundary Conditions (at z=0)

$$u=w=0 \tag{5}$$

$$T=1 \tag{6}$$

Boundary Conditions (at z=h(x,t))

$$p = \frac{2}{Re}\frac{(1-\lambda T)}{1+h_x^2}[h_x^2 u_x + w_z - h_x(u_z + w_x)] - \frac{(We - MaT)h_{xx}}{(1+h_x^2)^{3/2}} \quad \text{(normal component of force)} \tag{7}$$

$$[-4h_x u_x + (1-h_x^2)(u_z + w_x)]\frac{(1-\lambda T)}{1+h_x^2} = \frac{-MaRe(T_x + h_x T_z)}{(1+h_x^2)^{1/2}} \quad \text{(tangential component of force)} \tag{8}$$

$$(1+\Lambda T)(T_z - h_x T_x) = -BiT[1+h_x^2]^{1/2} \tag{9}$$

$$w = h_t + uh_x \tag{10}$$

These 10 equations provide a mathematical model of our problem based on the conservation of mass, momentum and energy.

Although these equations are too complex to be solved analytically, in the next chapter we consider two special cases where an analysis is possible. The first special case is where we set λ=0 and Λ=0, which corresponds to assuming no variation in the viscosity and thermal conductivity. The second case is where we set Bi=0. This corresponds to no heat transfer across the surface (in other words the surface is assumed to be thermally insulated).

## CHAPTER 3 – Linear Stability Analysis

We will employ a linear stability analysis of the full equations to investigate the instability of the steady and uniform flow given by:

$T_s=T_s(z)$, $u_s=u_s(z)$, $w_s=0$, $p_s=p_s(z)$, $h_s=1$,

where the functions $T_s(z)$, $u_s(z)$ and $p_s(z)$ are the solutions to the following problems:

Temperature From (4), (6), (9):

$$[(1+\Lambda T_s)T_{sz}]_z = 0 \tag{11}$$

$$(1+\Lambda T_s)T_{sz} = -BiT_s \qquad \text{(at z=1)} \tag{12}$$

$$T_s = 1 \qquad \text{(at z=0)} \tag{13}$$

Velocity From (2), (5), (8):

$$[(1-\lambda T_s)u_{sz}]_z + 3(1-\alpha T_s) = 0 \tag{14}$$

$$u_{sz} = 0 \qquad \text{(at z=1)} \tag{15}$$

$$u_s = 0 \qquad \text{(at z=0)} \tag{16}$$

Pressure From (3), (7):

$$Rep_{sz} = -3\cot\beta(1-\alpha T_s) \tag{17}$$

$$p_s = 0 \qquad \text{(at z=1)} \tag{18}$$

We introduce a small perturbation into our variables (denoted by ~). This will result in a perturbed flow, as follows:

$$u = u_s(z)+\tilde{u}(x,z,t)$$
$$w = \tilde{w}(x,z,t)$$
$$p = p_s(z)+\tilde{p}(x,z,t)$$
$$T = T_s(z)+\tilde{T}(x,z,t)$$
$$h(x,t) = 1+\eta(x,t)$$

We then substitute these equations into our governing equations ((1)-(10)) and linearize with respect to the perturbations: $\tilde{u},\tilde{w},\tilde{p},\tilde{T},\eta$. Our linearized system is as follows:

Linearized Perturbation Equations:

$$\tilde{u}_x + \tilde{w}_z = 0 \tag{19}$$

$$Re(\tilde{u}_t + u_s\tilde{u}_x + \tilde{w}u_{sz}) = -Re\tilde{p}_x - 3\alpha\tilde{T} + (1-\lambda T_s)\tilde{u}_{xx} + [(1-\lambda T_s)\tilde{u}_z]_z - \lambda[u_{sz}\tilde{T}]_z - \lambda T_{sz}\tilde{w}_x \tag{20}$$

$$\mathrm{Re}(\tilde{w}_t + u_s\tilde{w}_x) = -\mathrm{Re}\tilde{p}_z + 3\alpha\cot\beta\tilde{T} + (1-\lambda T_s)\tilde{w}_{xx} + [(1-\lambda T_s)\tilde{w}_z]_z - \lambda\tilde{T}_x u_{sz} - \lambda T_{sz}\tilde{w}_z \qquad (21)$$

$$\mathrm{PrRe}\cdot(1 + S/\Delta T_r + 2ST_s)(\tilde{T}_t + u_s\tilde{T}_x + \tilde{w}T_{sz}) = (1+\Lambda T_s)\tilde{T}_{xx} + [(1+\Lambda T_s)\tilde{T}]_{zz} \qquad (22)$$

<u>Conditions at the Interfaces (at z=0)</u>

$$\tilde{u} = \tilde{w} = 0 \qquad (23)$$

$$\tilde{T} = 0 \qquad (24)$$

<u>Conditions at the Interfaces (at z=1)</u>

$$\tilde{p} = -\eta p_{sz} + \frac{2}{\mathrm{Re}}(1-\lambda T_s)\tilde{w}_z - (\mathrm{We} - \mathrm{Ma}T_s)\eta_{xx} \qquad (25)$$

$$(1-\lambda T_s)(\eta u_{szz} + \tilde{u}_z + \tilde{w}_x) = -\mathrm{MaRe}(\tilde{T}_x + T_{sz}\eta_x) \qquad (26)$$

$$[(1+\Lambda T_s)\tilde{T}]_z + \eta[(1+\Lambda T_s)T_{sz} + \mathrm{Bi}T_s]_z = -\mathrm{Bi}\tilde{T} \qquad (27)$$

$$\tilde{w} = \eta_t + u_s\eta_x \qquad (28)$$

These 10 Equations ((19)-(28)) form our linearized perturbation equations. Just like the nonlinear governing equations, they are too complex to be solved analytically. However, as we had noted earlier, there are two special cases where these linearized equations can be solved analytically. The first special case is where we allow no heat transfer across the surface (the surface is assumed to be thermally insulated, Bi=0). The second special case is where we allow temperature variation in only the specific heat, surface tension and mass density (λ=0 and Λ=0).

## <u>A. Full Equations, Special Case with Bi=0</u>

This special case corresponds to no heat transfer across the free surface and amounts to the free surface being perfectly insulated, resulting in a uniform temperature of the equilibrium flow. However, small perturbations to this temperature can occur resulting in perturbations in the temperature dependent fluid properties.

We start by finding explicit solutions for $u_s, p_s, T_s$, the base flow, by solving (11)-(18).

<u>Temperature:</u>

$$[(1+\Lambda T_s)T_{sz}]_z = 0$$

$$(1+\Lambda T_s)T_{sz} = 0 \qquad \text{(at z=1)}$$

$$T_s = 1 \qquad \text{(at z=0)}$$

Solving this problem gives us $T_s(z) \equiv 1$.

Velocity (using $T_s(z) \equiv 1$):

$$(1-\lambda)u_{szz} + 3(1-\alpha) = 0$$

$$u_{sz} = 0 \qquad \text{(at z=1)}$$

$$u_s = 0 \qquad \text{(at z=0)}$$

Solving this problem gives us $u_s(z) = 3\dfrac{1-\alpha}{1-\lambda}(z - \dfrac{z^2}{2})$.

Pressure (using $T_s(z) \equiv 1$):

$$Rep_{sz} = -3\cot\beta(1-\alpha)$$

$$p_s = 0 \qquad \text{(at z=1)}$$

Solving this problem gives us $p_s(z) = \dfrac{3(1-\alpha)\cot\beta}{Re}(1-z)$.

We then proceed to the next step of the linear analysis, which is to introduce normal modes and substitute them into our linearized governing equations. Our perturbations are transformed as follows:

$$(\tilde{u}, \tilde{w}, \tilde{p}, \tilde{T}, \eta) = (\hat{u}(z), \hat{w}(z), \hat{p}(z), \hat{T}(z), \hat{\eta})e^{ik(x-ct)}$$

Making this substitution, our perturbation equations now take the following form:

$$\hat{w}_z + ik\hat{u} = 0 \tag{29}$$

$$Re\left[-ik(c-u_s)\hat{u} + u_{sz}\hat{w}\right] = -ikRe\hat{p} - 3\alpha\hat{T} - k^2\hat{u} + \lambda k^2\hat{u} + (1-\lambda)\hat{u}_{zz} - \lambda[u_{sz}\hat{T}]_z \tag{30}$$

$$ikRe(u_s - c)\hat{w} = -Re\hat{p}_z - (1-\lambda)k^2\hat{w} + (1-\lambda)\hat{w}_{zz} - ik\lambda u_{sz}\hat{T} + 3\alpha\cot\beta\hat{T} \tag{31}$$

$$-ikPrRe(c-u_s)(1 + \frac{S}{\Delta T_r} + 2S)\hat{T} = -k^2(1+\Lambda)\hat{T} + (1+\Lambda)\hat{T}_{zz} \tag{32}$$

Conditions at the Bottom (at z=0)

$$\hat{u} = \hat{w} = 0 \tag{33}$$

$$\hat{T} = 0 \tag{34}$$

Conditions at the Surface (at z=1)

$$\hat{p} = \frac{3(1-\alpha)\cot\beta}{Re}\hat{\eta} + \frac{2}{Re}(1-\lambda)\hat{w}_z + k^2(We - Ma)\hat{\eta} \tag{35}$$

$$(1-\lambda)(-3\frac{1-\alpha}{1-\lambda}\hat{\eta} + \hat{u}_z + ik\hat{w}) = -MaReik\hat{T} \tag{36}$$

$$(1+\Lambda)\hat{T}_z = 0 \tag{37}$$

$$\hat{w} = ik\left(-c + (\frac{3}{2})\frac{1-\alpha}{1-\lambda}\right)\hat{\eta} \tag{38}$$

This system constitutes an eigenvalue problem, with c being the parameter to which characteristic values must be assigned to obtain a non-trivial solution (i.e. the eigenvalue). Since we are dealing with long waves, we assume that the variables can be expanded using the wavenumber k, which is very small (since the waves are very long). Thus, the problem can be solved asymptotically as $k \to 0$. Our long-wave expansions are as follows:

$$c = c_0 + kc_1 + k^2 c_2 + O(k^3)$$
$$\hat{u}(z) = u_0(z) + ku_1(z) + k^2 u_2(z) + O(k^3)$$
$$\hat{w}(z) = w_0(z) + kw_1(z) + k^2 w_2(z) + O(k^3)$$
$$\hat{p}(z) = p_0(z) + kp_1(z) + k^2 p_2(z) + O(k^3)$$
$$\hat{T}(z) = T_0(z) + kT_1(z) + k^2 T_2(z) + O(k^3)$$
$$\hat{\eta} = \eta_0 + k\eta_1 + k^2 \eta_2 + O(k^3)$$

We normalize the eigenvalue problem by taking $\eta_0 = 1, \eta_1 = \eta_2 = 0$. We assume that all parameters are of O(1), except for the Weber number, which is large for most fluids. We let $W = k^2 We = O(1)$.

Now we proceed to solve for each of the terms in the long-wave expansions:

<u>The Order 1 Problem:</u>

<u>Temperature:</u>

$$T_{0zz} = 0$$
$$T_{0z} = 0 \qquad \text{(at z=1)}$$
$$T_0 = 0 \qquad \text{(at z=0)}$$

Solving this problem gives us $T_0(z) \equiv 0$.

<u>Continuity:</u>

$$w_{0z} = 0$$
$$w_0 = 0 \qquad \text{(at z=0)}$$

Solving this problem gives us $w_0(z) \equiv 0$.

<u>Velocity:</u>

$$(1-\lambda)u_{0zz} = 0$$
$$u_{0z} = 3\frac{1-\alpha}{1-\lambda} \qquad \text{(at z=1)}$$
$$u_0 = 0 \qquad \text{(at z=0)}$$

Solving this problem gives us $u_0(z) = 3\dfrac{1-\alpha}{1-\lambda}z$.

Pressure:

$$\mathrm{Re}\, p_{0z} = 0$$

$$p_0 = \frac{3\cot\beta(1-\alpha)}{\mathrm{Re}} + W \qquad \text{(at z=1)}$$

Solving this problem gives us $p_0(z) = \dfrac{3(1-\alpha)\cot\beta}{\mathrm{Re}} + W$ .

The Order k Problem:

Continuity:

$$w_{1z} = -iu_0$$

$$w_1 = 0 \qquad \text{(at z=0)}$$

Solving this problem gives us $w_1(z) = -\dfrac{3}{2} i \dfrac{1-\alpha}{1-\lambda} z^2$ .

Kinematic Condition:

$$w_1 = i(-c_0 + \frac{3}{2}\frac{1-\alpha}{1-\lambda}) \quad \text{(at z=1)}$$

Solving this equation gives us $c_0 = 3\dfrac{1-\alpha}{1-\lambda}$ (we will need $c_0$ in a later calculation).

Temperature:

$$T_{1zz} = 0$$

$$T_{1z} = 0 \qquad \text{(at z=1)}$$

$$T_1 = 0 \qquad \text{(at z=0)}$$

Solving this problem gives us $T_1(z) \equiv 0$ .

x-Momentum (Velocity):

$$(1-\lambda)u_{1zz} = i\mathrm{Re}\, p_0 - i\mathrm{Re}(c_0 - u_s)u_0 + \mathrm{Re}\, u_{sz} w_1$$

$$u_{1z} = 0 \qquad \text{(at z=1)}$$

$$u_1 = 0 \qquad \text{(at z=0)}$$

Solving this problem gives us

$$u_1(z) = \frac{i\mathrm{Re}}{2(1-\lambda)}\left(\frac{3(1-\alpha)\cot\beta}{\mathrm{Re}} + W\right)(z^2 - 2z) + i\mathrm{Re}\, 9\frac{(1-\alpha)^2}{(1-\lambda)^3}\left(\frac{-z^3}{6} + \frac{z^4}{24} + \frac{z}{3}\right).$$

The Order $k^2$ Problem:

Continuity:

$$w_{2z} = -iu_1$$

$$w_2 = 0 \qquad \text{(at z=0)}$$

Solving this problem gives us:

$$w_2(z) = \frac{Re}{2(1-\lambda)}\left(\frac{3(1-\alpha)\cot\beta}{Re} + W\right)\left(\frac{z^3}{3} - z^2\right) + 9Re\frac{(1-\alpha)^2}{(1-\lambda)^3}\left(\frac{-z^4}{24} + \frac{z^5}{120} + \frac{z^2}{6}\right).$$

<u>Kinematic Condition:</u>

$w_2 = -ic_1$ (at z=1)

This gives the expression for $c_1$ as $c_1 = iw_2(1)$.

Now that we have all the terms needed for the long-wave expansions, we go back to the normal modes and continue our analysis. Since c is a complex number, we separate it into its real and imaginary components:

$c = \Re(c) + \Im(c)i$

Now, the exponential factor can be written as:

$e^{ik(x-ct)} = e^{ikx}e^{\Im(c)kt}e^{-ik\Re(c)t} = e^{\Im(c)kt}e^{ik(x-\Re(c)t)}$,

where the first factor is the amplitude and the second factor is the shifted sinusoidal.

So we have $\Re(c)$=phase speed and $\Im(c)k$ =temporal growth rate. If $\Im(c)$<0 the perturbation is dampened and if $\Im(c)$>0 the perturbation grows in time and the flow is unstable. Setting $\Im(c)$=0 gives us a relationship between the perturbation wavenumber k and the flow parameters referred to as the neutral stability curve. Since $c_0$ is real, the neutral stability curve is given by $\Im(c_1) = 0$ which yields:

$$0 = \frac{-Re}{3(1-\lambda)}\left(\frac{3(1-\alpha)\cot\beta}{Re} + Wek^2\right) + \frac{6}{5}Re\frac{(1-\alpha)^2}{(1-\lambda)^3} \tag{39}$$

The minimum value of Re on this curve is the critical value, $Re_{CRIT}$, for the onset of instability for the flow. More specifically, for smaller values of Re all perturbations are dampened and the flow is stable. For larger values of Re some perturbations are amplified and the flow is unstable. An illustration of the neutral stability curve in the Re-k plane is given in Figure 3.1.

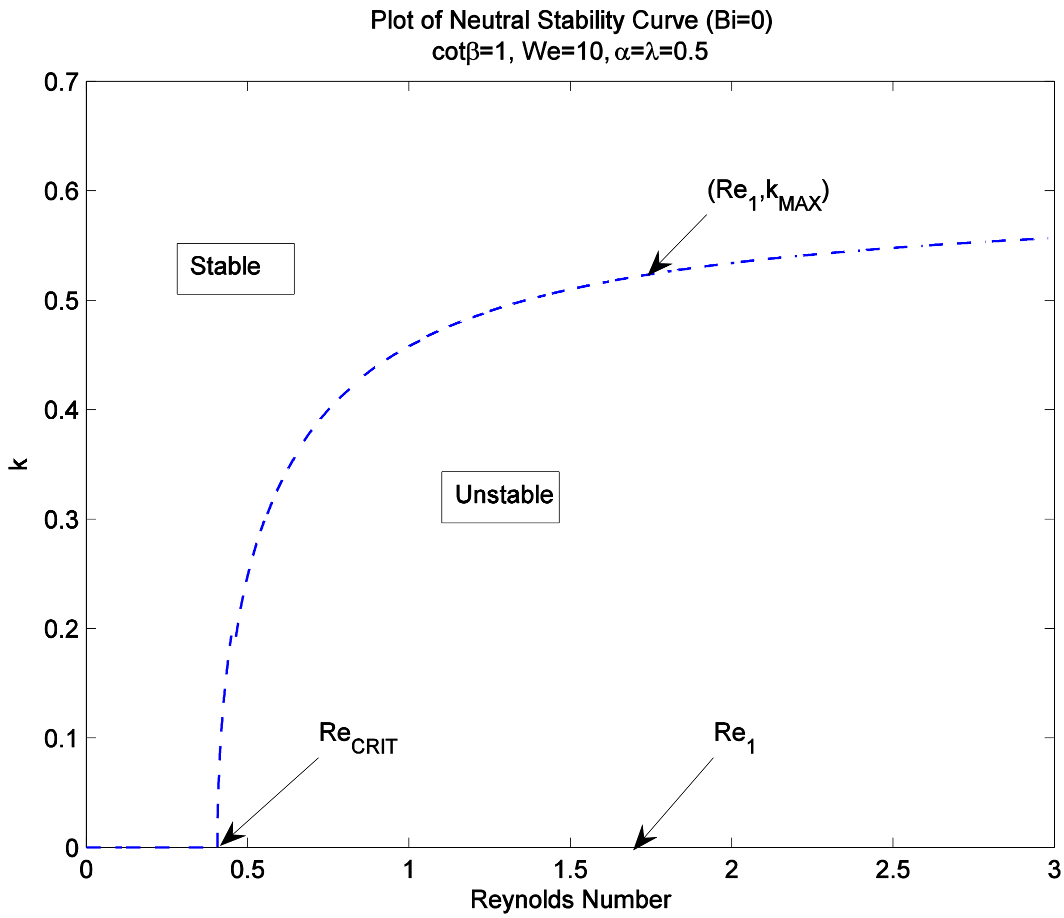


Figure 3.1

It is evident from this example, as well as from the formula in general, that $Re_{CRIT}$ corresponds to k=0. Furthermore, for a supercritical Re value, $Re_1$, perturbations with $k<k_{max}$ grow in time, while those with $k>k_{max}$ are dampened, where point $(Re_1,k_{max})$ lies on the neutral stability curve.

Letting k=0 in equation (39) we obtain the expression for $Re_{CRIT}$ in terms of the other flow parameters.

$$Re_{CRIT} = \frac{5}{6}\cot\beta\left(\frac{(1-\lambda)^2}{1-\alpha}\right)$$

Thus we have completed our analysis of the special case where Bi=0, i.e. when the surface is insulated. If we set all the temperature variations to zero the result for $Re_{CRIT}$ reduces to (5/6)cotβ, which is the well known result for isothermal flow (Benjamin (1957), Yih (1963)).

It is also interesting to note that the Marangoni number, S (the scaled gradient of specific heat capacity), and Λ (the scaled gradient of thermal conductivity) play no role in determining criticality. The independence of the Marangoni number is due to the fact the surface is perfectly insulated; thus stays at a constant temperature, neutralizing the Marangoni effect.

We point out that since the density and viscosity are decreasing functions of temperature, in order to maintain positive values for the prescribed temperature range, the values of the scaled gradients α and λ must be restricted to the interval [0,1). For these values the formula for $Re_{CRIT}$ indicates that increasing α stabilizes the flow (increases $Re_{CRIT}$) while increasing λ destabilizes the flow (decreases $Re_{CRIT}$). This is consistent with the expectation that a decrease in density stabilizes the flow due to reduced inertia, while a decrease in viscosity is a stabilizing factor.

It's also interesting to note that if α and λ are related such that $(1-\lambda)^2=1-\alpha$ or $\alpha=1-(1-\lambda)^2=2\lambda-\lambda^2$ then the effect of temperature variation effects cancel and the threshold for instability is the same as that for isothermal flow.

We now move on to our other special case where we allow temperature variation in only the specific heat, surface tension and mass density (λ=0 and Λ=0).

## B. Full Equations, Special Case with λ=0 and Λ=0

We now begin our analysis of another special case, which can be solved analytically, the case where both λ=0 and Λ=0. Going back to our steady state problems ((11)-(18)), we find explicit solutions for $u_s, p_s, T_s$.

Temperature:

$T_{szz} = 0$

$T_{sz} = -BiT_s$ (at z=1)

$T_s = 1$ (at z=0)

Solving this problem gives us $T_s(z) = 1 - \frac{Bi}{1+Bi} z$.

Velocity:

$u_{szz} + 3(1-\alpha T_s) = 0$

$u_{sz} = 0$ (at z=1)

$u_s = 0$ (at z=0)

Solving this problem gives us $u_s(z) = \frac{-1}{2} \frac{z(3z+3Biz+\alpha z^2 Bi-3\alpha z-3\alpha zBi-6-6Bi+3\alpha Bi+6\alpha)}{1+Bi}$.

Pressure:

$Re\, p_{sz} = -3\cot\beta(1-\alpha T_s)$

$p_s = 0$ (at z=1)

Solving this problem gives us $p_s(z) = \frac{-3}{2} \frac{\cot\beta(z-1)(\alpha zBi+2+2Bi-2\alpha-\alpha Bi)}{Re(1+Bi)}$.

As in the previous special case, we introduce the same normal modes and our perturbation equations now take the following form:

$$\hat{w}_z + ik\hat{u} = 0 \tag{40}$$

$$\mathrm{Re}\left[-ik(c-u_s)\hat{u} + u_{sz}\hat{w}\right] = -ik\,\mathrm{Re}\,\hat{p} - 3\alpha\hat{T} - k^2\hat{u} + \hat{u}_{zz} \tag{41}$$

$$ik\,\mathrm{Re}(u_s - c)\hat{w} = -\mathrm{Re}\,\hat{p}_z - k^2\hat{w} + \hat{w}_{zz} + 3\alpha\cot\beta\hat{T} \tag{42}$$

$$\Pr\mathrm{Re}(-ik(c-u_s)\hat{T} + T_{sz}\hat{w})(1 + \frac{S}{\Delta T_r} + 2ST_s) = -k^2\hat{T} + \hat{T}_{zz} \tag{43}$$

<u>Boundary Conditions (at z=0)</u>

$$\hat{u} = \hat{w} = 0 \tag{44}$$

$$\hat{T} = 0 \tag{45}$$

<u>Boundary Conditions (at z=1)</u>

$$\hat{p} = \frac{3\cot\beta(1+\mathrm{Bi}-\alpha)}{\mathrm{Re}(1+\mathrm{Bi})}\hat{\eta} + \frac{2}{\mathrm{Re}}\hat{w}_z + k^2(\mathrm{We} - \mathrm{Ma}T_s)\hat{\eta} \tag{46}$$

$$\frac{-3(1+\mathrm{Bi}-\alpha)}{1+\mathrm{Bi}}\hat{\eta} + \hat{u}_z + ik\hat{w} = -\mathrm{Ma}\,\mathrm{Re}\,ik(\hat{T} + T_{sz}\hat{\eta}) \tag{47}$$

$$\hat{T}_z = -\mathrm{Bi}\hat{T} + \frac{\mathrm{Bi}^2}{1+\mathrm{Bi}}\hat{\eta} \tag{48}$$

$$\hat{w} = -ik\left(c - u_s\right)\hat{\eta} \tag{49}$$

We then introduce the same long wave expansions as in the previous case and proceed to solve for each of the sub-variables in the long-wave expansions:

<u>The Order 1 Problem:</u>

<u>Temperature:</u>

$$T_{0zz} = 0$$

$$T_{0z} = -\mathrm{Bi}T_0 + \frac{\mathrm{Bi}^2}{1+\mathrm{Bi}} \qquad \text{(at z=1)}$$

$$T_0 = 0 \qquad \text{(at z=0)}$$

Solving this problem gives us $T_0(z) = \dfrac{\mathrm{Bi}^2}{(1+\mathrm{Bi})^2}z$.

<u>Continuity:</u>

$$w_{0z} = 0$$

$$w_0 = 0 \qquad \text{(at z=0)}$$

Solving this problem gives us $w_0(z) \equiv 0$.

Velocity:

$u_{0zz} = 3\alpha T_0$

$u_{0z} = \frac{3(1-\alpha+Bi)}{1+Bi}$ (at z=1)

$u_0 = 0$ (at z=0)

Solving this problem gives us $u_0(z) = \frac{1}{2}\frac{z(\alpha Bi^2 z^2 + 6 + 12Bi + 6Bi^2 - 6\alpha - 6\alpha Bi - 3\alpha Bi^2)}{(1+Bi)^2}$.

Pressure:

$\mathrm{Re}\, p_{0z} = 3\alpha \cot\beta T_0$

$p_0 = \frac{3\cot\beta(1-\alpha+Bi)}{\mathrm{Re}(1+Bi)} + W$ (at z=1)

Solving this problem gives us

$$p_0(z) = \frac{\left(\begin{array}{l} 3\cot\beta\alpha Bi^2 z^2 + 6\cot\beta + 12\cot\beta Bi + 6\cot\beta Bi^2 - 6\cot\beta\alpha - 6\cot\beta\alpha Bi + 2W\,\mathrm{Re} + 4W\,\mathrm{Re}\,Bi \\ + 2W\,\mathrm{Re}\,Bi^2 - 3\cot\beta\alpha Bi^2 \end{array}\right)}{2\mathrm{Re}(1+Bi)^2}$$

The Order k Problem:

Continuity:

$w_{1z} = -iu_0$

$w_1 = 0$ (at z=0)

Solving this problem gives us $w_1(z) = -\frac{1}{8}i\frac{z^2(\alpha Bi^2 z^2 - 12\alpha + 12 + 24Bi + 12Bi^2 - 12\alpha Bi - 6\alpha Bi^2)}{(1+Bi)^2}$.

Kinematic Condition:

$w_1 = i(-c_0 + u_s)$ (at z=1)

Solving this equation gives us $c_0 = \frac{-1}{8}\frac{(9\alpha Bi^2 + 24\alpha - 24 - 48Bi - 24Bi^2 + 28\alpha Bi)}{(1+Bi)^2}$ (we will need $c_0$ in a later calculation).

Temperature:

$T_{1zz} = \Pr \mathrm{Re}(-i(c_0 - u_s)T_0 + T_{sz}w_1)(1 + \frac{S}{\Delta T_r} + 2ST_s)$

$T_{1z} = -BiT_1$ (at z=1)

$T_1 = 0$ (at z=0)

Solving this problem gives us $T_1(z)$. However, the expression for $T_1$ was very long and complicated and there is no point in presenting it.

x-Momentum (Velocity):

$u_{1zz} = i\mathrm{Re}\,p_0 - i\mathrm{Re}(c_0 - u_s)u_0 + \mathrm{Re}\,u_{sz}w_1 + 3\alpha T_1$

$u_{1z} = -i\mathrm{Ma}\,\mathrm{Re}(T_0 + T_{sz})$ (at z=1)

$u_1 = 0$ (at z=0)

Solving this problem gives us $u_1(z)$. However, the expression for $u_1$ was very long and complicated and there is no point in presenting it.

The Order $k^2$ Problem:

Continuity:

$w_{2z} = -iu_1$

$w_2 = 0$ (at z=0)

Solving this problem gives us $w_2(z)$. However, the expression for $w_2$ was very long and complicated and there is no point in presenting it.

Kinematic Condition:

$w_2 = -ic_1$ (at z=1)

As in the previous case, we separate c into its real and imaginary components and set the imaginary component to zero, resulting in the neutral stability curve:

$$
\begin{aligned}
0 = {} & \\
& 725760R\Delta T_r Bi^2 - 40320\Delta T_r \cot\beta + 967680\,R\Delta T_r Bi^3 + 725760\,R\Delta T_r Bi^4 + 40320\Delta T_r \cot\beta\alpha \\
& - 13440\Delta T_r Wek^2 R + 20160 MaReBi\Delta T_r - 806400\Delta T_r \cot\beta Bi^3 - 604800\Delta T_r \cot\beta Bi^4 \\
& - 241920\Delta T_r \cot\beta Bi^5 - 40320\Delta T_r \cot\beta Bi^6 - 241920\Delta T_r \cot\beta Bi - 604800\Delta T_r \cot\beta Bi^2 \\
& + 3780\alpha^2 PrReBi^4\Delta T_r + 3780\alpha^2 PrReBi^4 S + 1740\alpha^2 PrReBi^3 S\Delta T_r + 3738\alpha^2 PrReBi^5 S\Delta T_r \\
& + 3075\alpha^2 PrReBi^5 S + 9195\alpha^2 PrReBi^4 S\Delta T_r + 7548\alpha PrReBi^3 S\Delta T_r - 18648\alpha^2 PrReBi^2 S\Delta T_r \\
& - 22320\alpha^2 PrReBiS\Delta T_r - 18045\alpha^2 PrReBi^2\Delta T_r - 18045\alpha^2 PrReBi^2 S - 13440\Delta T_r Wek^2 ReBi^6 \\
& + 447552\Delta T_r \cot\beta\alpha Bi^3 + 268128\Delta T_r \cot\beta\alpha Bi^4 + 84672\Delta T_r \cot\beta\alpha Bi^5 - 268800\Delta T_r Wek^2 ReBi^3 \\
& - 201600\Delta T_r Wek^2 ReBi^4 - 80640\Delta T_r Wek^2 ReBi^5 + 11088\Delta T_r \cot\beta\alpha Bi^6 + 414288\Delta T_r \cot\beta\alpha Bi^2 \\
& + 201600\Delta T_r \cot\beta\alpha Bi - 80640\Delta T_r Wek^2 ReBi - 201600\Delta T_r Wek^2 ReBi^2 - 1575\alpha PrReBi^6\Delta T_r \\
& + 11160\alpha PrReBiS + 48384\,Re\,\Delta T_r\alpha^2 + 48384\,Re\,\Delta T_r Bi^6 + 290304\,Re\,\Delta T_r Bi^5 - 96768\,Re\,\Delta T_r\alpha \\
& + 290304\,Re\,\Delta T_r Bi + 48384\,Re\,\Delta T_r - 485712\,Re\,\Delta T_r\alpha Bi + 80640 MaReBi^2\Delta T_r \\
& + 80640 MaReBi^4\Delta T_r + 120960 MaReBi^3\Delta T_r + 20160 MaReBi^5\Delta T_r + 53313\,Re\,\Delta T_r\alpha^2 Bi^5 \\
& - 250092\,Re\,\Delta T_r\alpha Bi^5 - 37863\,Re\,\Delta T_r\alpha Bi^6 + 7285\,Re\,\Delta T_r\alpha^2 Bi^6 + 195408\,Re\,\Delta T_r\alpha^2 Bi \\
& + 313621\,Re\,\Delta T_r\alpha^2 Bi^3 - 718506\,Re\,\Delta T_r\alpha Bi^4 - 1130364\,Re\,\Delta T_r\alpha Bi^3 + 333783\,Re\,\Delta T_r\alpha^2 Bi^2 \\
& + 172890\,Re\,\Delta T_r\alpha^2 Bi^4 - 1013031\,Re\,\Delta T_r\alpha Bi^2 + 11160\alpha PrReBi\Delta T_r - 5715\alpha^2 PrReBi^3\Delta T_r
\end{aligned}
$$

$$
\begin{aligned}
&-5715\alpha^2 \Pr \mathrm{Re}Bi^3 S + 465\alpha^2 \Pr \mathrm{Re}Bi^6 \Delta T_r - 1596\alpha \Pr \mathrm{Re}Bi^6 S\Delta T_r - 1575\alpha \Pr \mathrm{Re}Bi^6 S \\
&+ 486\alpha^2 \Pr \mathrm{Re}Bi^6 S\Delta T_r + 465\alpha^2 \Pr \mathrm{Re}Bi^6 S + 22320\alpha \Pr \mathrm{Re}BiS\Delta T_r + 3075\alpha^2 \Pr \mathrm{Re}Bi^5 \Delta T_r \\
&+ 40968\alpha \Pr \mathrm{Re}Bi^2 S\Delta T_r - 10620\alpha \Pr \mathrm{Re}Bi^5 S\Delta T_r - 20124\alpha \Pr \mathrm{Re}Bi^4 S\Delta T_r - 9000\alpha \Pr \mathrm{Re}Bi^5 S \\
&- 9000\alpha \Pr \mathrm{Re}Bi^5 \Delta T_r + 29205\alpha \Pr \mathrm{Re}Bi^2 \Delta T_r + 19080\alpha \Pr \mathrm{Re}Bi^3 \Delta T_r + 29205\alpha \Pr \mathrm{Re}Bi^2 S \\
&+ 19080\alpha \Pr \mathrm{Re}Bi^3 S - 11160\alpha^2 \Pr \mathrm{Re}Bi\Delta T_r - 11160\alpha^2 \Pr \mathrm{Re}BiS - 6390\alpha \Pr \mathrm{Re}Bi^4 \Delta T_r \\
&- 6390\alpha \Pr \mathrm{Re}Bi^4 S
\end{aligned}
$$

To determine the Critical Reynolds Number at the onset of instability, we once again set k, the wavenumber to zero (k=0).

$$
\mathrm{Re}_{CRIT} = \frac{-\left(1008\cot\beta\Delta T_r(1+Bi)^4(-40Bi^2 + 11\alpha Bi^2 + 40\alpha Bi - 80Bi + 40\alpha - 40)\right)}{\left(\begin{aligned}
&-96768\Delta T_r\alpha + 48384\Delta T_r\alpha^2 - 10620\alpha \Pr Bi^5 S\Delta T_r + 3780\alpha^2 \Pr Bi^4 \Delta T_r + 1740\alpha^2 \Pr Bi^3 S\Delta T_r \\
&- 6390\alpha \Pr Bi^4 S - 37863\Delta T_r\alpha Bi^6 - 250092\Delta T_r\alpha Bi^5 + 53313\Delta T_r\alpha^2 Bi^5 + 7285\Delta T_r\alpha^2 Bi^6 \\
&- 485712\Delta T_r\alpha Bi + 20160 MaBi\Delta T_r - 1013031\Delta T_r\alpha Bi^2 + 195408\Delta T_r\alpha^2 Bi + 313621\Delta T_r\alpha^2 Bi^3 \\
&- 718506\Delta T_r\alpha Bi^4 - 1130364\Delta T_r\alpha Bi^3 + 172890\Delta T_r\alpha^2 Bi^4 + 333783\Delta T_r\alpha^2 Bi^2 - 9000\alpha \Pr Bi^5 S \\
&+ 3075\alpha^2 \Pr Bi^5 \Delta T_r + 22320\alpha \Pr BiS\Delta T_r + 3780\alpha^2 \Pr Bi^4 S + 486\alpha^2 \Pr Bi^6 S\Delta T_r - 1575\alpha \Pr Bi^6 S \\
&+ 465\alpha^2 \Pr Bi^6 \Delta T_r - 18648\alpha^2 \Pr Bi^2 S\Delta T_r + 967680\Delta T_r Bi^3 + 725760\Delta T_r Bi^2 + 290304\Delta T_r Bi \\
&+ 120960 MaBi^3 \Delta T_r + 80640 MaBi^4 \Delta T_r + 80640 MaBi^2 \Delta T_r + 20160 MaBi^5 \Delta T_r - 5715\alpha^2 \Pr Bi^3 \Delta T_r \\
&- 5715\alpha^2 \Pr Bi^3 S - 6390\alpha \Pr Bi^4 \Delta T_r + 19080\alpha \Pr Bi^3 \Delta T_r + 29205\alpha \Pr Bi^2 \Delta T_r + 40968\alpha \Pr Bi^2 S\Delta T_r \\
&- 22320\alpha^2 \Pr BiS\Delta T_r + 11160\alpha \Pr BiS + 48384\Delta T_r + 725760\Delta T_r Bi^4 + 7548\alpha \Pr Bi^3 S\Delta T_r \\
&+ 11160\alpha \Pr Bi\Delta T_r - 9000\alpha \Pr Bi^5 \Delta T_r - 1596\alpha \Pr Bi^6 S\Delta T_r + 19080\alpha \Pr Bi^3 S - 1575\alpha \Pr Bi^6 \Delta T_r \\
&+ 3075\alpha^2 \Pr Bi^5 S + 465\alpha^2 \Pr Bi^6 S + 48384\Delta T_r Bi^6 + 290304\Delta T_r Bi^5 + 3738\alpha^2 \Pr Bi^5 S\Delta T_r \\
&+ 9195\alpha^2 \Pr Bi^4 S\Delta T_r + 29205\alpha \Pr Bi^2 S - 11160\alpha^2 \Pr Bi\Delta T_r - 11160\alpha^2 \Pr BiS - 18045\alpha^2 \Pr Bi^2 \Delta T_r \\
&- 18045\alpha^2 \Pr Bi^2 S - 20124\alpha \Pr Bi^4 S\Delta T_r
\end{aligned}\right)}
$$

Thus we have completed our analysis of the special case where both λ=0 and Λ=0. If we set all the temperature variations, with the exception of the surface tension, to zero, the result for $\mathrm{Re}_{CRIT}$ reduces to $\frac{10(1+Bi)^2\cot\beta}{5MaBi + 12(1+Bi)^2}$, which is the $\mathrm{Re}_{CRIT}$ for the basic non-isothermal problem obtained by D'Alessio et al. (2010) and coincides with that obtained by Trevelyan et al. (2007) if the difference in scaling is taken into account.

It is also interesting to note that S, the scaled gradient of specific heat capacity appears to be coupled with α, the scaled gradient of mass density. In other words, whenever there is an S in the critical Reynolds number formula, it is always multiplied by an α.

However, there are less occurrences of S than there are of α. So it seems that in the absence of viscosity and conductivity variation, when we are only left with S and α, α seems to play the dominant role. So the mass density seems to play a larger role than the specific heat capacity of the fluid when it comes to determining criticality, in the absence of the other two temperature variations.

## C. Another Approach

While it is beneficial that we were able to solve the full equations of our system for the two special cases we discussed above, we would prefer to have solutions that work for more general cases. As was discussed earlier, this is not possible due to the complicated nature of the system of equations. However, we now consider an approach pioneered by Benney (1966).

This approach again exploits the fact that the inclined flow is subject to long-wave instability. However, the idea is to apply long-wave expansions directly to the nonlinear equations of motion. To facilitate this, in scaling the equations we assume the lateral length scale to be much larger than the vertical one. This gives rise to a small aspect ratio that provides a small parameter necessary for a perturbation expansion. Benney applied such an expansion and derived a *single* nonlinear evolution equation for the position of the free surface.

We begin our analysis with the equations derived in Chapter 2. However, we employ one key difference; we will use one scale for length and *another* for depth (two scales).

### C1. Long Wave Expansions

Continuing from Chapter 2, Part F, we will make our equations non-dimensional by scaling them. We will be using two different scales for the horizontal and vertical lengths, z and x. We will scale x by L, which is assumed to be much larger than H (the scale for z). This introduces a new small parameter in the equations, δ, which is the ratio of the two scales (δ = H/L).

Scaling all the other variables as before and including the new small parameter δ, and discarding terms of $O(\delta^2)$ and higher (and dropping the *s) we end up with the following equations:

$$u_x + w_z = 0 \tag{50}$$

$$\mathrm{Re}\,\delta(u_t + uu_x + wu_z) = -\mathrm{Re}\,\delta p_x + 3(1-\alpha T) + [(1-\lambda T)u_z]_z \tag{51}$$

$$0 = -\mathrm{Re}\,p_z - 3\cot\beta(1-\alpha T) + \delta[(1-\lambda T)w_z]_z + \delta(1-\lambda T)_x u_z + \delta(1-\lambda T)_z w_z \tag{52}$$

$$\delta \Pr \mathrm{Re}\cdot \frac{D}{Dt}[(1+ {}^{S}\!/_{\Delta T_r})T + ST^2] = [(1+\Lambda T)T_z]_z \tag{53}$$

Conditions at the Bottom (at z=0)

$u=w=0$ (54)

$T=1$ (55)

Conditions at the Surface (at z=h(x,t))

$$p+\frac{2\delta}{Re}(1-\lambda T)u_x+\delta^2 We h_{xx}=0 \tag{56}$$

$$(1-\lambda T)u_z+Ma\,Re\,\delta(T_x+h_xT_z)=0 \tag{57}$$

$$(1+\Lambda T)T_z=-BiT \tag{58}$$

$$w=h_t+uh_x \tag{59}$$

It should be noted that although we are discarding $\delta^2$ terms, in equation (56), we kept the $\delta^2$ We term. This is because We is so large that despite $\delta^2$ being so small, the term still makes an impact in the equations. We will introduce a new parameter W = $\delta^2$ We, to aid in the computations of the various sub-order problems.

## C2. Benney Equation for the special case (Λ=0 and λ=0)

The Benney approach involves finding an asymptotic solution to the equations (50)-(59) and substituting these results into the kinematic condition (59). This becomes the Benney equation.

Our first step is to derive the Benney equation. Although this system of equations ((50)-(59)) is simpler than our full equations from Chapter 2, it is still too complex to be solved exactly. So we first consider a simpler case, where Λ=λ=0.

We now introduce the long-wave expansions:

$$u(x,z,t)=u_0(x,z,t)+\delta u_1(x,z,t)+O(\delta^2)$$
$$w(x,z,t)=w_0(x,z,t)+\delta w_1(x,z,t)+O(\delta^2)$$
$$p(x,z,t)=p_0(x,z,t)+\delta p_1(x,z,t)+O(\delta^2)$$
$$T(x,z,t)=T_0(x,z,t)+\delta T_1(x,z,t)+O(\delta^2)$$

We then substitute these expansions into the 10 governing equations above ((50)-(59)) and solve the various order problems.

The Order 1 Problem:

Temperature:

$T_{0zz}=0$

$T_{0z}=-BiT_0$ (at z=h(x,t))

$T_0=1$ (at z=0)

Solving this problem gives us $T_0(x,z,t)=1-\dfrac{Bi}{1+Bih}z$.

Pressure (z-Momentum):

$$\mathrm{Re}\, p_{0z} = \frac{-3\cot\beta(1-\alpha T_0)}{\mathrm{Re}}$$

$$p_0 = -Wh_{xx} \qquad \text{(at z=h(x,t))}$$

Solving this problem gives us

$$p_0(x,z,t) = \frac{-1}{2\mathrm{Re}(1+\mathrm{Bi}h)}\begin{pmatrix} 6\cot\beta z + 6\cot\beta z\mathrm{Bi}h + 3\cot\beta z^2\alpha\mathrm{Bi} - 6\cot\beta z\alpha - 6\cot\beta z\alpha\mathrm{Bi}h + 2W\mathrm{Re}h_{xx} + \\ 2W\mathrm{Re}h_{xx}\mathrm{Bi}h - 6\cot\beta h - 6\cot\beta h^2\mathrm{Bi} + 3\cot\beta h^2\alpha\mathrm{Bi} + 6\cot\beta h\alpha \end{pmatrix}$$

Velocity (x-Momentum):

$$u_{0zz} + 3(1-\alpha T_0) = 0$$

$$u_{0z} = 0 \qquad \text{(at z=h(x,t))}$$

$$u_0 = 0 \qquad \text{(at z=0)}$$

Solving this problem gives us

$$u_0(x,z,t) = \frac{-1}{2}\frac{z(3z + 3\mathrm{Bi}hz + \alpha z^2\mathrm{Bi} - 3\alpha z - 3\alpha z\mathrm{Bi}h - 6h - 6\mathrm{Bi}h^2 + 3\alpha\mathrm{Bi}h^2 + 6\alpha h)}{1+\mathrm{Bi}h}$$

Continuity (w):

$$w_{0z} = -u_{0x}$$

$$w_0 = 0 \qquad \text{(at z=0)}$$

Solving this problem gives us

$$w_0(x,z,t) = \frac{1}{8}\frac{z^2h_x(12\alpha\mathrm{Bi}h + 6\alpha\mathrm{Bi}^2h^2 + 12\alpha - 12 - 24\mathrm{Bi}h - 12\mathrm{Bi}^2h^2 - \alpha z^2\mathrm{Bi}^2)}{(1+\mathrm{Bi}h)^2}$$

The Order δ Problem:

Temperature:

$$T_{1zz} = \Pr\mathrm{Re}([(1+S/\Delta T_r)T_0 + ST_0^{\,2}]_t + u_0[(1+S/\Delta T_r)T_0 + ST_0^{\,2}]_x + w_0[(1+S/\Delta T_r)T_0 + ST_0^{\,2}]_z)$$

$$T_{1z} = -\mathrm{Bi}T_1 \qquad \text{(at z=h(x,t))}$$

$$T_1 = 0 \qquad \text{(at z=0)}$$

Solving this problem gives us $T_1(x,z,t)$. However, the expression for $T_1$ was very long and complicated and there is no point in presenting it.

Velocity (x-Momentum):

$$u_{1zz} = \mathrm{Re}(u_{0t} + u_0u_{0x} + w_0u_{0z} + p_{0x}) + 3\alpha T_1$$

$$u_{1z} = -\mathrm{Ma}\,\mathrm{Re}(T_{0x} + T_{0z}h_x) \qquad \text{(at z=h(x,t))}$$

$$u_1 = 0 \qquad \text{(at z=0)}$$

Solving this problem gives us $u_1(x,z,t)$. However, the expression for $u_1$ was very long and complicated and there is no point in presenting it.

Continuity (w):

$w_{1z} = -u_{1x}$

$w_1 = 0$ (at z=0)

Solving this problem gives us $w_1(x,z,t)$. However, the expression for $w_1$ was very long and complicated and there is no point in presenting it.

Now that we have solutions for the required sub-variables, we can acquire the Benney equation. This is done by introducing the velocity expansions into the kinematic condition (59). This becomes:

$$h_t + u_0 h_x - w_0 + \delta(u_1 h_x - w_1) = 0$$

This equation describes our fluid flow down the heated ramp. But we want to test for criticality, i.e. under what conditions does the flow become unstable. So we move to a linear analysis to find the critical Reynolds number.

**Linear Analysis:**

We now introduce the perturbed solution $h(x,t) = 1 + \xi(x,t)$ into the equation (where $h \equiv 1$ is the base solution to the Benney equation) and linearize in terms of $\xi$. We then introduce normal modes (we allow $\xi(x,t) = \hat{\xi} \cdot e^{\sigma t + ikx}$). This leads to a characteristic equation, which was solved for σ. Setting the real part of σ (the growth rate) to zero gives the neutral stability curve. Both this formula, and consequently that for $Re_{CRIT}$, are identical to the ones obtained from the analysis of the full equations. This exact agreement serves to verify the validity of the linear stability analysis of the Benney equation.

**C3. Benney Equations (Expansions with respect to Temperature Variations):**

One of the disappointments with the previous approach was that we had to let λ=Λ=0 in order to make any analytic progress with our scaled equations. Removing 2 of 4 of our new temperature variations almost defeats the purpose of including temperature variations into our system. We now explore an alternative approach that would allow us to retain all the temperature dependent properties and still make analytic progress by means of making additional approximations.

We start just as we did in the previous section. We employ the same two-scale equations ((50)-(59)) and employ the same long wave expansions. However, this time we will solve the resulting hierarchy of problems by implementing additional asymptotic expansions with respect to the temperature variation parameters α, λ, Λ and S up to and including quadratic terms. In other words we assume cubic terms to be $O(\delta^2)$ and hence the temperature variations to be $O(\delta^{2/3})$.

The Order 1 Problem as δ→0 :

Temperature:

$[(1+\Lambda T_0)T_{0z}]_z = 0$

$(1+\Lambda T_0)T_{0z} = -BiT_0$ (at z=h(x,t))

$T_0 = 1$ (at z=0)

We will solve this problem using a perturbation expansion in Λ as Λ→0, and thus let $T_0 = T_{00} + \Lambda T_{01} + \Lambda^2 T_{02} + O(\Lambda^3)$.

Now, because of how our system (50-59) is set up, we recognize that $T_0$ appears only in the $u_0$ problem as $\lambda T_0$ and $\alpha T_0$. It is also used in the calculation of $p_0$, but $p_0$ is used to determine $u_1$, which appears in the O(δ) term. Thus we only need to expand $T_0$ as $T_0=T_{00}+\Lambda T_{01}$.

So our original problem for $T_0$ above, with the perturbation expansion takes the following form:

Temperature:

$[(1+\Lambda T_{00})(T_{00z} + \Lambda T_{01z})]_z = 0$

$(1+\Lambda T_{00})(T_{00z} + \Lambda T_{01z}) = -Bi(T_{00} + \Lambda T_{01z})$ (at z=h(x,t))

$T_{00} = 1$ (at z=0)

$T_{01} = 0$ (at z=0)

Just as we broke up equations of motion into sub-problems for the various powers of δ, we do the same now for this sub-problem in powers of Λ.

The Order 1 Temperature Problem as Λ→0:

$T_{00zz} = 0$

$T_{00z} = -BiT_{00}$ (at z=h(x,t))

$T_{00} = 1$ (at z=0)

Solving this problem gives us $T_{00}(x,z,t) = 1 - \frac{Bi}{Bih+1} z$.

The Order Λ Temperature Problem:

$[T_{00}T_{00z} + T_{01z}]_z = 0$

$T_{00}T_{00z} + T_{01z} = -BiT_{01}$ (at z=h(x,t))

$T_{01} = 0$ (at z=0)

Solving this problem gives us $T_{01}(x,z,t) = \frac{-1}{2}\left(1 - \frac{Biz}{Bih+1}\right)^2 + \frac{(-2Bi^2h - Bi^3h^2)z}{2 + 6Bih + 6Bi^2h^2 + 2Bi^3h^3} + \frac{1}{2}$.

Putting these two together, our solution for $T_0$ is:

$$T_0(x,z,t) = 1 - \frac{Biz}{Bih+1} + \Lambda\left[\frac{-1}{2}\left(1-\frac{Biz}{Bih+1}\right)^2 + \frac{(-2Bi^2h - Bi^3h^2)z}{2+6Bih+6Bi^2h^2+2Bi^3h^3} + \frac{1}{2}\right]$$

Notice that if we allow $\Lambda$=0, we get the same $T_0$ as was previously calculated (and also notice how much more complicated $T_0$ becomes by adding $\Lambda$; and this is only an asymptotic solution, not an exact one). Now that we have solved for $T_0$, we go back to the original δ expansion and solve for $u_0$.

<u>Velocity (x-Momentum):</u>

$u_{0zz} - \lambda[T_0 u_{0z}]_z + 3(1-\alpha T_0) = 0$

$u_{0z} = 0$ (at z=h(x,t))

$u_0 = 0$ (at z=0)

We will solve this problem using a perturbation expansion with respect to α, λ and Λ. We let $u_0 = u_{00} + \alpha u_{01} + \lambda u_{02} + \Lambda u_{03} + \alpha\lambda u_{04} + \alpha\Lambda u_{05} + \lambda\Lambda u_{06} + \lambda^2 u_{07} + \alpha^2 u_{08} + \Lambda^2 u_{09}$ and use $T_0$ (calculated earlier). This will give us 9 ODE problems to solve; the O(1), α, λ, Λ, αλ, αΛ, λΛ, $\alpha^2$, $\lambda^2$ and $\Lambda^2$ problems.

<u>The Order 1 Velocity Problem as α, λ, Λ→0:</u>

$u_{00zz} = -3$

$u_{00z} = 0$ (at z=h(x,t))

$u_0 = 0$ (at z=0)

Solving this problem gives us $u_{00}(x,z,t) = 3hz - \frac{3}{2}z^2$

<u>The α problem for velocity:</u>

$u_{01zz} = 3T_{00}$

$u_{01z} = 0$ (at z=h(x,t))

$u_{01} = 0$ (at z=0)

Solving this problem gives us $u_{01}(x,z,t) = \frac{-1}{2}\frac{z(Biz^2 - 3z - 3zBih + 3Bih^2 + 6h)}{(1+Bih)}$

<u>The λ problem for velocity:</u>

$u_{02zz} - [T_{00}u_{00z}]_z = 0$

$u_{02z} = 0$ (at z=h(x,t))

$u_{02} = 0$ (at z=0)

Solving this problem gives us $u_{02}(x,z,t) = \frac{1}{2}\frac{z(2Biz^2 - 3z - 6zBih + 6Bih^2 + 6h)}{(1+Bih)}$

The Λ problem for velocity:

$u_{03zz} = 0$

$u_{03z} = 0$ (at z=h(x,t))

$u_{03} = 0$ (at z=0)

Solving this problem gives us $u_{03}(x,z,t) \equiv 0$

The αλ problem for velocity:

$u_{04zz} - [T_{00} u_{01z}]_z = 0$

$u_{04z} = 0$ (at z=h(x,t))

$u_{04} = 0$ (at z=0)

Solving this problem gives us

$$u_{04}(x,z,t) = \frac{-3}{8} \frac{z(-Bi^2 z^3 + 4Biz^2 + 4Bi^2 z^2 h - 4z - 12zBih - 6zBi^2 h^2 + 12Bih^2 + 4Bi^2 h^3 + 8h)}{(1+Bih)^2}$$

The αΛ problem for velocity:

$u_{05zz} = 3T_{01}$

$u_{05z} = 0$ (at z=h(x,t))

$u_{05} = 0$ (at z=0)

Solving this problem gives us

$$u_{05}(x,z,t) = \frac{-1}{8} \frac{Biz(2Bi^2 h^4 - 2Bi^2 z^2 h^2 + Bi^2 z^3 h + 8Bih^3 - 4Biz^2 h + Biz^3 + 12h^2 - 4z^2)}{1 + 3Bih + 3Bi^2 h^2 + h^3 Bi^3}$$

The λΛ problem for velocity:

$u_{06zz} - [T_{01} u_{00z}]_z = 0$

$u_{06z} = 0$ (at z=h(x,t))

$u_{06} = 0$ (at z=0)

Solving this problem gives us

$$u_{06}(x,z,t) = \frac{1}{8} \frac{Biz^2(6Bi^2 h^3 - 8Bi^2 zh^2 + 3Bi^2 z^2 h + 12Bih^2 - 12Bizh + 3Biz^2 - 8z + 12h)}{1 + 3Bih + 3Bi^2 h^2 + h^3 Bi^3}$$

The $\lambda^2$ problem for velocity:

$u_{07zz} - [T_{00} u_{02z}]_z = 0$

$u_{07z} = 0$ (at z=h(x,t))

$u_{07} = 0$ (at z=0)

Solving this problem gives us

$$u_{07}(x,z,t) = \frac{1}{4} \frac{z(-3Bi^2 z^3 + 8Biz^2 + 12Bi^2 z^2 h - 6z - 24zBih - 18zBi^2 h^2 + 12h + 24Bih^2 + 12Bi^2 h^3)}{(1+Bih)^2}$$

The $\alpha^2$ problem for velocity:

$u_{08zz} = 0$

$u_{08z} = 0$ (at z=h(x,t))

$u_{08} = 0$ (at z=0)

Solving this problem gives us $u_{08}(x,z,t) \equiv 0$

The $\Lambda^2$ problem for velocity:

$u_{09zz} = 0$

$u_{09z} = 0$ (at z=h(x,t))

$u_{09} = 0$ (at z=0)

Solving this problem gives us $u_{09}(x,z,t) \equiv 0$

Piecing together the solutions from these 9 sets of ODE problems, we arrive at our solution for $u_0(x,z,t)$
$\left(u_0 = u_{00} + \alpha u_{01} + \lambda u_{02} + \Lambda u_{03} + \alpha\lambda u_{04} + \alpha\Lambda u_{05} + \lambda\Lambda u_{06} + \lambda^2 u_{07} + \alpha^2 u_{08} + \Lambda^2 u_{09}\right)$.

Pressure (z-Momentum):

$\mathrm{Re}\, p_{0z} = -3\cot\beta(1 - \alpha T_0)$

$p_0 = -Wh_{xx}$ (at z=h(x,t))

We will solve this problem using a perturbation expansion with respect to α, λ and Λ. However, since $p_0$ is only involved in the O(δ) problem, we only need to consider a linear expansion for $p_0$.

We let $p_0 = p_{00} + \alpha p_{01} + \lambda p_{02} + \Lambda p_{03}$ and use $T_0$ (calculated earlier). This will give us 4 ODE problems to solve; the O(1), α, λ and Λ problems.

The Order 1 Pressure Problem as α, λ, Λ→0:

$\mathrm{Re}\, p_{00z} = -3\cot\beta$

$p_{00} = -Wh_{xx}$ (at z=h(x,t))

Solving this problem gives us $p_{00}(x,z,t) = \dfrac{3\cot\beta}{\mathrm{Re}}(h - z) - Wh_{xx}$

The α problem for pressure:

$\mathrm{Re}\, p_{01z} = 3\cot\beta T_{00}$

$p_{01} = 0$ (at z=h(x,t))

Solving this problem gives us $p_{01}(x,z,t) = \dfrac{-3\cot\beta(\mathrm{Bi}z^2 - 2z - 2z\mathrm{Bi}h + \mathrm{Bi}h^2 + 2h)}{2\mathrm{Re}(1 + \mathrm{Bi}h)}$

The λ problem for pressure:

$$\mathrm{Re}\,p_{02z} = 0$$

$$p_{02} = 0 \qquad \text{(at z=h(x,t))}$$

Solving this problem gives us $p_{02}(x,z,t) \equiv 0$

The Λ problem for pressure:

$$\mathrm{Re}\,p_{03z} = 0$$

$$p_{03} = 0 \qquad \text{(at z=h(x,t))}$$

Solving this problem gives us $p_{03}(x,z,t) \equiv 0$

Piecing together the solutions from these 4 sets of ODE problems, we arrive at our solution for $p_0$.

$$p_0(x,z,t) = \frac{3\cot\beta(h-z)}{\mathrm{Re}} - Wh_{xx} + \alpha\left(\frac{-3\cot\beta(Biz^2 - 2z - 2zBih + Bih^2 + 2h)}{2\mathrm{Re}(1+Bih)}\right)$$

Continuity (w):

$$w_{0z} = -u_{0x}$$

$$w_0 = 0 \qquad \text{(at z=0)}$$

Having already obtained $u_0$, we merely take its derivative with respect to x and integrate with respect to z to calculate $w_0(x,z,t)$.

$$w_{0i} = \int_0^z -u_{0ix}\,dz \qquad \text{for i=0..9}$$

Piecing together the expansion for $w_0(x,z,t)$ $\left(w_0 = w_{00} + \alpha w_{01} + \lambda w_{02} + \Lambda w_{03} + \alpha\lambda w_{04} + \alpha\Lambda w_{05} + \lambda\Lambda w_{06} + \lambda^2 w_{07} + \alpha^2 w_{08} + \Lambda^2 w_{09}\right)$, we get

$$w_0(x,z,t) = -\frac{3}{2}h_x z^2 + \alpha\left(\frac{1}{8}\frac{z^2h_x(12Bih + 6Bi^2h^2 + 12 - Bi^2z^2)}{(1+Bih)^2}\right) + \lambda\left(\frac{-1}{4}\frac{z^2h_x(-2Biz + 6 + 12Bih + 6Bi^2h^2 - Bi^2z^2)}{(1+Bih)^2}\right)$$

$$+\alpha\lambda\left(\frac{1}{40}\frac{z^2h_x(-15Bi^2z^2 - 15Bi^3z^2h - 20Biz + 90Bi^2h^2 + 30h^3Bi^3 + 60 + 120Bih + 6Bi^3z^3)}{(1+Bih)^3}\right)$$

$$+\alpha\Lambda\left(\frac{1}{80}\frac{Biz^2h_x\begin{pmatrix}10Bi^3h^4 + 5z^2Bi^3h^2 - 4Bi^3z^3h + 40Bi^2h^3 + 10z^2Bi^2h - 4Bi^2z^3 + 60Bih^2\\ +20z^2Bi + 120h\end{pmatrix}}{(1+Bih)(1+3Bih+3Bi^2h^2+h^3Bi^3)}\right)$$

$$+\lambda\Lambda\left(\frac{-1}{40}\frac{Biz^3h_x\left(10Bi^3zh^2 - 6Bi^3z^2h + 10Bi^2h^2 + 10zBi^2h - 6Bi^2z^2 + 15Biz + 20\right)}{(1+Bih)(1+3Bih+3Bi^2h^2+h^3Bi^3)}\right)$$

$$+\lambda^2\left(\frac{-1}{20}\frac{z^2h_x\left(-5Bi^2z^2 - 15Bi^3z^2h - 20zBi^2h - 20Biz + 90Bih + 90Bi^2h^2 + 30h^3Bi^3 + 30 + 6Bi^3z^3\right)}{(1+Bih)^3}\right)$$

Thus we have solved all of the variables for the order 1 problem ($T_0$, $u_0$, $p_0$, $w_0$). However, before we can proceed to the order δ problem we recognize that we will need some time derivatives. To get these time derivatives we will use the kinematic condition and the chain rule.

The Order 1 Kinematic Condition as $\delta \to 0$:

$h_t = w_0 - u_0 h_x$ (at z=h(x,t))

Since the dependence of $u_0$ on t is only through h, we can use the chain rule to calculate $u_{0t}$: $u_{0t} = h_t u_{0h}$ (and $h_t$ we get from the kinematic condition above).

We thus obtain $u_{0t} = [w_{00} + \alpha w_{01} + \lambda w_{02} - h_x u_{00} - h_x u_{01}\alpha - h_x u_{02}\lambda]_{z=h} * (u_{00h} + \alpha u_{01h} + \lambda u_{02h})$

Since $u_{0t}$ only appears in the O(δ) problem, we only need linear terms; and since $u_{03}$=0, no Λ terms are present in this expansion. This perturbation expansion will give us 3 terms; the O(1), α and λ terms.

The Order 1 Time Derivative as α, $\lambda \to 0$:

$u_{00t} = [w_{00} - h_x u_{00}]_{z=h} u_{00h}$

Substituting into this equation gives us $u_{00t} = -9zh_x h^2$

The α terms in the time derivative:

$u_{01t} = [w_{00} - h_x u_{00}]_{z=h} u_{01h} + [w_{01} - h_x u_{01}]_{z=h} u_{00h}$

Substituting into this equation gives us $u_{01t} = \dfrac{3zh_x h^2(52Bih + 21Bi^2h^2 + 48 - 4Bi^2z^2)}{8(1+Bih)^2}$

The λ terms in the time derivative:

$u_{02t} = [w_{00} - h_x u_{00}]_{z=h} u_{02h} + [w_{02} - h_x u_{02}]_{z=h} u_{00h}$

Substituting into this equation gives us $u_{02t} = \dfrac{-3zh_x h^2(44Bih + 21Bi^2h^2 + 24 - 6Biz - 4Bi^2z^2)}{4(1+Bih)^2}$

Piecing together the solutions from these 3 terms, we arrive at our solution for $u_{0t}$ ($u_{0t} = u_{00t} + \alpha u_{01t} + \lambda u_{02t} + \Lambda u_{03t}$).

$$u_{0t} = -9zh_x h^2 + \alpha\left(\frac{3zh_x h^2(52Bih + 21Bi^2h^2 + 48 - 4Bi^2z^2)}{8(1+Bih)^2}\right) +$$
$$\lambda\left(\frac{-3zh_x h^2(44Bih + 21Bi^2h^2 + 24 - 6Biz - 4Bi^2z^2)}{4(1+Bih)^2}\right)$$

The Order δ Problem:

Temperature:

$$[(1+\Lambda T_0)T_1]_{zz} = \mathrm{Re}\,\mathrm{Pr}\frac{D}{Dt}[\left(1+\frac{S}{\Delta T_r}\right)T_0 + ST_0^2]$$

$$T_{1z} + \Lambda(T_0T_1)_z = -\mathrm{Bi}T_1 \qquad \text{(at z=h(x,t))}$$

$$T_1 = 0 \qquad \text{(at z=0)}$$

We will solve this problem using a perturbation expansion in Λ. However, we recognize that $T_1$ appears only in the $u_1$ problem as $\lambda T_1$ and $\alpha T_1$ and $u_1$ is multiplied by δ. In other words, since we are only keeping terms up to $\delta^2$ and the temperature variation parameters are $O(\delta^{2/3})$ we only need to expand $T_1$ as $T_1=T_{10}$.

So the $T_1$ problem becomes:

$$T_{10zz} = \mathrm{Re}\,\mathrm{Pr}(T_{00t} + u_{00}T_{00x} + w_{00}T_{00z})$$

$$T_{10z} = -\mathrm{Bi}T_{10} \qquad \text{(at z=h(x,t))}$$

$$T_{10} = 0 \qquad \text{(at z=0)}$$

where, $T_{00t} = T_{00h}\cdot h_t = \dfrac{-3\mathrm{Bi}^2 z h_x h^2}{(1+\mathrm{Bih})^2}$

Solving this problem gives us

$$T_1(x,z,t) = T_{10}(x,z,t) =$$

$$\frac{(8h^5\mathrm{Bi}^2 - 20z^2\mathrm{Bi}^2h^3 + 15z^3\mathrm{Bi}^2h^2 - 3z^4\mathrm{Bi}^2h + 10h^4\mathrm{Bi} - 20z^2\mathrm{Bi}h^2 + 20z^3\mathrm{Bi}h - 3\mathrm{Bi}z^4 + 5z^3 - 20h^3)}{40(1+3\mathrm{Bi}h + 3\mathrm{Bi}^2h^2 + \mathrm{Bi}^3h^3)}\mathrm{PrReBi}zh_x$$

It turns out that under our assumption that the temperature variations of the fluid properties are $O(\delta^{2/3})$, S, as well as the parameter $\Delta T_r$ do not figure in this problem and will not play a role in our analysis. Their effect will only appear in the $O(\delta^2)$ terms.

Velocity (x-Momentum):

$$u_{1zz} - [\lambda T_0 u_{1z} + \lambda T_1 u_{0z}]_z = \mathrm{Re}\,p_{0x} + 3\alpha T_1 + \mathrm{Re}(u_{0t} + u_0u_{0x} + w_0u_{0z})$$

$$u_{1z} - \lambda T_0 u_{1z} - \lambda T_1 u_{0z} = -\mathrm{Ma}\,\mathrm{Re}(T_{0x} + h_xT_{0z}) \qquad \text{(at z=h(x,t))}$$

$$u_1 = 0 \qquad \text{(at z=0)}$$

We will solve this problem using a perturbation expansion with respect to α, λ and Λ. However, since $u_1$ is only involved in the O(δ) term, we only need to consider a linear expansion for $u_1$.

Letting $u_1 = u_{10} + \alpha u_{11} + \lambda u_{12} + \Lambda u_{13}$ gives us 4 ODE problems to solve; the O(1), α, λ and Λ problems.

The Order 1 Velocity Problem as α, λ, Λ→0:

$u_{10zz} = \mathrm{Re}\, p_{00x} + \mathrm{Re}(u_{00t} + u_{00}u_{00x} + w_{00}u_{00z})$

$u_{10z} = -\mathrm{Ma}\,\mathrm{Re}(T_{00x} + h_x T_{00z})$ (at z=h(x,t))

$u_{10} = 0$ (at z=0)

Solving this problem gives us $u_{10}$(x,z,t). However, the expression for $u_{10}$ was very long and complicated and there is no point in presenting it.

The α problem for velocity:

$u_{11zz} = \mathrm{Re}\, p_{01x} + 3T_{10} + \mathrm{Re}(u_{01t} + u_{00}u_{01x} + u_{01}u_{00x} + w_{00}u_{01z} + w_{01}u_{00z})$

$u_{11z} = 0$ (at z=h(x,t))

$u_{11} = 0$ (at z=0)

Solving this problem gives us $u_{11}$(x,z,t). However, the expression for $u_{11}$ was very long and complicated and there is no point in presenting it.

The λ problem for velocity:

$u_{12zz} - [T_{00}u_{10z} + T_{10}u_{00z}]_z = \mathrm{Re}(u_{02t} + u_{00}u_{02x} + u_{02}u_{00x} + w_{00}u_{02z} + w_{02}u_{00z})$

$u_{12z} - T_{00}u_{10z} - T_{10}u_{00z} = 0$ (at z=h(x,t))

$u_{12} = 0$ (at z=0)

Solving this problem gives us $u_{12}$(x,z,t). However, the expression for $u_{12}$ was very long and complicated and there is no point in presenting it.

The Λ problem for velocity:

$u_{13zz} = 0$

$u_{13z} = -\mathrm{Ma}\,\mathrm{Re}(T_{01x} + h_x T_{01z})$ (at z=h(x,t))

$u_{13} = 0$ (at z=0)

Solving this problem gives us $u_{13}$(x,z,t). However, the expression for $u_{13}$ was very long and complicated and there is no point in presenting it.

Piecing together the solutions from these 4 sets of ODEs, we arrive at our solution for $u_1$(x,z,t), ($u_1 = u_{10} + \alpha u_{11} + \lambda u_{12} + \Lambda u_{13}$).

Continuity (w):

$w_{1z} = -u_{1x}$

$w_1 = 0$ (at z=0)

Having already solved $u_1$, we merely take its derivative with respect to x and integrate with respect to z to calculate $w_1$(x,z,t):

$$w_{1i} = \int_0^z -u_{1ix}\,dz \qquad \text{for i=0..3}$$

The expansion is then given by $w_1(x,z,t) = w_{10} + \alpha w_{11} + \lambda w_{12} + \Lambda w_{13}$.

Now that we have solutions for the required sub-variables, we can acquire the Benney equation, just as we had done on the previous section. This is done by substituting the velocity expansions into the kinematic condition (59):

$$h_t + u_0 h_x - w_0 + \delta(u_1 h_x - w_1) = 0$$

This equation governs the thickness of the fluid flow down the heated ramp. But we want to test for criticality, i.e. under what conditions does the flow become unstable. So we move to a linear analysis to find the critical Reynolds number, just as we had done in the previous chapter.

**Linear Analysis:**

As before, we introduce perturbations into the equation by letting $h(x,t) = 1 + \xi(x,t)$ and linearize in terms of $\xi$. We then introduce normal modes (we allow $\xi(x,t) = \hat{\xi} \cdot e^{\sigma t + ikx}$). This leads to a characteristic equation which was solved for σ. Setting the real part of σ (the growth rate) to zero gives the neutral stability curve.

$$k = \left( \frac{\begin{array}{l} -70W\,Re(4Bi^2 + 3\lambda Bi^2 + 8Bi + 7\lambda Bi + 4 + 4\lambda) \times \\ \left( \begin{array}{l} 13440\cot\beta Bi^4 - 16128 Re Bi^4 - 6720 Ma Re Bi^3 + 43680\lambda\cot\beta Bi^3 - 246570\lambda Re Bi^2 \\ + 10080\lambda\cot\beta Bi^4 - 6720 Ma Re Bi + 97392\alpha Re Bi - 154596\lambda Re Bi^3 - 40320\alpha\cot\beta Bi \\ - 13440 Ma Re Bi^2 + 58122\alpha Re Bi^3 - 37002\lambda Re Bi^4 - 3696\alpha\cot\beta Bi^4 - 20832\alpha\cot\beta Bi^3 \\ + 12621\alpha Re Bi^4 + 50400\lambda\cot\beta Bi - 96768 Re Bi^2 - 44016\alpha\cot\beta Bi^2 + 110637\alpha Re Bi^2 \\ + 70560\lambda\cot\beta Bi^2 - 13440\alpha\cot\beta - 177360\lambda Re Bi - 64512 Re Bi^3 + 32256\alpha Re \\ + 53760\cot\beta Bi^3 - 48384\lambda Re + 13440\lambda\cot\beta - 16128 Re + 53760\cot\beta Bi - 6720\lambda Ma Re Bi \\ + 954\lambda Pr Re Bi^2 + 6720\Lambda Ma Re Bi - 2295\alpha Pr Re Bi^2 - 1140\lambda Pr Re Bi^3 + 525\alpha Pr Re Bi^4 \\ - 3720\alpha Pr Re Bi - 3360\Lambda Ma Re Bi^3 - 4480\lambda Ma Re Bi^3 + 1950\alpha Pr Re Bi^3 - 11200\lambda Ma Re Bi^2 \\ - 6720\Lambda Ma Re Bi^2 + 1632\lambda Pr Re Bi - 462\lambda Pr Re Bi^4 + 13440\cot\beta - 64512 Re Bi \\ + 80640\cot\beta Bi^2 \end{array} \right) \end{array}}{280W\,Re(4Bi^2 + 3\lambda Bi^2 + 8Bi + 7\lambda Bi + 4 + 4\lambda)(1 + Bi)} \right)^{\frac{1}{2}}$$

To get the critical Reynolds number, we want the minimum Reynolds number where instability occurs. If we plot this neutral stability relationship in the Re-k plane we realize this occurs when k, the wavenumber, is zero. So to get the critical Reynolds number we set k=0.

$$\mathrm{Re}_{CRIT} = \frac{336\cot\beta\begin{pmatrix}-40Bi^4-130\lambda Bi^3-30\lambda Bi^4+120\alpha Bi+11\alpha Bi^4+62\alpha Bi^3-150\lambda Bi+131\alpha Bi^2-210\lambda Bi^2\\+40\alpha-160Bi^3-40\lambda-160Bi-40-240Bi^2\end{pmatrix}}{\begin{pmatrix}-16128+110637\alpha Bi^2+58122\alpha Bi^3+12621\alpha Bi^4+97392\alpha Bi-154596\lambda Bi^3-37002\lambda Bi^4\\-246570\lambda Bi^2-177360\lambda Bi-64512Bi^3-96768Bi^2-16128Bi^4-48384\lambda+32256\alpha+954\lambda \Pr Bi^2\\-6720\lambda MaBi-2295\alpha \Pr Bi^2+6720\Lambda MaBi-13440MaBi^2-6720MaBi-6720MaBi^3\\-1140\lambda \Pr Bi^3+1950\alpha \Pr Bi^3-4480\lambda MaBi^3-3360\Lambda MaBi^3-3720\alpha \Pr Bi+525\alpha \Pr Bi^4\\-11200\lambda MaBi^2-6720\Lambda MaBi^2-462\lambda \Pr Bi^4+1632\lambda \Pr Bi-64512Bi\end{pmatrix}}$$

Thus we have obtained a solution while allowing for variation in all temperature dependent fluid properties. We point out that this solution reduces to the familiar $\frac{10(1+Bi)^2\cot\beta}{5MaBi+12(1+Bi)^2}$, if we let α=λ=Λ=0. This is exactly what should happen, and helps to validate our result.

## CHAPTER 4 – Depth Integrated Model:

We now consider another approach to solving our system, the depth integrated model. Ruyer-Quil and Manneville (2000) used a modified IBL approach, where they combined a gradient expansion with a weighted residual technique using polynomial test functions. This approach predicts the correct instability threshold. Trevelyan et al. (2007) extended this approach for the basic non-isothermal problem. This is the approach we will now take with our system of equations.

We begin with the two-scale equations derived in Chapter 3 ((50)-(59)). The general idea behind this approach is to simplify the long-wave equations by depth-integrating (integrating with respect to z), and thus reducing the space dimensionality.

The first two equations, the Continuity Equation (50) and the z-Momentum equation (52) can be integrated directly. We depth integrate (with respect to z, from zero to h) the continuity equation (50). This gives us our first equation:

$$h_t + q_x = 0 \tag{60}$$

Where $q = \int_0^h u\,dz$ is defined as the flow rate. Since u is the direction of the flow, by integrating u from z=0 to z=h, we are capturing the whole volume of u at that particular cross-section; giving us the 'flow rate'.

From the z-Momentum equation (52) we get p (pressure), which is substituted into the x-Momentum equation (51). This gives us:

$$\begin{aligned}\mathrm{Re}\,\delta(u_t + uu_x + wu_z) = 3(1-\alpha T) + [(1-\lambda T)u_z]_z + 3\delta\cot\beta\alpha\theta h_x + \delta^3 \mathrm{We} h_{xxx} - 3\delta\cot\beta h_x \\ - 3\delta\cot\beta\alpha\int_h^z T_x\,dz\end{aligned} \tag{61}$$

where θ denotes the surface temperature, i.e. θ(x,t) = T(x,h(x,t),t).

We recognize that our new x-Momentum equation (61) and our temperature equation (53) cannot be converted to our new variables h, q and θ via direct integration. We will thus proceed by employing a weighted residual method. The general approach is to expand u and T in linear combinations of z dependent test functions. The coefficients are then determined by equating the weighted residuals to zero.

For the basic non-isothermal problem, Trevelyan et al. (2007) extended the method proposed by Ruyer-Quil and Manneville (2000) for the isothermal flow and assumed the following profiles for u and T:

$$u(x,z,t) = \frac{3}{2h^3(x,t)} q(x,t) b_0(x,z,t) + \frac{\delta Ma Re\, \theta_x(x,t)}{4h(x,t)} b_1(x,z,t) \tag{62}$$

$$T(x,z,t) = 1 + \frac{\theta(x,t) - 1}{h(x,t)} z \tag{63}$$

where $b_0(x,z,t) = 2hz-z^2$ , and $b_1(x,z,t) = 2hz-3z^2$.

Conditions at the Interfaces:

It should be pointed out that the velocity profile (62) satisfies the condition $q = \int_0^h u dz$ , the bottom conditions (54) (u=w=0) as well as the surface tangential force condition (57) which for the basic non-isothermal problem is given by $u_z=-MaRe\delta\theta_x$ at z=h(x,t).

The temperature profile (63) satisfies the bottom condition (55) (T=1) but not the surface condition (58) $T_z=-Bi\theta$ at z=h(x,t). However this condition can be incorporated into the residual by implementing integration by parts in the integration process, as will be explained below.

For the isothermal problem Ruyer-Quil and Manneville (2002) formally analyzed the accuracy of employing the velocity profile (62) (with Ma=0). They show that employing more elaborate expansions leads to formulations which ultimately converge to the modified IBL equations. For the basic non-isothermal problem, Trevelyan et al. (2007) demonstrate the efficacy of the temperature profile (63). Indeed, the linear stability analysis of the modified IBL equations predicts the correct critical Reynolds number for the onset of instability as obtained from the full equations for the isothermal problem and the basic non-isothermal problem.

For the current problem with variation in all fluid properties we again resort to the profiles (62) and (63). No adjustment is made to the profiles to account for the extra temperature variations. Various options were considered but none improved the agreement with the full equations, they only complicated the governing equations.

It turns out that for the current problem the profile for u in (62) does not satisfy the surface tangential force condition which can be stated as $u_z=(-MaRe\delta\theta_x)/(1-\lambda\theta)$ at z=h(x,t). However, like with the temperature profile, we can include the correct condition into the integrated momentum equation.

The Modified IBL Equations:

In accordance with the Galerkin method, test functions are used to weight the residuals. The temperature equation (53) is weighted with z and then depth integrated. Since we would like to satisfy the boundary condition (58), we apply integration by parts to the term arising from $[(1+\Lambda T)T_z]_z$ and include the boundary condition as follows:

$$\int_0^h z[(1+\Lambda T)T_z]_z dz = [z(1+\Lambda T)T_z]_{z=0}^{z=h} - \int_0^h (1+\Lambda T)T_z dz = -Bih\theta - \int_0^h (1+\Lambda T)T_z dz$$

Continuing with the rest of equation (53) and including this result we obtain an equation in h, q and θ:

$$h\left(3S\theta+S+2\left(1+S/\Delta T_r\right)\right)\theta_t=\frac{3}{10}Sq_x-\frac{6}{5}Sq\theta_x-\frac{21}{5}Sq\theta\theta_x-\frac{2}{5}S\theta^2q_x+\frac{1}{10}S\theta q_x$$
$$+\frac{7}{20}\left(1+S/\Delta T_r\right)(1-\theta)q_x-\frac{27}{10}\left(1+S/\Delta T_r\right)q\theta_x+\frac{6\,(1-\theta\,(1+Bih))}{Pr\,Re\,\delta\,h}-\frac{3\,\Lambda(\theta^2-1)}{Pr\,Re\,\delta\,h} \qquad (64)$$

The x-Momentum equation (61) is weighted with $b_0$ and then depth integrated. Since the profile for u in (62) does not satisfy the surface tangential force condition we apply integration by parts twice to the term arising from $[(1-\lambda T)u_z]_z$ and include the boundary conditions as follows:

$$\int_0^h b_0[(1-\lambda T)u_z]_z dz=b_0[(1-\lambda T)u_z]_{z=0}^{z=h}-b_{0z}[(1-\lambda T)u]_{z=0}^{z=h}+\int_0^h[b_{0z}(1-\lambda T)]_z u dz$$

Applying the correct interface conditions yields:

$$\int_0^h b_0[(1-\lambda T)u_z]_z dz=Ma\,Re\,\delta\theta_x b_0(z=h)+\int_0^h[b_{0z}(1-\lambda T)]_z u dz$$

Continuing with the rest of the x-Momentum equation (61) and including this result we obtain an equation in h, q and θ:

$$\delta q_t-\delta\left(\begin{array}{l}\frac{9}{7}\frac{q^2h_x}{h^2}-\frac{17}{7}\frac{q_xq}{h}-\frac{5}{4}Ma\theta_x-\frac{5}{48}\lambda Ma\theta_x\theta+\frac{5}{48}\lambda Ma\theta_x-\frac{5}{2}\frac{h\cot\beta h_x}{Re}\\+\frac{29}{16}\frac{h\cot\beta h_x\alpha\theta}{Re}+\frac{5}{6}hWh_{xxx}+\frac{11}{16}\frac{h^2\cot\beta\alpha\theta_x}{Re}+\frac{11}{16}\frac{h\cot\beta\alpha h_x}{Re}\end{array}\right)$$
$$-\frac{5}{2}\frac{h}{Re}+\frac{25}{16}\frac{h\alpha\theta}{Re}+\frac{15}{16}\frac{h\alpha}{Re}-\frac{5}{8}\frac{\lambda q\theta}{h^2Re}-\frac{15}{8}\frac{\lambda q}{h^2Re}+\frac{5}{2}\frac{q}{h^2Re}=0 \qquad (65)$$

These three equations (60, 64 and 65) govern the unknowns h, q and θ and constitute “modified” IBL equations. If we allow α=Λ=λ=S=0, these equations reduce to those used by Trevelyan et al. (2007) for the basic non-isothermal problem.

## A. Linear Stability Analysis (for Modified IBL):

As in the case of the long-wave expansions, we carry out a linear stability analysis on these equations to arrive at a critical Reynolds number that we can compare to the other methods used earlier.

For the first step of our stability analysis, we compute the steady state, by setting all time derivatives to zero. From the continuity equation (60) we realize that q is a constant. Thus, our steady state solution is h=1, $q=q_s$=constant and $\theta=\theta_s$=constant, where $q_s$ and $\theta_s$ are obtained from: $8-5\alpha\theta_s-3\alpha+2\lambda\theta_sq_s+6\lambda q_s-8q_s=0$ and $-2+2\theta_sBi+2\theta_s+\Lambda\theta_s^2-\Lambda=0$.

We then introduce a perturbed steady flow into the modified IBL equations by letting $h = 1 + \tilde{h}$, $q = q_s + \tilde{q}$ and $\theta = \theta_s + \tilde{\theta}$, and then linearize with respect to the perturbations. Into these linearized equations, we introduce the normal modes: $(\tilde{h}, \tilde{q}, \tilde{\theta}) = (\hat{h}, \hat{q}, \hat{\theta})e^{\sigma t + ikx}$. This results in a 3x3 system of linear (homogeneous) equations for $\hat{h}$, $\hat{q}$ and $\hat{\theta}$. Solving the characteristic equation we get a dispersion relation, which we solve for σ. For neutral stability, the growth rate is zero. Setting $\Re(\sigma) = 0$ gives us our neutral stability curve in the Re-k plane. The critical Reynolds number is the intercept of this curve with the Re axis, so we set k=0 and solve for Re to get the critical Reynolds number. Thus giving us the critical Reynolds number and the conditions under which interfacial instability will occur.

All calculations were done analytically using Maple. However, the expressions for the neutral stability curve and the critical Reynolds number are too long to give. It is however apparent from this formula that $Re_{CRIT}$ reduces to the familiar $\frac{10(1+Bi)^2 \cot\beta}{5MaBi + 12(1+Bi)^2}$, if we let α=λ=Λ=S=0. This is exactly what should happen, and helps to validate our result. Also, if we set Bi=0 we get the same expression as that from the full equations: $\frac{5}{6}\cot\beta\left(\frac{(1-\lambda)^2}{1-\alpha}\right)$.

Another interesting observation is that this formula for the critical Reynolds number is much more complicated than the one we obtained using the Benney approach. It also contains the S, specific heat variation parameter, in several of the terms, which was missing in the Benney version.

We now compare our three different techniques for solving this problem.

**B. Comparisons and Discussion:**

As we cannot analyze the full equations for the general case, we have to resort to an approximation. We have two approximation methods:

1) "The Approximate Benney Equation" – The Benney Equation we obtained by applying asymptotic expansions as α, λ, Λ, S → 0
2) The Modified IBL equations

We would like to determine the accuracy of these approximation methods. As was pointed out earlier, for the special case where Bi=0, the formula for $Re_{CRIT}$ given by the modified IBL was identical to that from the full equations.

Since we were also able to solve the full equations for the special case when λ=Λ=0, we will plot the solutions from both approximation methods and compare them to the solution we obtained from the full equations.

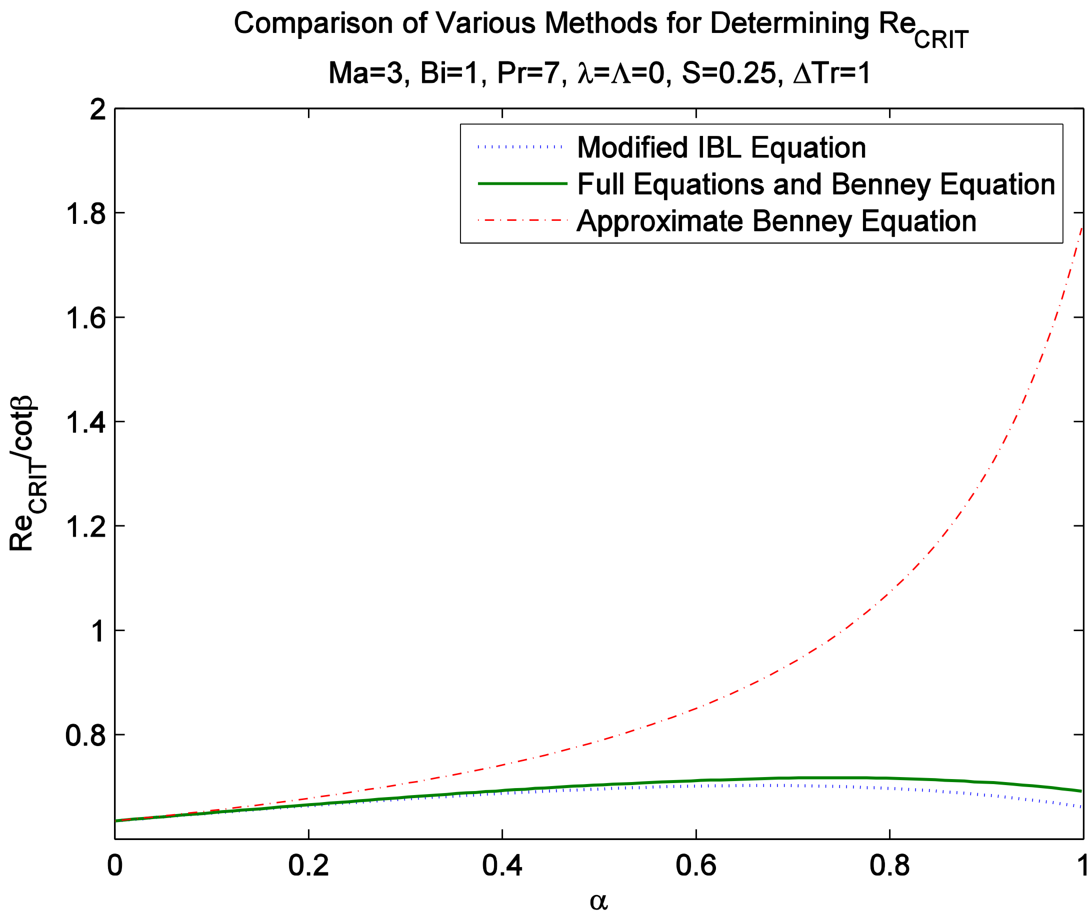


Figure 4.1

In Figure 4.1 we find is that the modified IBL equation does a very good job of approximating the solution. However, our approximate Benney equation only does a good job of approximating our solution when α is between 0 and 0.2. For larger values the solution increases rapidly, when it should not. This is expected since the approximation is based on asymptotic expansions as $\alpha \rightarrow 0$. The approximate Benney solution was also missing the S term. So information was clearly lost here. Despite the erroneous results for higher values of α, our solution seems to be correct for lower values of α.

We also notice that the modified IBL is very accurate for a much wider range of α and is reasonable for all relevant values of α.

In order to gauge the accuracy of the modified IBL equations for nonzero values of λ and Λ, we can only compare it to the approximate Benney equation.

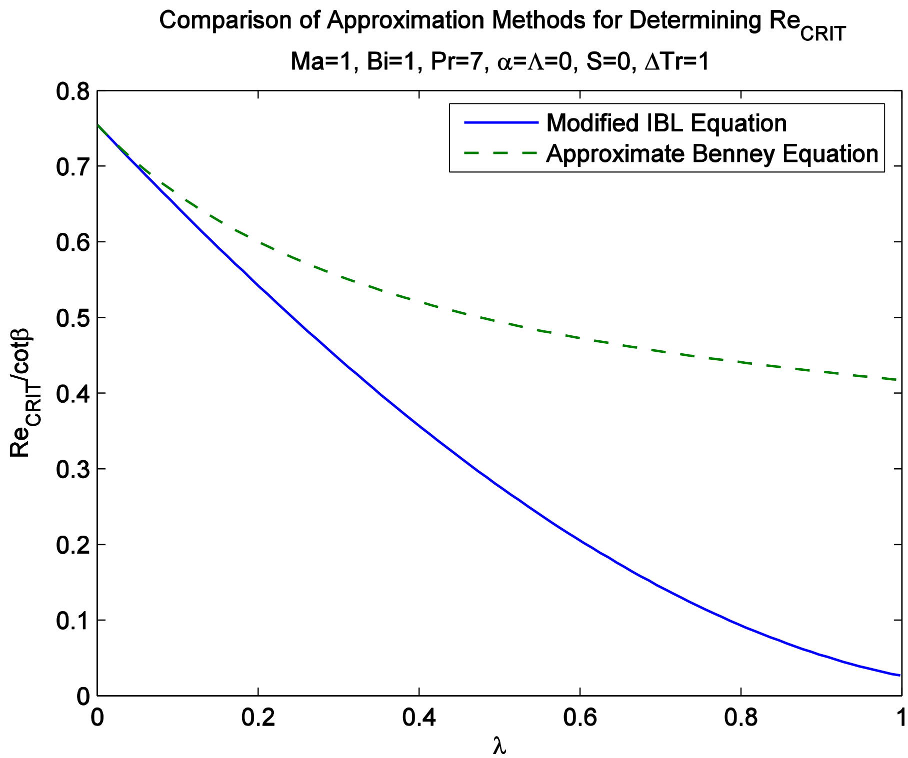


Figure 4.2

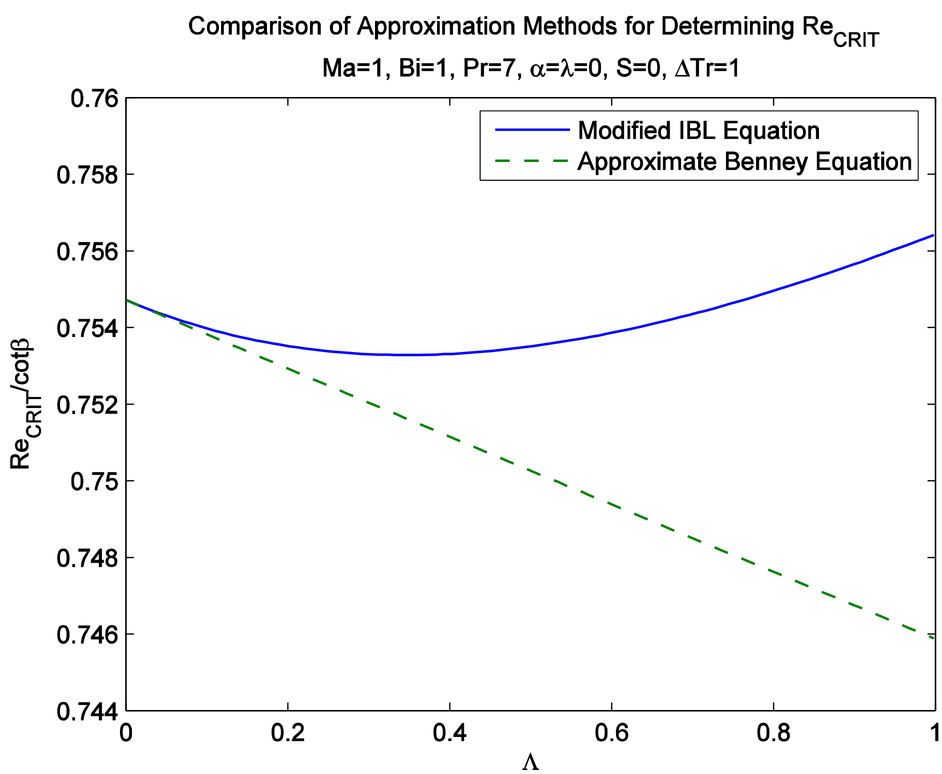


Figure 4.3

We have an excellent agreement with the approximate Benney equation for small values of $\lambda$ and $\Lambda$, where we are confident that the approximate Benney equation is accurate. For larger values we have no direct means of checking the modified IBL equations approximation. We will validate the results by physical interpretation.

## CHAPTER 5 – Discussion of Results from the Linear Stability Analysis

Now that we have some solutions to our system of equations (i.e. we can pinpoint criticality and can tell under what conditions the flow is unstable), we would like to see how criticality is affected by various fluid properties and how these fluid properties interact with each other when determining criticality. For our analysis we will be using the plots of the critical Reynolds number as determined by the modified IBL equations.

It turns out that the expression for the critical Reynolds number is independent of the Weber number, as is the case for the isothermal and basic nonisothermal problems. We point out that the Weber number is the scaled surface tension at the reference temperature $T_b$ (the prescribed temperature of the bottom surface). The Marangoni number, on the other hand, is the scaled gradient of the surface tension with temperature, and measures the effect of thermocapillary forces, which do affect the onset of instability.

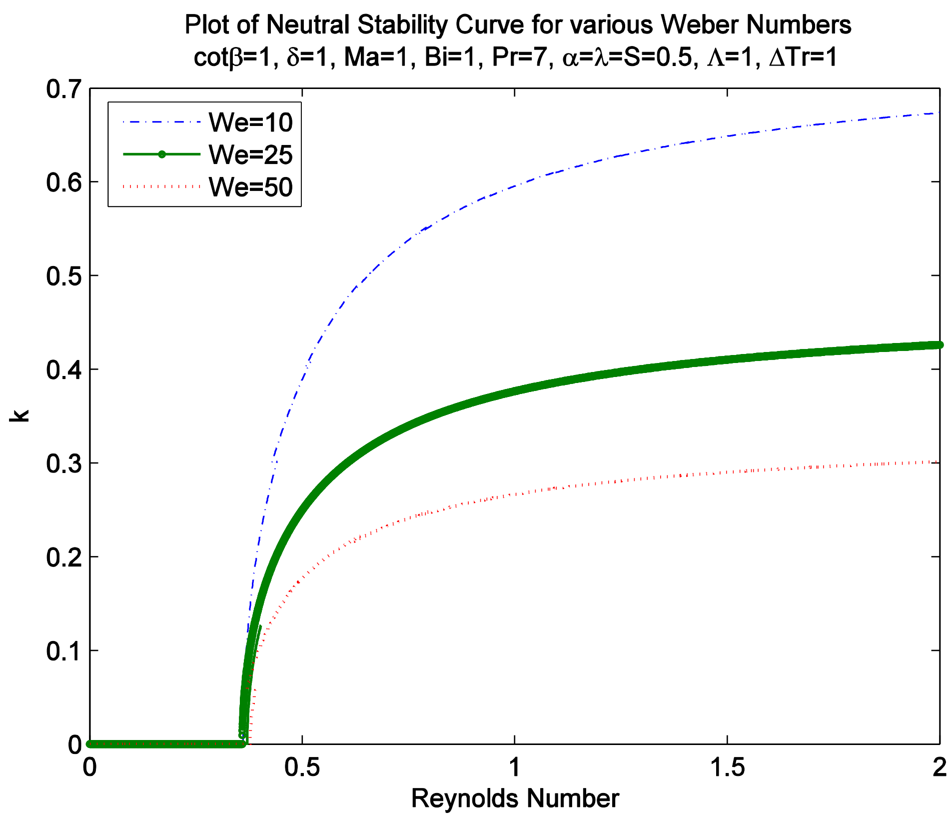


Figure 5.0

In Figure 5.0 we present the neutral stability curve for different values of the Weber number. It can be seen that the minimum Reynolds number (i.e. critical Reynolds number) is the same for all the curves, so the critical Reynolds number is independent of the Weber number. For a supercritical Reynolds number, as the Weber number is decreased the range of unstable perturbations (region below the neutral stability curve) is increased as shorter perturbations (i.e. with larger k) become unstable.

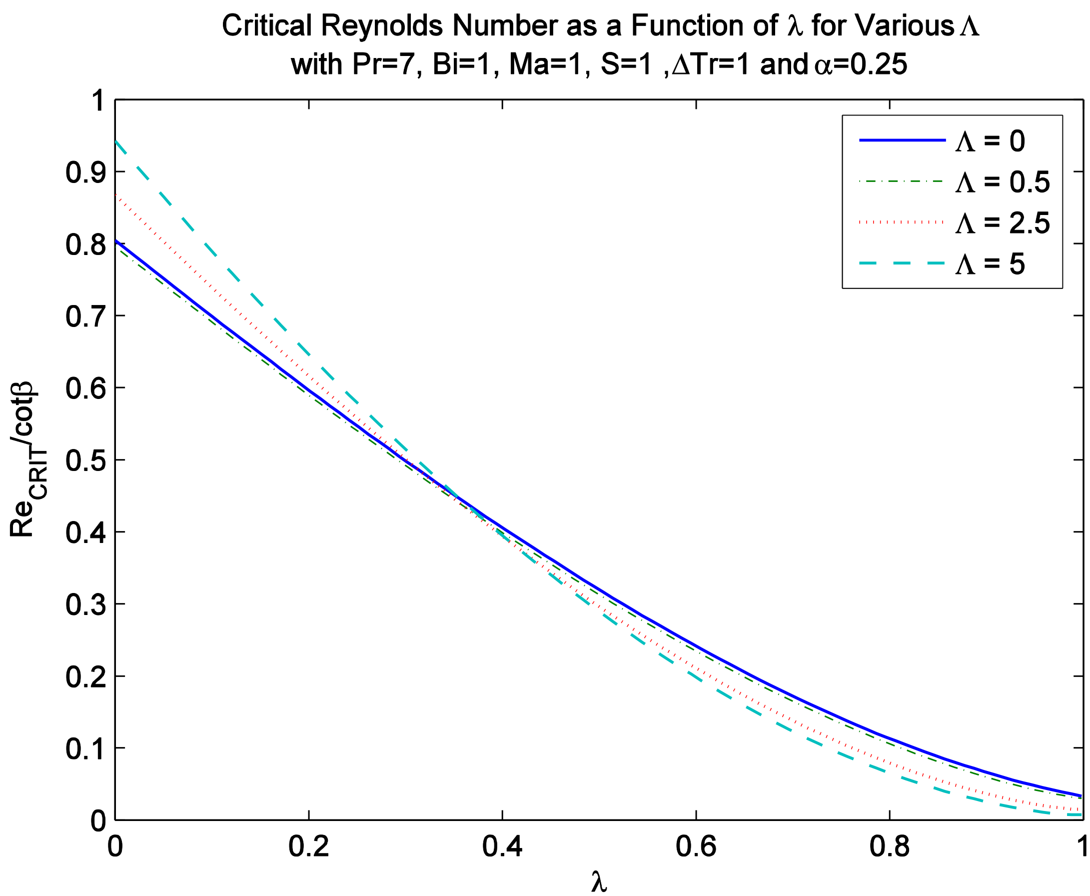

Figure 5.1

We now consider the effect of the viscosity variation parameter λ. It should be pointed out that since we assume the viscosity to decrease with temperature we must restrict the range of λ to nonnegative values less than 1 in order to maintain a positive value for the viscosity. In Figures 5.1 and 5.2 we illustrate the dependence of $Re_{CRIT}$ on λ for various values of other parameters. In all cases $Re_{CRIT}$ decreases with λ, thus indicating that increasing λ destabilizes the flow. This is to be expected since for a heated fluid, increasing λ lowers the viscosity, and viscosity is a stabilizing factor since it counteracts the effect of inertia. In Figure 5.1 we present the distribution of $Re_{CRIT}$ with λ for different values of the thermal conductivity variation parameter, Λ. It is apparent that for $\lambda \gtrsim 0.35$, increasing Λ has a destabilizing effect, while for $0.35 \gtrsim \lambda$, increasing Λ from zero to 0.5 lowers $Re_{CRIT}$ and beyond that the opposite happens. Now, the anticipation is that an increased thermal conductivity stabilizes the flow since it smoothes out temperature differences and thus weakens the thermal effects. However, it should be pointed out that, while the conductivity increases with Λ for a fixed temperature difference, as conductivity increases the temperature gradient is actually being reduced thus lowering the increment to the conductivity. Therefore, the exact dependence of thermal conductivity on Λ is difficult to ascertain, and there is a complicated non-monotonic dependence of $Re_{CRIT}$ on Λ as is illustrated by the results in Figure 5.1.

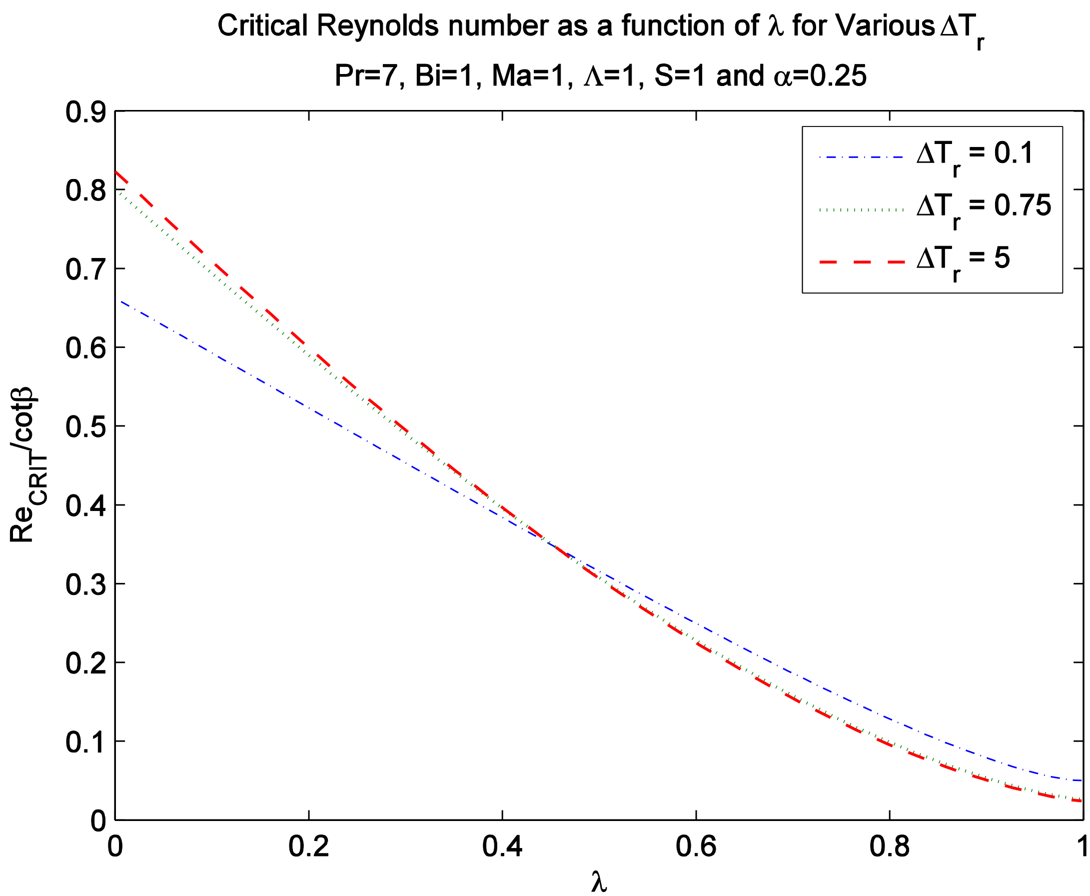

Figure 5.2

Figure 5.2 contains the distribution of $Re_{CRIT}$ with λ for different values of the relative difference between the temperature prescribed at the bottom and that of the ambient, $\Delta T_r$. It can be seen that as $\Delta T_r$ is increased there is a greater variation in $Re_{CRIT}$ over the interval $0 \leq \lambda \leq 1$. This is due to the fact that increased temperature differences amplify the variation in viscosity as λ is increased. The results in Figure 5.2 also indicate that $Re_{CRIT}$ quickly approaches a constant value as $\Delta T_r$ increases. More specifically, it can be seen that for a fixed λ the change in $Re_{CRIT}$ is very small as $\Delta T_r$ increases beyond 0.75.

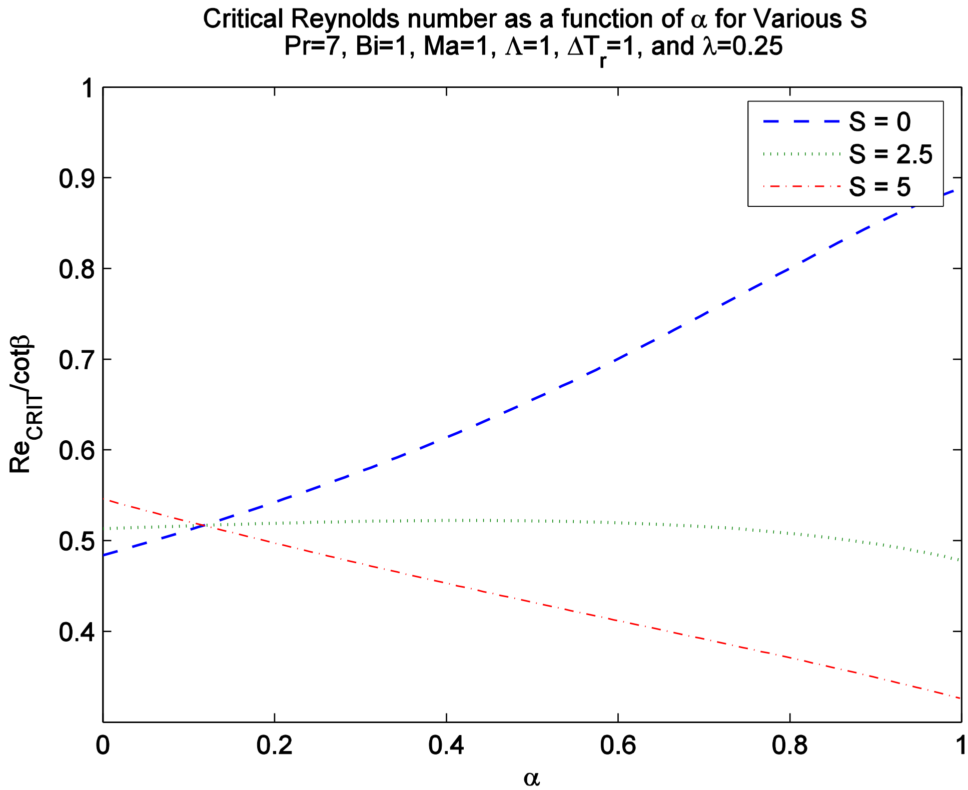


Figure 5.3

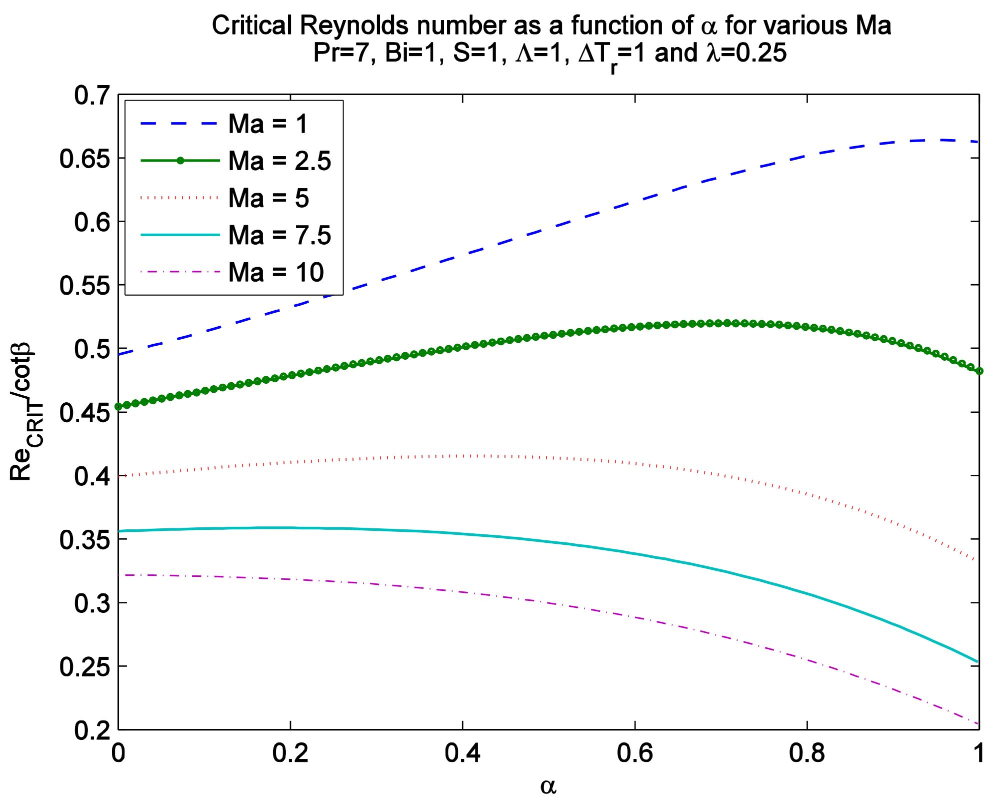


Figure 5.4

In Figures 5.3 and 5.4 we display the effect of the density variation parameter, α, on the stability of the flow. The considered values of α are less than 1 in order to obtain positive values for the density. It is evident from the results that, unlike with λ, depending on the value of other parameters, increasing α can result in an increase or decrease in $Re_{CRIT}$. In general, a decrease in mass density reduces inertia and stabilizes the flow. However, for our problem the vertical temperature gradient in the fluid results in a top-heavy density stratification. The density differences associated with depth fluctuations resulting from surface waves can combine with thermocapillary forces and destabilize the flow. In Figure 5.3 we see that if the specific heat variation parameter S is sufficiently large, then the density variation acts to destabilize the flow. This is explained by the fact that an increase in the specific heat of the fluid decreases thermal diffusivity and thus steepens temperature gradients and consequently accentuates the density stratification. The results in Figure 5.4 reveal that the same effect occurs if Ma is sufficiently large. In other words, with substantial thermocapillary action, increasing the density variation destabilizes the flow.

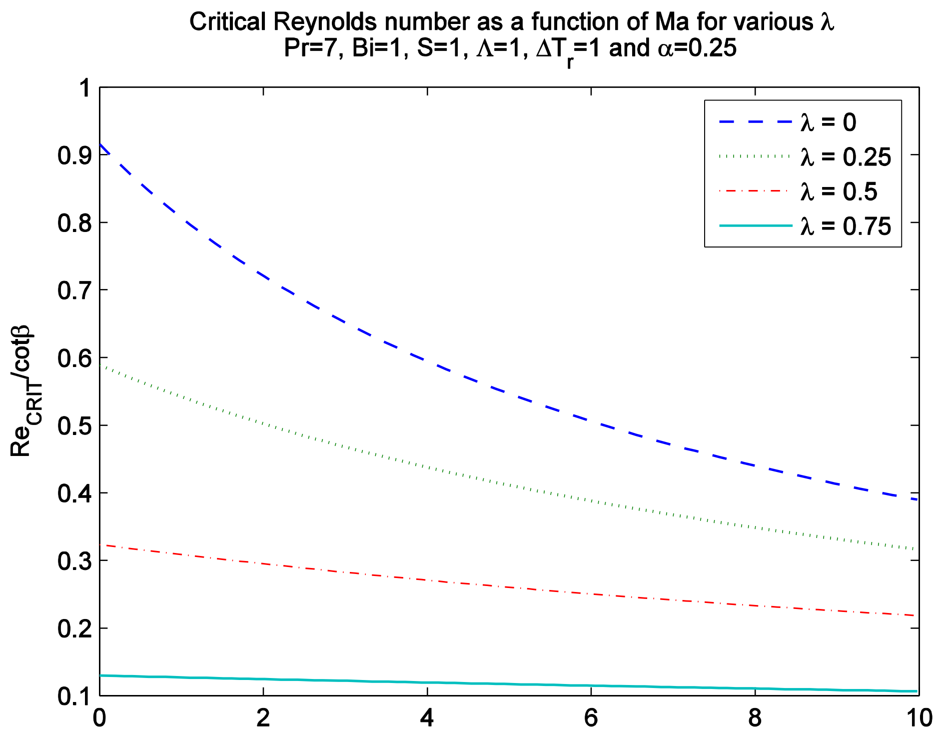

Figure 5.5

In Figure 5.5 we present the variation of $Re_{CRIT}$ with Ma for different values of λ. In all cases $Re_{CRIT}$ decreases with Ma in accordance with the expectation that strengthening the thermocapillary effects acts to destabilize the flow. Another interesting observation however, is that as λ increases there is less variation in $Re_{CRIT}$ with Ma. Indeed, in the case with λ = 0.75, $Re_{CRIT}$ is essentially independent of Ma. We can then conclude that if the viscosity is sufficiently reduced the resulting increase to flow inertia is the dominant instability mechanism and the contribution from the Marangoni effect is negligible.

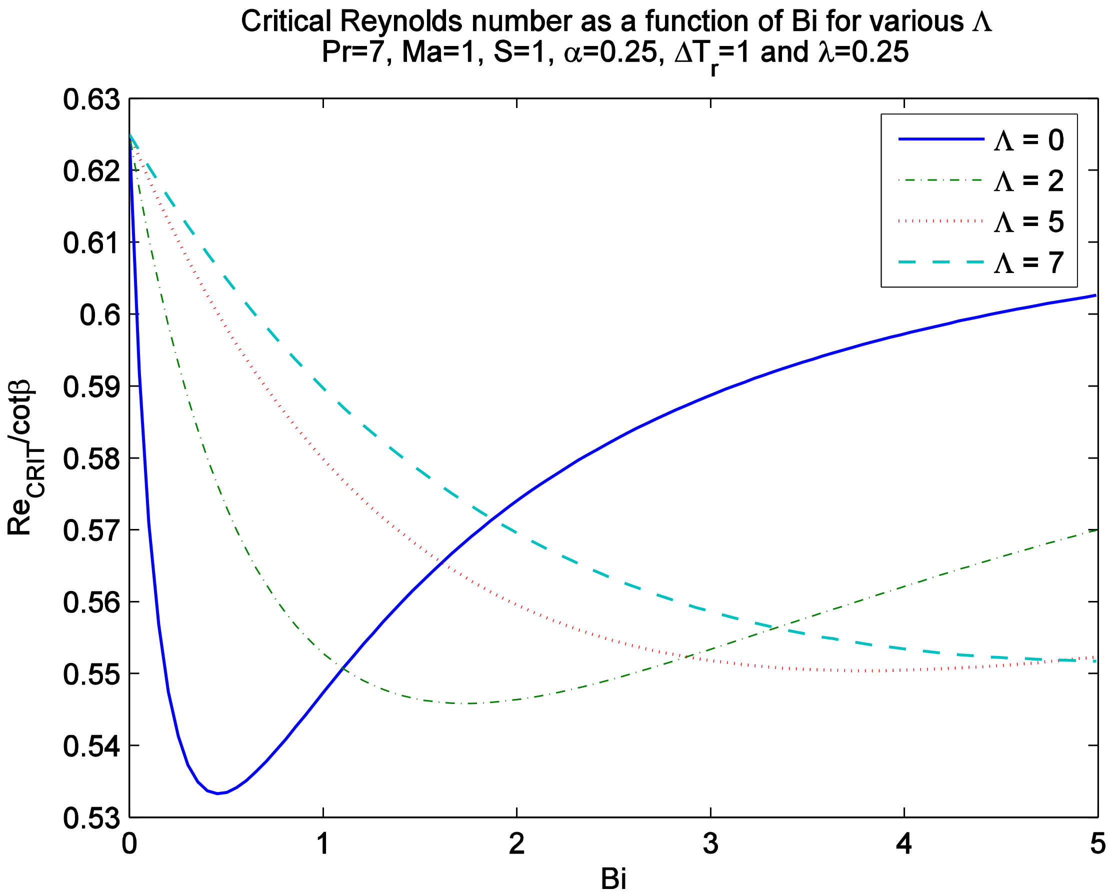


Figure 5.6

In Figure 5.6 we illustrate the variation of $Re_{CRIT}$ with the Biot number for several values of Λ. With Bi = 0 there is no heat transfer across the fluid-air interface and as a result the surface remains at a constant temperature. In the absence of temperature variation along the surface the Marangoni effect is neutralized. As the Biot number is increased from zero, $Re_{CRIT}$ decreases as the destabilizing thermocapillary effects become more significant. However, as Bi approaches infinity Newton's Law of Cooling requires that the temperature of the surface must approach that of the ambient medium and thus the temperature variations along the interface vanish. Consequently as Bi approaches infinity $Re_{CRIT}$ asymptotically approaches the value at Bi = 0. Therefore, the $Re_{CRIT}$ distribution with Bi has a minimum at a positive value of Bi. This critical value is a complicated function of the various temperature variation parameters. The results in Figure 5.6, for example, suggest that the critical value of Bi increases with Λ for fixed values of the other parameters.

## CHAPTER 6 – Nonlinear Simulations:

In this final chapter we obtain numerical simulations of the fully nonlinear modified IBL solutions. While we could not solve the nonlinear modified IBL equations analytically, we were able to solve them using numerical analysis techniques; namely the MacCormack Method and the Crank-Nicolson scheme via the Fractional Step method. The nonlinear simulations are used to verify the predictions for the onset of instability made by the linear theory and also determine the evolution of unstable flows.

### A. Numerical Analysis

We begin with our modified IBL equations (60, 64, and 65), as were derived previously in chapter 4. In order to solve this system numerically, we express these equations in a form suitable for applying numerical methods. Our first step is to discard terms of $O(\delta S^2)$ from the temperature equation (64). Dividing through the equation by 2, we get the following factor for the $\theta_t$ coefficient:

$$\Theta \equiv 1 + S\left(\frac{1}{\Delta T_r} + \frac{1}{2}\right) + \frac{3}{2}S\theta$$

Dividing through by $\Theta$, we obtain $\Theta^{-1}$ on the right-hand side of the equation, which we approximate by a truncated binomial series as $\Theta^{-1} = 1 - S\left(\frac{1}{\Delta T_r} + \frac{1}{2}\right) - \frac{3}{2}S\theta + O(S^2)$ in the $\delta$ terms. Discarding the $O(\delta S^2)$ terms we arrive at a new temperature equation:

$$\delta\theta_t = \delta\left\{\frac{7(1-\theta)q_x}{40h} - \frac{27q\theta_x}{20h} + S\left[\frac{q_x}{16h} - \frac{\theta q_x}{8h} + \frac{\theta^2 q_x}{16h} + \frac{3q\theta_x}{40h} - \frac{3q\theta\theta_x}{40h}\right]\right\} + \frac{3[1-(1+\mathrm{Bi}h)\theta]}{\mathrm{Pr}\,\mathrm{Re}\,h^2\Theta} - \frac{3\Lambda(\theta^2-1)}{2\,\mathrm{Pr}\,\mathrm{Re}\,h^2\Theta} \qquad (66)$$

With these three equations (60), (65) and (66), we make a further modification. Instead of working with θ, we introduce Φ, related to θ by the equation $\Phi=(\theta-1)h$. From the relation $(T-1)h=(\theta-1)z$, it follows that the variable Φ is related to T through:

$$\int_0^h (T-1)\,dz = \frac{\Phi}{2}$$

and thus, Φ is proportional to the lineal heat content stored in the fluid layer.

We introduce Φ into the x-Momentum equation and rearrange:

$$q_t + \frac{\partial}{\partial x}\left[\frac{9}{7}\frac{q^2}{h} + \frac{5}{4}\frac{\cot\beta}{\mathrm{Re}}(1-\alpha)h^2 + \frac{5}{4}\mathrm{Ma}\frac{\Phi}{h} + \frac{5}{96}\lambda\mathrm{Ma}\frac{\Phi^2}{h^2} - \frac{11\alpha\cot\beta}{16\mathrm{Re}}h\Phi\right] = \frac{qq_x}{7h} + \frac{5}{6}\delta^2\mathrm{We}hh_{xxx} + \frac{7\alpha\cot\beta}{16\mathrm{Re}}h_x\Phi + \frac{5}{2\mathrm{Re}\delta}\left(h - \frac{q}{h^2}\right) - \frac{5\alpha}{16\mathrm{Re}\delta}(8h+5\Phi) + \frac{5\lambda}{8\mathrm{Re}\delta}\frac{q}{h^3}(4h+\Phi) \qquad (67)$$

We introduce Φ into the Temperature equation (making use of the fact $h_t$=-$q_x$):

$$\Phi_t + \frac{\partial}{\partial x}\left[\frac{27}{20}\frac{\Phi q}{h} + \frac{3}{80}S\frac{q\Phi^2}{h^2}\right] = \frac{7}{40}\frac{\Phi q_x}{h} + \frac{S}{10}\frac{\Phi^2 q_x}{h^2} - \frac{3(Bih + Bi\Phi + \Phi/h)}{\delta Pr Re h\Theta} - \frac{3\Lambda(2\Phi h + \Phi^2)}{2\delta Pr Re h^3\Theta} \qquad (68)$$

where $\Theta = 1 + \frac{S}{\Delta Tr} + 2S + \frac{3}{2}S\frac{\Phi}{h}$

Our equations are now in a form suitable for numerical analysis.

We will use the Fractional-Step Method for this problem (LeVeque (2002)). The idea is to split the equation into two steps (decouple the advective and diffusive components) that can be solved in an alternating manner, using our numerical methods. For our purposes we will use MacCormack's method for the first step and for the second we will use Crank-Nicolson.

For the first step we discard the diffusive terms and solve:

$$h_t + q_x = 0$$

$$q_t + \frac{\partial}{\partial x}\left[\frac{9}{7}\frac{q^2}{h} + \frac{5}{4}\frac{\cot\beta}{Re}(1-\alpha)h^2 + \frac{5}{4}Ma\frac{\Phi}{h} + \frac{5}{96}\lambda Ma\frac{\Phi^2}{h^2} - \frac{11\alpha\cot\beta}{16Re}h\Phi\right] =$$
$$\frac{5}{2Re\delta}(h - \frac{q}{h^2}) - \frac{5\alpha}{16Re\delta}(8h + 5\Phi) + \frac{5\lambda}{8Re\delta}\frac{q}{h^3}(4h + \Phi)$$

$$\Phi_t + \frac{\partial}{\partial x}\left[\frac{27}{20}\frac{\Phi q}{h} + \frac{3}{80}S\frac{q\Phi^2}{h^2}\right] = -\frac{3(Bih + Bi\Phi + \Phi/h)}{\delta Pr Re h\Theta} - \frac{3\Lambda(2\Phi h + \Phi^2)}{2\delta Pr Re h^3\Theta}$$

over a time step Δt. In the second step we focus on the diffusive terms and solve

$$q_t = \frac{qq_x}{7h} + \frac{5}{6}\delta^2 Wehh_{xxx} + \frac{7\alpha\cot\beta}{16Re}h_x\Phi$$
$$\Phi_t = \frac{7}{40}\frac{\Phi q_x}{h} + \frac{S}{10}\frac{\Phi^2 q_x}{h^2}$$

using the solution obtained from the first step as the initial condition. The second step then returns the solution for q and Φ at the new time t + Δt, with the solution for h being that obtained in the first step.

The system considered in the first step consists of nonlinear hyperbolic conservation equations with source terms, which we re-write in a more compact form using vectors, as follows:

$$\frac{\partial}{\partial t}U + \frac{\partial}{\partial x}F(U) = B(U), \text{ where } U = \begin{bmatrix} h \\ q \\ \Phi \end{bmatrix},$$

$$F(U) = \begin{bmatrix} q \\ \frac{9}{7}\frac{q^2}{h} + \frac{5}{4}\frac{\cot\beta}{\mathrm{Re}}(1-\alpha)h^2 + \frac{5}{4}\mathrm{Ma}\frac{\Phi}{h} + \frac{5}{96}\lambda\mathrm{Ma}\frac{\Phi^2}{h^2} - \frac{11\alpha\cot\beta}{16\mathrm{Re}}h\Phi \\ \frac{27}{20}\frac{\Phi q}{h} + \frac{3}{80}S\frac{q\Phi^2}{h^2} \end{bmatrix} \text{ and}$$

$$B(U) = \begin{bmatrix} 0 \\ \frac{5}{2\mathrm{Re}\,\delta}(h - \frac{q}{h^2}) - \frac{5\alpha}{16\mathrm{Re}\,\delta}(8h + 5\Phi) + \frac{5\lambda}{8\mathrm{Re}\,\delta}\frac{q}{h^3}(4h+\Phi) \\ -\frac{3(\mathrm{Bi}h + \mathrm{Bi}\Phi + \Phi/h)}{\delta\mathrm{Pr}\,\mathrm{Re}\,h\Theta} - \frac{3\Lambda(2\Phi h + \Phi^2)}{2\delta\mathrm{Pr}\,\mathrm{Re}\,h^3\Theta} \end{bmatrix}$$

While there are several methods available to solve this system, its complicated eigenstructure makes the use of eigen-based methods impractical. We resort to MacCormack's method since it can be applied component-wise and does not require the eigenstructure of the system. MacCormack's method is a conservative second-order accurate finite difference scheme, which correctly captures discontinuities and converges to the physical weak solution of the problem. LeVeque & Yee (1990) extended MacCormack's method to include source terms via the explicit predictor–corrector scheme

Predictor Step: $\quad U_j^* = U_j^n - \frac{\Delta t}{\Delta x}\left[F(U_{j+1}^n) - F(U_j^n)\right] + \Delta t B(U_j^n)$

Corrector Step: $\quad U_j^{n+1} = \frac{1}{2}(U_j^n + U_j^*) - \frac{\Delta t}{2\Delta x}\left[F(U_j^*) - F(U_{j-1}^*)\right] + \frac{\Delta t}{2}B(U_j^*)$

where $U_j^n \equiv U(x_j, t_n)$ and the x-t plane is discretized such that the mesh width is denoted as $\Delta x$ and the time step is denoted as $\Delta t$.

In the second step, we have a coupled system of generalized one-dimensional nonlinear diffusion equations. Discretizing using the Crank–Nicolson scheme and using the output from the first step as an initial condition, leads to a nonlinear system of algebraic equations, which was solved iteratively. A robust algorithm, taking advantage of the structure and sparseness of the resulting linearized systems, was used to speed up the iterative process. It was found that convergence was reached quickly, typically in less than five iterations.

This process yields a numerical solution for our system.

## B. Nonlinear Stability Analysis

To perform a stability analysis of the flow we begin by solving our equations on a periodic spatial domain, from 0 to L. As the initial condition, we use the base flow, with a small amplitude sinusoidal perturbation of length L added to h:

$$h = 1 + 0.0001\sin\left(\frac{2\pi}{L}x\right)$$

$$q = q_s$$

$$\theta = \theta_s$$

With this small perturbation now in our system, we calculate the evolution and determine if it is amplified or dampened. If the evolution of the wave is amplified, then the system is unstable. By iterating over Re we can determine the value for which a perturbation with wavenumber k=2π/L is neutrally stable and thus by considering different L values we obtain points on the neutral stability curve. For longer L values (L>3) we use a mesh width of Δx=0.02 which required, for numerical stability, a time step of $\Delta t=10^{-4}$. For the shorter L values we used Δx=0.005 and $\Delta t=7 \times 10^{-6}$. In Table 6.1 we compare these results with those from the linear analysis. The indication is that there is excellent agreement between the two. So we conclude that the linear analysis is effective in predicting neutral stability.

| L | k=2π/L | Re (Linear) | Re (Nonlinear; Simulation) | Difference | Percentage Difference (%) |
|---|---|---|---|---|---|
| 1 | 6.2832 | 0.35215 | 0.35550 | 0.00335 | 0.9423 |
| 3 | 2.0944 | 0.28346 | 0.28375 | 0.00029 | 0.1020 |
| 5 | 1.2566 | 0.27911 | 0.27935 | 0.00024 | 0.0859 |
| 10 | 0.6283 | 0.27731 | 0.27745 | 0.00014 | 0.0504 |
| 15 | 0.4189 | 0.27699 | 0.27715 | 0.00016 | 0.0577 |
| 20 | 0.3142 | 0.27689 | 0.27700 | 0.00011 | 0.0397 |
| 50 | 0.1257 | 0.27678 | 0.27690 | 0.00012 | 0.0433 |
| 100 | 0.0628 | 0.27675 | 0.27685 | 0.00010 | 0.0361 |

Table 6.1: – Comparison of Linear and Nonlinear Results (with α=λ=Λ=S=0.25, $\Delta T_r=1$, We=10, δ=0.05 and cotβ=0.5)

For supercritical conditions the nonlinear simulations on a sufficiently long domain can be used to determine the evolution of the unstable flow. The advantage of the nonlinear simulations is that they include the nonlinear interactions of the perturbations and thus capture the entire instability mechanism of the flow. Furthermore, for unstable flows, the temporal evolution can be continued until the growth of the disturbances reaches saturation. An illustration of the evolution of an unstable film flow is given in Figure 6.1. Notice that at time t=40 our small amplitude sinusoidal perturbation makes very small waves in our system. As time goes on that small perturbation becomes larger and larger, until finally we have a permanent wave structure at time t=140. In other words these solitary waves will not subside or grow in time and will propagate with a constant speed.

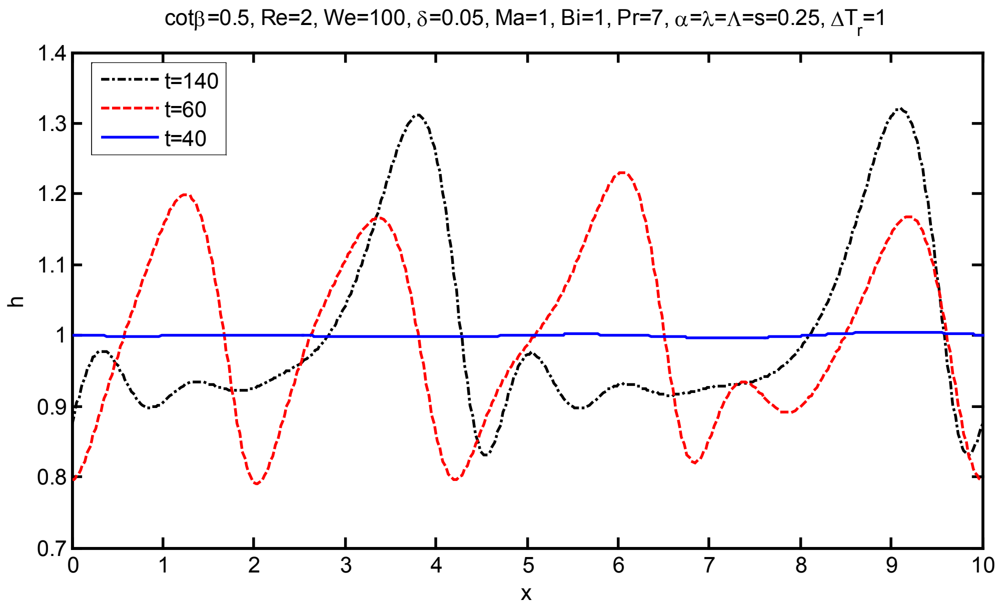

Figure 6.1: Evolution of an Unstable Film Flow

In Figure 6.2 we compare the permanent surface profiles for unstable flows with different Reynolds numbers. Notice that for the larger Re values the instability leads to large solitary-wave structures with the height increasing with Re. However, for smaller Re values the flow is “less unstable” with the interfacial deflection being almost sinusoidal with small amplitude.

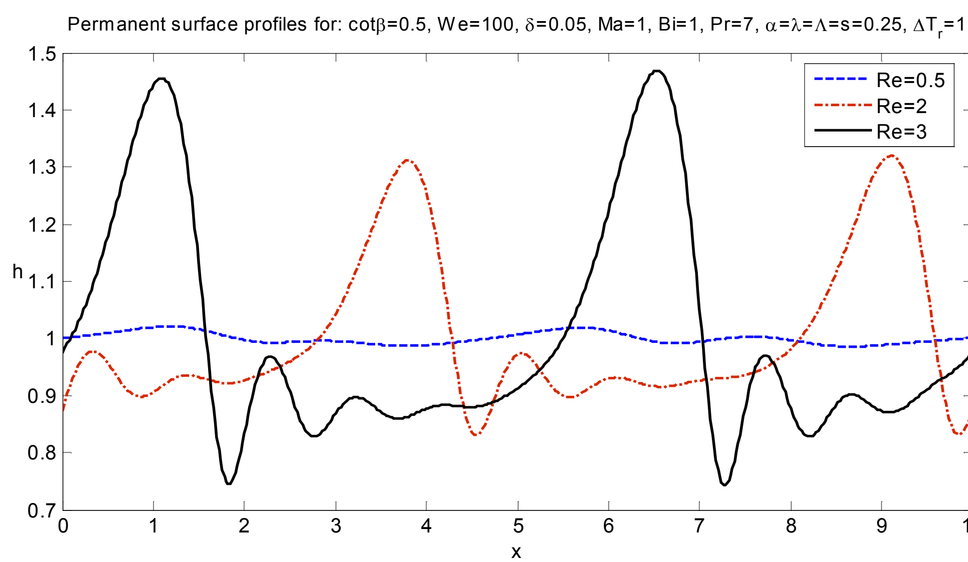

Figure 6.2: Permanent Surface Profile for an Unstable Flow

In Figure 6.3 we consider another plot showing the relation between surface temperature and wave height for a permanent solution. We notice that at the crest of the waves the surface is cooler; and at the troughs the surface is warmer. This makes sense as increasing the distance from the surface of the fluid to the heated ramp will cool the fluid's surface and vice versa.

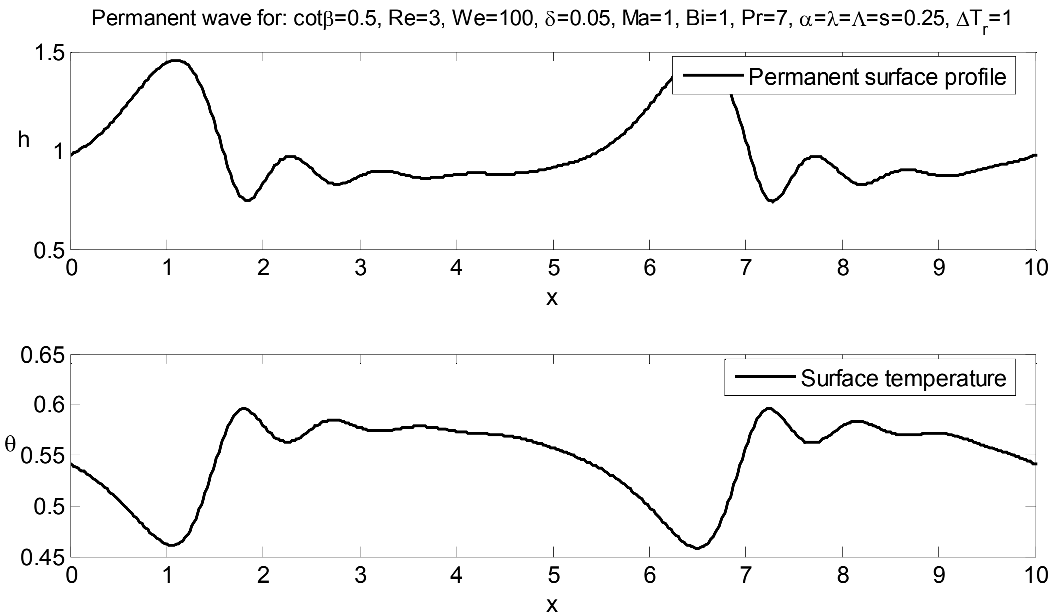


Figure 6.3: Relation Between Surface Temperature and Wave Height

## CONCLUSION/CLOSING REMARKS

The purpose of this thesis was to examine how the flow of a fluid film down a heated inclined plane is affected by temperature dependent fluid properties. The effects of five different temperature dependent fluid properties were examined: surface tension, mass density, dynamic viscosity, thermal conductivity and specific heat capacity. Each of these can be significantly affected by changes in temperature and can have either stabilizing or destabilizing effects on the fluid flow.

The investigation utilized a theoretical model based on the conservation of mass, momentum and energy and included the physically appropriate Newton's Law of Cooling to incorporate temperature changes on the surface of the film. This system was too complicated to be solved analytically, so a linear stability analysis was carried out on this system, where the equations were linearized with respect to the perturbations that were introduced into the system. The linearized perturbation equations were still too complex to solve, with the exception of two special cases: the case where Bi=0 and the case where $\lambda=\Lambda=0$. Each of these cases was examined.

In the case where Bi=0 (a perfectly insulated surface), the critical Reynolds number was found to be $\frac{5}{6}\cot\beta\left(\frac{(1-\lambda)^2}{1-\alpha}\right)$. For a fluid with a perfectly insulated surface, the temperature variations in specific heat capacity and thermal conductivity play no role in determining criticality for the stability of the flow. The critical Reynolds number was also found to be independent of the Marangoni number, for this special case.

In the second case we set $\lambda=\Lambda=0$. The critical Reynolds number for this case showed coupling between the specific heat and the mass density variations. The formula revealed a stronger dependence on the variation in mass density than on that in specific heat. By allowing all other temperature dependent fluid properties (except surface tension) to be zero, the critical Reynolds number reduced to the same one obtained by Trevelyan et al (2007) for the basic non-isothermal problem.

In an attempt to make analytic progress in the general case we exploited the assumed shallowness of the fluid layer by creating a two-scale model for our problem where height and length were scaled differently. A ratio of the two scales ($\delta$ = Height/Length) was introduced into the system. $\delta$ is small, so $O(\delta^2)$ terms were discarded from the system. As per the approach used by Benney (1966) an asymptotic solution (as $\delta\to 0$) to this system was attempted. Although this system was still too complex, we were able to generate a single nonlinear evolution equation for the position of the free surface for the special case where $\lambda=\Lambda=0$. The neutral stability curve and critical Reynolds number were found to be identical to those from the analysis of the full equations when we allowed $\lambda=\Lambda=0$. This exact agreement verified our use of the Benney equation for our system with temperature dependent fluid properties.

In an effort to include temperature variation in all the fluid properties (no special cases) a Benney equation was attempted using perturbation expansions with respect to the temperature variations, as ($\alpha$, $\lambda$, $\Lambda$, S)$\rightarrow$0. This approximation was found to be accurate for small values of the variation parameters $\alpha$, $\lambda$, $\Lambda$ and S.

A depth-integrated model for our problem was also considered. Using the approach employed Ruyer-Quil and Manneville (2000) and the extension by Trevelyan et al. (2007) for the basic non-isothermal problem, we established modified IBL equations for the flow. This approximation was found to be accurate in comparison with the full equations for the special cases when the full equations could be analyzed.

Using the results from the modified IBL approach, the effects of the various temperature dependent fluid properties on the criticality of the fluid flow were examined and discussed.

We also compared our linear analysis of the modified IBL equations to the results from nonlinear simulations. The fully nonlinear modified IBL equations were solved by decoupling the advective and diffusive components and using MacCormack's method for the advective part and the Crank-Nicolson scheme for the diffusive part. The agreement between the linear stability analysis and the nonlinear analysis was found to be excellent. The nonlinear simulations were also used to determine the evolution of the unstable flow.

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