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A Predator-Prey Model In Deterministic And Stochastic Environments

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A PREDATOR-PREY MODEL IN DETERMINISTIC AND STOCHASTIC ENVIRONMENTS

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Abstract

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We investigate the phase portraits, the uniqueness of limit cycles and the Hopf bifurcations in the Holling-Tanner models in deterministic and stochastic environments. We provide the conditions on the parameters to assure saddle, focus and node. We use numerical simulations to demonstrate our results in the deterministic cases. We also explore the Holling-Tanner model in a stochastic environment by using numerical simulations. We generalize and improve some new results on Holling-Tanner model from Lotka-Volterra model on real ecological systems.

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Finally, my deepest thanks go to my parents for their unconditional love and supports. My work is dedicated to my beloved wife, Iraint, and to our children Iksa and Mundhum, for all the joy and hope they bring to my life.

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Introduction

1.1 Background

In recent years, there are many new challenging and exciting problems arising in mathematical biology and ecology. The horizon of this research is widening as well as deepening day by day due to the exotic usefulness and importance for human beings. There has also been a growing concern about the preservation of the ecological balance in nature, which plays an important role in the harmony among different species. As a result, mathematical models for populations with interaction between species have become popular among biologists and scientists in general. The remarkable exhibition of a variety of dynamical behaviours can be seen in many plant, insect and animal species. This has stimulated a great interest in the study of the dynamical systems of populations or ecosystem. For instance, predator-prey models are widespread and come in many flavours in population ecology.

The Holling-Tanner models are well known models for studying the interactions of predator and prey species. The stability of the two interacting species may depend upon the intrinsic predator and prey growth rates. Some general assumptions are considered in the case of interaction of prey-predator systems:

(*i*) If the two populations inhabit in the same area then their densities are directly proportional to their numbers. (*ii*) There is no time lag in the responses of either populations to changes. (*iii*) There is an abundance of food supply for prey. (*iv*) The prey is the sole source of food for the predator [33].

One of the basic relations between the prey and the predator is given by the ecological and the social models. This is also the base block of more complicated food chain, food web as well as biochemical structure.

The Holling-Tanner model was studied for its accuracy in describing the real ecological systems like mite/spider mite, Canadian lynx/ snowshoe hare, sparrow/ sparrow hawk and more by Tanner [33].

The dynamical properties of the Holling-Tanner model equation have been widely studied. May [28] applied Kolmogorov's theorem [23] to provide the criterion for the stability of the interior equilibrium or the existence of stable limit cycles. May's method was simplified by Tanner [33]. Murray [29] and Hsu and Huang [19] provided some general conditions under which the interior equilibrium is a stable node or focus or an unstable node or focus, respectively, and under which the Holling-Tanner model equation possesses a stable limit cycle. However, under these conditions, one can determine that neither the interior equilibrium is a node nor it is a focus. In addition, an open question left in [19] is whether the stable limit cycle obtained under a certain condition is unique. We refer to [15, 19, 24] for the study of the global stability of the Holling-Tanner equation and to [14] for the study of the existence of the solution. The references [8, 19, 29] discuss the Holling-Tanner models with delays and other similar systems.

In this thesis, we study the phase portraits, the Hopf bifurcations, and the existence and uniqueness of the stable limit cycles near the interior equilibrium of the equation.

Statement of problems

The main problems are:

1. Extinction can be a natural occurrence caused by an unpredictable catastrophe, chronic environmental stress or ecological interactions such as competition, disease or predation.

2. Many species of animals and plants are at the risk of extinction. Some of them are wiped out from the world.

3. We need to figure the condition of critical equilibrium point with suitable values of parameters, so that we can keep preservable size of endangered species.

Goals

The main objectives of the thesis are the following:

1. To investigate the phase portraits, the uniqueness of limit cycles and Hopf bifurcations in the Holling-Tanner models.

2. To find different types of equilibria on changing suitable values of parameters.

3. To use numerical simulations to demonstrate our results in the deterministic cases.

4. To figure out critical equilibrium size of populations so that it helps to establish and sustain efficient conservation of species which are at the edge of extinction as well as it also supports for development of recovery strategies such as collection of base line information about the threats to a species' survival and identification of the critical habitat of species.

Methodology

The Holling-Tanner model is nonlinear first order differential equation. First of all, we will convert our model in nondimensionlization form, so we can reduce six parameters to three parameters. Then, it will be easy to analyse different types of equilibria. Furthermore, we use linearization technique to obtain Jacobian matrix and to investigate the Hopf bifurcation in the Holling-Tanner models. Finally, we include some graphical approaches with help of numerical stochastic simulations by using MATLAB.

1.2 Lotka-Volterra system

Volterra (1926) first proposed a simple model for the predation of one species by another, to explain the oscillatory levels of certain fish catches in the Adriatic. If $X(T)$ is the prey population and $Y(T)$ that of the predator at time T, then Volterra's model [29] is:

$$\begin{cases} \dfrac{dX}{dT} &= a_1X - a_2XY = X(a_1 - a_2Y), \\ \dfrac{dY}{dT} &= b_1XY - b_2Y = Y(b_1X - b_2), \end{cases} \tag{1.1}$$

where a_1, a_2, b_1 and b_2 are positive constants.

Figure 1.1: An eagle is predator and fish is prey.

Figure 1.2: A lion is predator and a zebra is prey.

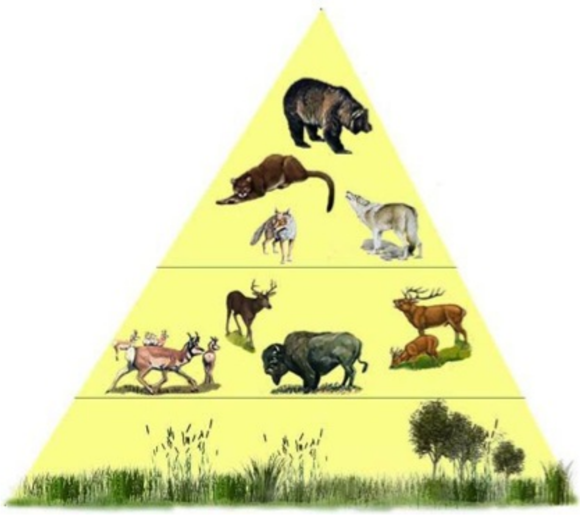

Figure 1.3: This is the ecological pyramid in which the upper part is always predator and the lower part is always prey.

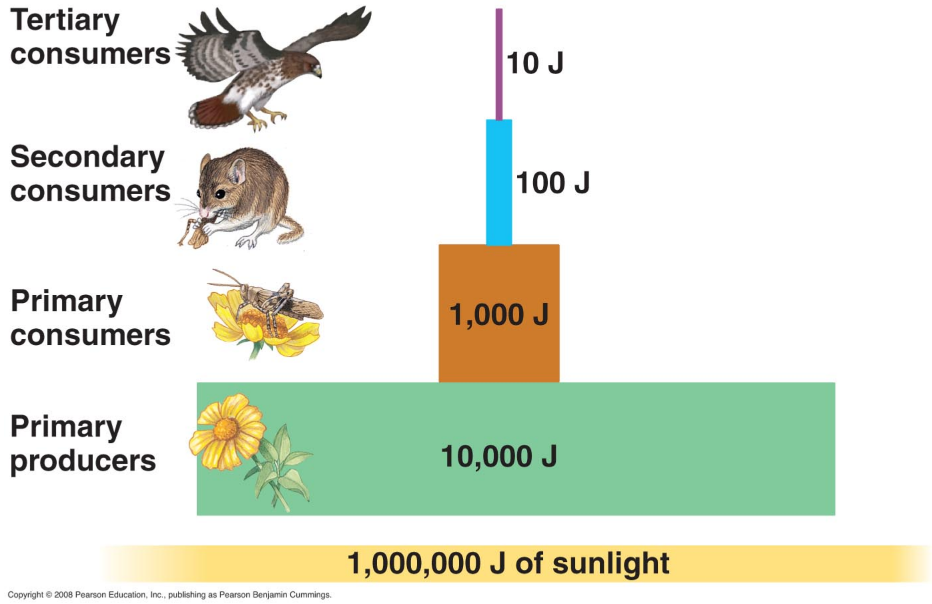

Figure 1.4: The energy distribution in ecological pyramid shows that the amount of energy consumption is decreasing from predator to prey.

The assumptions in the model are given below: (i) The prey in the absence of any predation grows unboundedly in Malthussian way; this is the a_1X term in the first part of (1.1). (ii) The effect of the predation is to reduce the prey's per capita growth rate by a term proportional to the prey and predator populations; this is the $-a_2XY$ term. (iii) In the absence of any prey for sustenance the predator's death rate results in exponential decay, that is, the $-b_2Y$ term in the second part of (1.1). (iv) The prey's contribution to the predator growth rate is b_1XY; that is, it is proportional to the available prey as well as to the size of the predator population. The XY term can be thought of as representing the conversion of energy from one source to another: a_2XY is taken from the prey and b_1XY accrues to the predators.

The model (1.1) is the well known Lotka-Volterra model since the same equation was derived by Lotka ($1920-1925$) from a hypothetical chemical reaction [29] when he could exhibit periodic oscillations in chemical concentrations.

As a first step in analysing the Lotka-Volterra model, we nondimensionalise the system by writing

$$u(\tau) = \frac{b_1X(t)}{b_2}, v(\tau) = \frac{a_2Y(t)}{a_1}, \tau = at, \alpha = \frac{b_2}{a_1}, \tag{1.2}$$

and it becomes

$$\frac{du}{d\tau} = u(1-v), \frac{dv}{d\tau} = \alpha v(u-1). \tag{1.3}$$

In the (u,v) phase plane, we obtain

$$\frac{dv}{du} = \alpha\frac{v(u-1)}{u(1-v)} \tag{1.4}$$

which has the singular points at $u=v=0$ and $u=v=1$.

1.3 Basic definitions and notations

1.3.1 Definitions of limit cycle, globally stable, equilibria and linear or non-linear differential equations

Definition 1.3.1. We wish to study the behaviour of solutions of a two-dimensional system.

$$\begin{cases} \dfrac{dx}{dt} = f(x, y), \\ \dfrac{dy}{dt} = g(x, y). \end{cases} \tag{1.5}$$

by studying the phase portrait in the phase plane [10]. Let (x(t), y(t)) be a solution of (1.5) that is bounded as $t \to \infty$. The positive C^+ of this solution is defined to be the set of points (x(t), y(t)) for $t \geq 0$ in the (x, y)- plane. The limit set $L(C^+)$ of the semi-orbit is defined to be the set of all points $(\bar{x}, \bar{y})$ such that there is a sequence of times $t_n \to \infty$ with $x(t_n) \to \bar{x}$, $y(t_n) \to \bar{y}$ as $n \to \infty$. For example, if the solution (x(t), y(t)) tends to an equilibrium (x^*, y^*) as $t \to \infty$, then the limit set consists of the equilibrium (x^*, y^*). If (x(t), y(t)) is a *periodic solution*, so that the semi-orbit C^+ is a closed curve, then the limit set $L(C^+)$ consists of all points of the semi-orbit C^+. The $Poincaré - Benedixon$ theorem, states that if C^+ is a bounded semi-orbit whose limit set $L(C^+)$ contains no equilibrium points, then either C^+ is a periodic orbit and $L(C^+) = C^+$, or $L(C^+)$ is a periodic orbit, called a *limit cycle*, which C^+ approaches spirally (either from inside or outside).

Definition 1.3.2. *Globally asymptotically stable*:

If a limit set contains more than one equilibrium, then it must also contain orbits joining these equilibria. In essence, we can say that a bounded solution tends either to an equilibrium or to limit cycle, overlooking such "unlikely coincidences" as the possibility of a running from a saddle point to itself. Thus, if we can show that, for a given system, all solutions are bounded but there are no asymptotically stable equilibrium points, we can deduce that there must be at least one periodic orbit. This situation will arise in our study of predator-prey systems. If there is only one periodic orbit, then it must be *globally asymptotically stable* [10] in the sense that every orbit tends to it. If there is more than one periodic orbit, each must be asymptotically stable from at least one side: orbits may spiral toward it from the inside, from the outside, or both.

Definition 1.3.3. An *equilibrium* is a solution (x^*, y^*) of the pair of equations

$$\begin{cases} f(x^*, y^*) = 0, \\ g(x^*, y^*) = 0. \end{cases} \tag{1.6}$$

Thus an equilibrium of (1.5) is a constant solution (or critical points) of the system of differential equations.

Definition 1.3.4. [2] The n-th order differential equation written as $\frac{d^n x}{dt^n} + a_1(t)\frac{d^{n-1}x}{dt^{n-1}} + ... + a_{n-1}(t)\frac{dx}{dt} + a_n(t)x = g(t)$, is said to be *linear* if all the coefficients a_i, $i = 1, ..., n$, and g do not depend on x or any of its derivatives. Otherwise it is said to be *non-linear*.

Definition 1.3.5. The system (1.5) is called a *planar autonomous* system. The term autonomous means self-governing, justified by the absence of the time variable t in the functions $f(x, y)$ and $g(x, y)$.

It is assumed that f, g are continuously differentiable in some region D in the xy plane. A graph which contains all the equilibria and the typical trajectories or orbits of a planar autonomous system (1.5) is called a *phase portrait* [10].

Definition 1.3.6. [2], The *Hopf bifurcation* is defined as the appearance or disappearance of a periodic orbit through a local change in the stability properties of a steady point.

It is named in the memory of the mathematician Eberhard Hopf . In a dynamical system, Hopf bifurcation loses stability in the form of complex conjugate eigenvalues of the linearization around the fixed point which crosses the imaginary axis of the complex plane.

The *Hopf bifurcation* theorem states sufficient conditions for the existence of periodic solutions. As one parameter is varied, the dynamics of the system changes from a stable spiral to a center to an unstable spiral. The eigenvalues of the linearized system change from having a negative real part to a zero real part to positive real part. Under certain conditions, there exist periodic solutions.

Consider a system of autonomous differential equations given by

$$\begin{aligned} \frac{dx}{dt} &= f_1(x, y, r) \\ \frac{dy}{dt} &= g_1(x, y, r), \end{aligned} \tag{1.7}$$

where the functions f_1 and g_1 depend on the bifurcation parameter r. Suppose there exists an equilibrium $(x^*(r), y^*(r))$ of system (1.7) and the Jacobian matrix evaluated at this equilibrium has eigenvalues $\alpha(r) \pm i\beta(r)$. In addition, suppose that a change in stability occurs at the value of $r = r^*$, where $\alpha(r^*) = 0$. If $\alpha(r) < 0$ for values of r close to r^* but $r < r^*$ or if $\alpha(r) > 0$ for values of r close to r^* but $r > r^*$(also $\beta(r^*) \neq 0$) then the equilibrium changes from a stable spiral to an unstable spiral as r passes through r^*. The Hopf Bifurcation theorem states that there exists a periodic orbit near $r = r^*$ for any neighbourhood of the equilibrium in the xy plane. The parameter r is the bifurcation parameter and r^*

is the bifurcation value. The theorem is valid when the bifurcation parameter has values close to the bifurcation value.

We consider Hopf bifurcation and limit cycles at the equilibrium $(0,0)$ in the following system:

$$\begin{cases} \dot{u} = au + bv + p(u,v) := f_1(u,v), \\ \dot{v} = cu + dv + q(u,v) := g_1(u,v), \end{cases} \tag{1.8}$$

where $p(u,v) = \sum_{i+j=2}^{\infty} a_{ij} u^i v^j$ and $q(u,v) = \sum_{i+j=2}^{\infty} b_{ij} u^i v^j$. The Jacobian matrix of f_1 and g_1 at the equilibrium $(0,0)$ is

$$A(0,0) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{1.9}$$

If $\Delta = |A(0,0)| = ad - bc > 0$, then it follows from the formula $(3')$ of Section 4.4 in [31] that the Liapunov number, denoted by σ_2, of (1.8), is given by

$$\sigma_2 = \frac{-3\pi}{2b\Delta^{3/2}} \sum_{i=1}^{8} \xi_i, \tag{1.10}$$

where

$$\xi_1 = ac(a_{11}^2 + a_{11}b_{02} + a_{02}b_{11}),\ \xi_2 = ab(b_{11}^2 + a_{20}b_{11} + a_{11}b_{02}),$$

$$\xi_3 = c^2(a_{11}a_{02} + 2a_{02}b_{02}),\ \xi_4 = -2ac(b_{02}^2 - a_{20}a_{02}),\ \xi_5 = -2ab(a_{20}^2 - b_{20}b_{02}),$$

$$\xi_6 = -b^2(2a_{20}b_{20} + b_{11}b_{20}),\ \xi_7 = (bc - 2a^2)(b_{11}b_{02} - a_{11}a_{20}),$$

$$\xi_8 = -(a^2 + bc)[3(cb_{03} - ba_{30}) + 2a(a_{21} + b_{12}) + (ca_{12} - bb_{21})].$$

The following result is a special case of Theorem 1 and Remark 1 in the section 4.4 of [31] (also see pages 253, and 261-264 in [6]).

Lemma 1.3.7. *Assume that $\Delta > 0$ and $a + d = 0$ and. Then the following assertions hold [25]:*

1. *If* $\sigma_2 < 0$ *(or* $\sigma_2 > 0$*), then the equilibrium* $(0,0)$ *is a stable (or unstable) center or a stable (or unstable) focus with multiplicity one.*

2. *If* $\sigma_2 < 0$ *(or* $\sigma_2 > 0$*), then a supercritical (or sub critical) Hopf bifurcation occurs at* $(0,0)$ *of* (1.8) *at the bifurcation value* $\tau = a + d = 0$.

3. *If* $\sigma_2 < 0$ *(or* $\sigma_2 > 0$*), then a unique stable (or unstable) limit cycle bifurcates from* $(0,0)$ *of* (1.8) *as bifurcation value* $\tau = a + d = 0$ *increases from zero.*

Definition 1.3.8. [2] The *linearization*:

The local stability of an equilibrium is determined by the eigenvalues of the Jacobian matrix. The functions f and g are expanded using Taylor's formula about the equilibrium, (x^*, y^*), where $u = x - x^*$ and $v = y - y^*$. Assume that f and g have continuous second-order partial derivatives in an open set containing the point (x^*, y^*). Then,

$$\begin{aligned}\frac{du}{dt} &= f(x^*+u, y^*+v) = f(x^*, y^*) + f_x(x^*, y^*)u + f_y(x^*, y^*)v + f_{xx}(x^*, y^*)\frac{u^2}{2} \\ &\quad + f_{xy}(x^*, y^*)uv + f_{yy}(x^*, y^*)\frac{v^2}{2} + \ldots\ldots \\ &= f(x^*, y^*) + f_x(x^*, y^*)u + f_y(x^*, y^*)v + h_1(u, v).\end{aligned}$$

where $h_1(u, v) = f_{xx}(x^*, y^*)\frac{u^2}{2} + f_{xy}(x^*, y^*)uv + f_{yy}(x^*, y^*)\frac{v^2}{2} + \ldots\ldots.$

$$\begin{aligned}\frac{dv}{dt} &= g(x^*+u, y^*+v) = g(x^*, y^*) + g_x(x^*, y^*)u + g_y(x^*, y^*)v + g_{xx}(x^*, y^*)\frac{u^2}{2} \\ &\quad + g_{xy}(x^*, y^*)uv + g_{yy}(x^*, y^*)\frac{v^2}{2} + \ldots\ldots \\ &= g(x^*, y^*) + g_x(x^*, y^*)u + g_y(x^*, y^*)v + h_2(u, v).\end{aligned}$$

where $h_2(u, v) = g_{xx}(x^*, y^*)\frac{u^2}{2} + g_{xy}(x^*, y^*)uv + g_{yy}(x^*, y^*)\frac{v^2}{2} + \ldots\ldots.$

We assume that h_1 and h_2 take small values for small u, v. In addition, $h_1(u, v)$ and $h_2(u, v)$ tend to zero as u and v tend to zero.

The *linearization* of the system, obtained by using $f(x^*, y^*) = 0$, $g(x^*, y^*) = 0$ and neglecting the higher-order terms $h_1(u, v)$ and $h_2(u, v)$, is defined to be the two-dimensional linear system,

$$\frac{du}{dt} = f_x(x^*, y^*)u + f_y(x^*, y^*)v, \tag{1.11}$$

$$\frac{dv}{dt} = g_x(x^*, y^*)u + g_y(x^*, y^*)v. \tag{1.12}$$

Equations (1.11) and (1.12) can be expressed in the following vector form

$$\begin{bmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{bmatrix} = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} \times \begin{bmatrix} u \\ v \end{bmatrix}$$

or

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} \times \begin{bmatrix} u \\ v \end{bmatrix}$$

Finally, we can write $\frac{dZ}{dt} = JZ$, where $Z = \begin{bmatrix} u \\ v \end{bmatrix}$ and $J = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}$.

1.4 Equilibria of the system (1.5)

The solution to the non-linear system (1.6) is characterized by the eigenvalues of the Jacobian matrix J given below. These eigenvalues depend on the trace and on the determinant of J. The local stability of an equilibrium is obtained by studying the eigenvalues of the Jacobian matrix. We have used the fact that $f(x^*, y^*) = 0$ and $g(x^*, y^*) = 0$. Then the system linearised about the equilibrium (x^*, y^*) is $\frac{dZ}{dt} = JZ$, where $Z = (u, v)^T$ and J is the Jacobian matrix which is evaluated at the equilibrium:

$$A(x,y) = J(x,y) = \begin{pmatrix} f_x(x,y) & f_y(x,y) \\ g_x(x,y) & g_y(x,y) \end{pmatrix}\Bigg|_{x=x^*,y=y^*} \tag{1.13}$$

Following [2], the classification of the equilibria (node, saddle, spiral) for non-linear systems is developed from the classification scheme of linear systems, since the linearisation is only an approximation of the non-linear system.

The nature of the curve can be determined by the characteristic polynomial of the matrix J:

$$\lambda^2 - \mathrm{tr}\,(J)\lambda + \det\,(J) = 0 \Rightarrow\ \lambda^2 - \tau\lambda + \Delta = 0 \Rightarrow\ \lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}$$

where τ is the trace of the Jacobian matrix, Δ is the determinant of Jacobian matrix and λ_1, λ_2 are the eigenvalues.

I **Saddle point** If $\Delta < 0$, then the eigenvalues have opposite signs, that is, $\lambda_1 < 0 < \lambda_2$. Since the discriminant $\tau^2 - 4\Delta$ is positive, the eigenvalues are real. The equilibrium point is called a saddle .

II **Stable node** If $\Delta > 0$ and $\tau^2 - 4\Delta \geq 0$, then the equilibrium point is a stable node when $\tau < 0$, since both eigenvalues are real and negative.

III **Unstable node** If $\Delta > 0$ and $\tau^2 - 4\Delta \geq 0$, then the equilibrium point is unstable when $\tau > 0$, since both eigenvalues are real and positive.

IV **Stable focus** If $\Delta > 0$ and $\tau^2 - 4\Delta < 0$, then the equilibrium point is a stable focus when $\tau < 0$, since the eigenvalues are complex whose real parts are negative.

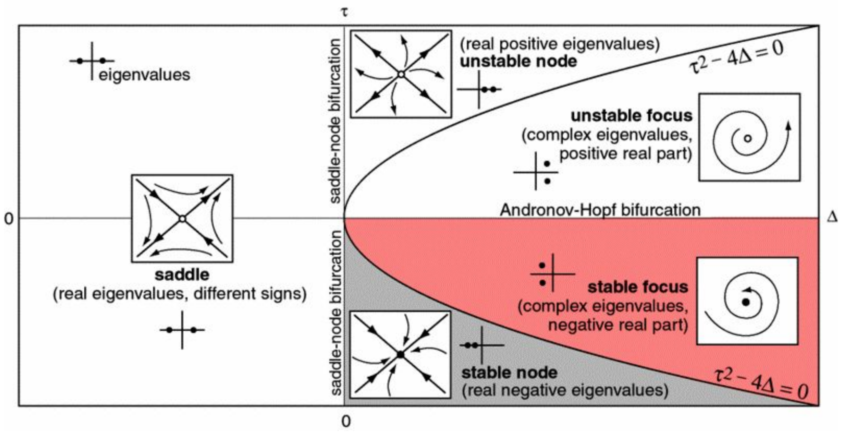

Figure 1.5: The global stability of the origin in the case of linearisation [13]

V **Unstable focus** If $\Delta > 0$ and $\tau^2 - 4\Delta < 0$, then the equilibrium point is an unstable focus when $\tau > 0$, since the eigenvalues are complex with positive real parts.

VI **Center or Focus** If $\Delta > 0$ and $\tau = 0$, then the equilibrium point is a center or a focus , since the eigenvalues are purely imaginary.

We give a brief introduction of the qualitative theory on phase portraits of planar systems near equilibria [18, 23] . We denote by $A(x^*, y^*)$ the Jacobian matrix of

f and g at (x^*, y^*), that is,

$$A(x,y) = \left(\begin{array}{cc} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \\ \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} \end{array} \right) \Bigg|_{x=x^*, y=y^*} \tag{1.14}$$

and by $|A(x^*, y^*)|$ and $\mathrm{tr}(A(x^*, y^*))$ the determinant and the trace of $A(x^*, y^*)$, respectively.

The following results follow from Theorems $3-5$ of Section 2.10 and Definition 2 of Section 2.6 in [20]. These results have been used in [24, 25] to study some prey-predator models.

Lemma 1.4.1. *Assume that* (x^*, y^*) *is an equilibrium point of* (1.5). *Then the following assertions hold:*
(i) If $|A(x^*, y^*)| < 0$, *then* (x^*, y^*) *is a saddle of* (1.5).
(ii) If $|A(x^*, y^*)| > 0$ *and* $(\mathrm{tr}(A(x^*, y^*)))^2 - 4|A(x^*, y^*)| \geq 0$, *then* (x^*, y^*) *is a node of* (1.5)*; it is stable if* $\mathrm{tr}(A(x^*, y^*)) < 0$ *and unstable if* $\mathrm{tr}(A(x^*, y^*)) > 0$.
(iii) If $|A(x^*, y^*)| > 0$, $(\mathrm{tr}(A(x^*, y^*)))^2 - 4|A(x^*, y^*)| < 0$ *and* $\mathrm{tr}(A(x^*, y^*)) \neq 0$, *then* (x^*, y^*) *is a focus of* (1.5)*; it is stable if* $\mathrm{tr}(A(x^*, y^*)) < 0$ *and unstable if* $\mathrm{tr}(A(x^*, y^*)) > 0$.
(iv) If $|A(x^*, y^*)| > 0$ *and* $\mathrm{tr}(A(x^*, y^*)) = 0$, *then* (x^*, y^*) *is a center or a focus of* (1.5).

Difference between the Lotka-Volterra model and our Holling-Tanner predator-prey model

Equation (1.1) is a simple model which incorporates four parameters. After nondimensionalisation, it has one parameter. Any small perturbation will move the solution onto another trajectory which does not lie everywhere close to the original trajectory. Consequently, solutions are not structurally stable which is the main drawback of the Lotka-Volterra model.

Equation (2.1) is our model, also known as the Holling-Tanner model. In this model, six parameters are reduced into three parameters after nondimensionalisation. This model is more realistic than the Lotka-Volterra model because we introduce the logistic growth rate K, per capita birth rate $r_1(1-\dfrac{X}{K})$, the predation rate and the responses to the number killed per predator per time $\dfrac{mX}{aY+X}$, the maximum number of prey that can be eaten per predator per time m, a measure of the quality of the prey as food for the predator h. Finally, we obtain stable (or unstable) focus, node, and center under suitable values of parameters.

Chapter 2

The predator-prey interactions

2.1 Formulation of the predator-prey model

In this section, we present the model which we shall analyze. The Lotka-Volterra equations are a pair of first order, non-linear differential equations which are used to describe the dynamics of biological systems for the interaction of two species: the prey and the predator. The Lotka-Volterra [29] predator-prey model was introduced by Alfred J. Lotka in the theory of chemical reactions in 1910. He further extended this theory to predator-prey interactions in 1925.

Vito-Volterra developed independently the same equations for statistical analysis of fish catches in the Adriatic in 1926. Furthermore, the model was developed by Holling-Tanner in [1959] to study the interactions of prey and predator species. The study of these models have been of interest to both applied mathematicians and ecologists. The Holling-Tanner models are governed by the following system of two first order differential equations with time T as the independent variable:

$$\begin{cases} \dfrac{dX}{dT} = r_1 X(1 - \dfrac{X}{K}) - \dfrac{mXY}{aY + X}, \\ \dfrac{dY}{dT} = r_2 Y(1 - \dfrac{Y}{hX}), \end{cases} \tag{2.1}$$

where $X(T)$ and $Y(T)$ are the sizes of the prey and of the predator, respectively. The parameters r_1 and r_2 are the intrinsic growth rates of the prey and the predator, respectively. The number K is the carrying capacity of the prey. In other words, the carrying capacity means the maximum population size of the prey, it depends on (i) the amount of resources available in the ecosystem, (ii) the size of the population and (iii) the amount of resources for each individual's consumption. The term $mX/(aY + X)$ is called the predation rate and represents the number killed per predator per time. It is also known as a Holling type II predator response [8, 18, 19, 28, 31]. The value m is the maximum number of prey that can be eaten per predator per time and the parameter a is the half saturation constant at which the predation rate achieves the value $m/2$, one half the maximum rate m. The parameter h is a measure of the quality of prey as food for predator and hX is the prey-dependent carrying capacity for predator (also see [8, 28, 33] for these interpretations).

In this thesis, we study the phase portraits, the Hopf bifurcations , and the existence and uniqueness of stable limit cycles near the interior equilibrium of equation (2.1). As in [5, 6, 28], we consider the nondimensional system of the equation (2.1). The new system involves three parameters α, β and δ (see equation (2.3) below). When $\alpha > \alpha_1$, or $\alpha \leq \alpha_1$ and $\alpha_1 = \dfrac{(1+\beta)^2}{1+2\beta}$, we provide the ranges of the three parameters under which the interior equilibrium can be justified to be a stable (an unstable) node or focus. Our results improve the known results in [6, 28], where the node or focus can be determined. When $\beta = \alpha$, we give the ranges of α and

β under which the Hopf bifurcation occurs and there is a unique limit cycle . In addition, our results confirm some known results observed and suggested from the real ecological systems, for example, in [19, 28].

The methodology used here for the local stability is to utilize the well-known results on the phase portraits of the planar systems near equilibria (see for example (1.4.1) [18, 23]), where determining the signs of the determinants and of the trace of the Jacobian matrix at the equilibria plays an important role. We shall provide the formulas of the determinants and the traces of the matrices in order to determine their signs. The key method employed here to show that the Hopf bifurcation occurs and the unique stable limit cycle exists is the well-known result on bifurcation theory [23, 33], where the negative Liapunov number is required. We shall prove that the Liapunov number involved in equation (2.1) is negative and the proof is not trivial.

2.1.1 Nondimensionalization and meanings of parameters in the predator-prey model

Let x and y represent the number of prey and predators, respectively. The derivatives $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$ indicate the growth rate of the two populations at time t. Following [19], we make the translations:

$$\begin{cases} t = r_1 T, \quad x(t) = \dfrac{X(t)}{K}, \quad y(t) = \dfrac{aY(t)}{K}, \\ \alpha = \dfrac{m}{r_1 a}, \ \delta = \dfrac{r_2}{r_1}, \ \beta = \dfrac{1}{ah}, \end{cases} \tag{2.2}$$

then

$$\frac{dX}{dT} = K\frac{dx}{dt}\frac{dt}{dT} = Kr_1\frac{dx}{dt}, \ \ \frac{dY}{dT} = \frac{K}{a}\frac{dy}{dt}\frac{dt}{dT} = \frac{K}{a}r_1\frac{dy}{dt}.$$

Substituting $\dfrac{dX}{dT}$, $\dfrac{dY}{dT}$, X and Y into equation (2.1), we get

$$
\begin{aligned}
Kr_1\frac{dx}{dt} &= r_1Kx(1-\frac{Kx}{K}) - \frac{mKx\frac{K}{a}y}{a\frac{K}{a}y+Kx},\\
\frac{dx}{dt} &= x(1-x) - \frac{m}{r_1a}\frac{xy}{y+x},\\
\frac{dx}{dt} &= x(1-x) - \alpha\frac{xy}{x+y}.
\end{aligned}
$$

$$
\begin{aligned}
\frac{K}{a}r_1\frac{dy}{dt} &= r_2\frac{K}{a}y(1-\frac{\frac{K}{a}y}{hKx}),\\
\frac{dy}{dt} &= \frac{r_2}{r_1}y(1-\frac{1}{ah}\frac{y}{x}),\\
\frac{dy}{dt} &= \delta y(1-\beta\frac{y}{x}).
\end{aligned}
$$

Then the equation (2.1) becomes

$$
\begin{cases}
\dfrac{dx}{dt} = x(1-x) - \alpha\dfrac{xy}{x+y} := f(x,y),\\
\dfrac{dy}{dt} = \delta y(1-\beta\dfrac{y}{x}) := g(x,y),
\end{cases}
\tag{2.3}
$$

where $\alpha, \beta, \delta > 0$.

The biological meaning of the parameters δ, α and β is explained as following where $\delta = \dfrac{r_2}{r_1} > 0$. This implies three cases: $0 < \dfrac{r_2}{r_1} < 1$, $0 < \dfrac{r_2}{r_1} = 1$ and $\dfrac{r_2}{r_1} > 1$. In other words, if $r_2 < r_1$ then the prey is produced faster than the predator obeys. If $r_2 = r_1$ then the prey and predator have to be cyclic and if $r_2 > r_1$ then they might be extinct.

If $\alpha = \dfrac{m}{r_1a} > 0$ then $0 < \dfrac{m}{r_1a} < 1$ or $0 < \dfrac{m}{r_1a} = 1$ or $0 < \dfrac{m}{r_1a} > 1$. So it is biologically meaningful to consider the case when $\alpha < 1$, both species are sustained since the growth rate of prey is greater than the number of prey killed by predators at a time. If we allow α to take values greater than or equal to 1 in some cases, then the species are eradicated due to the fact that the growth rate is less than or equal to the number of prey killed by predators at a time.

If $0 < \beta = \dfrac{1}{ah} < 1$, then the product of the maximum number of killed prey and the measure of the quality of prey as food for predator must be greater than one for the existence of the species. If $1 \geq ah$, then the species do not exist or they can be eradicated from nature.

2.1.2 Phase portraits near the interior equilibrium

It is obvious that $(1, 0)$ is an equilibrium of (2.3) and there exists a unique interior equilibrium (x^*, y^*) of (2.3).

Initially, Alfred J. Lotka extended the Lotka-Volterra system of equations, via the Kolmogorov model, to the predator-prey interactions, in his book on biomathematics, in 1925. S. Holling extended further this model in 1959. The Lotka-Volterra model and Holling's extensions have been used to analyse the moose and wolf populations in the Isle Royale National Park [20]. A. Mathi and S. Pathak [27] modified the Holling-Tanner model to introduce stochastic fluctuations. Lan and Zhu [24] also further modified the phase portraits, the Tanner models for predator-prey interactions of the two species.

The following result provides the exact expression of the interior equilibrium (x^*, y^*) which is useful in this thesis.

Lemma 2.1.1. *If $\beta > 0$, $\delta > 0$ and $1 + \beta > \alpha > 0$, then the system (2.3) has a unique equilibrium (x^*, y^*) with $x^*, y^* > 0$.*

Where $x^* = 1 - \dfrac{\alpha}{1+\beta}$ and $y^* = \dfrac{1}{\beta}(1 - \dfrac{\alpha}{1+\beta})$.

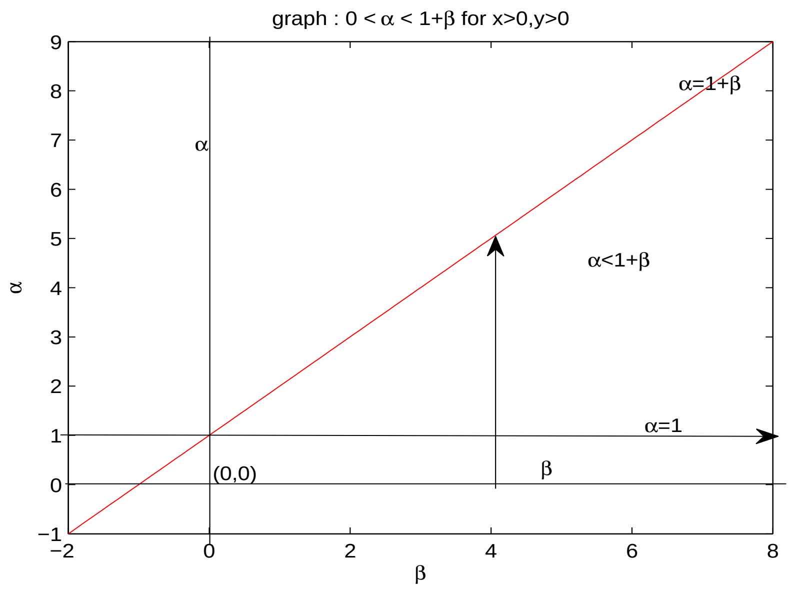

Figure 2.1: The region of equilibria lies below the line $\alpha = 1 + \beta$ and above the line $\alpha = 1$.

Proof. To find the equilibria of (2.3), let

$$\begin{cases} x(1-x) - \alpha \dfrac{xy}{x+y} = 0, \\ \delta y(1 - \beta \dfrac{y}{x}) = 0. \end{cases} \tag{2.4}$$

If $x \neq 0$ and $y \neq 0$, then (2.4) becomes

$$(1-x) - \alpha \frac{y}{x+y} = 0, \tag{2.5}$$

$$(1 - \beta \frac{y}{x}) = 0. \tag{2.6}$$

From the equation (2.6), we obtain $y = \dfrac{x}{\beta}$. We then substitute $y = \dfrac{x}{\beta}$ into the equation (2.5) and we obtain:

$$
\begin{aligned}
(1-x)-\frac{\alpha y}{x+y} &= 0,\\
1-x-\frac{\alpha\frac{x}{\beta}}{x+\frac{x}{\beta}} &= 0,\\
1-x-\frac{x\frac{\alpha}{\beta}}{x(1+\frac{1}{\beta})} &= 0,\\
1-x-\frac{\alpha}{1+\beta} &= 0,\\
x^* &= 1-\frac{\alpha}{1+\beta},\\
y^* &= \frac{1}{\beta}(1-\frac{\alpha}{1+\beta}).
\end{aligned}
$$

□

Finally, we get

$$
\begin{cases}
x^* = \dfrac{1}{1+\alpha},\\
y^* = \dfrac{x^*}{\alpha},
\end{cases}
\tag{2.7}
$$

when $\alpha=\beta>0$. From (2.3) and substituting the values of equilibrium point (x^*,y^*), we get

$$
\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\partial(x-x^2-\alpha\frac{xy}{x+y})}{\partial x} = 1-2x-\alpha[\frac{(x+y)y-xy}{(x+y)^2}]\\
&= 1-2x-\frac{\alpha y^2}{(x+y)^2} = 1-2(1-\frac{\alpha}{1+\beta})-\alpha\frac{(\frac{x^*}{\beta})^2}{(x^*+\frac{x^*}{\beta})^2}\\
&= -1+\frac{2\alpha}{1+\beta}-\frac{\alpha}{\beta^2}\frac{\beta^2}{(1+\beta)^2} = -1+\frac{2\alpha(1+\beta)-\alpha}{(1+\beta)^2}\\
&= -1+\frac{\alpha(2+2\beta-1)}{(1+\beta)^2} = -1+\frac{\alpha(1+2\beta)}{(1+\beta)^2},
\end{aligned}
$$

$$\frac{\partial f}{\partial y} = \frac{\partial(x - x^2 - \alpha\frac{xy}{x+y})}{\partial y} = -\alpha[\frac{(x+y)x - xy}{(x+y)^2}]$$
$$= -\frac{\alpha x^2}{(x+y)^2} = -\alpha\frac{x^{*2}}{(x^* + \frac{x^*}{\beta})^2} = -\frac{\alpha\beta^2}{(1+\beta)^2},$$

$$\frac{\partial g}{\partial x} = \frac{\partial(\delta y - \frac{\delta\beta y^2}{x})}{\partial x} = -\delta\beta y^2(\frac{-1}{x^2}) = \delta\beta\frac{y^2}{x^2} = \delta\beta\frac{x^{*2}}{\beta^2 x^{*2}} = \frac{\delta}{\beta},$$

$$\frac{\partial g}{\partial y} = \frac{\partial(\delta y - \frac{\delta\beta y^2}{x})}{\partial y} = \delta(1 - \frac{2\beta y}{x}) = \delta\big(1 - \frac{2\beta\frac{x^*}{\beta}}{x^*}\big) = \delta(1-2) = -\delta.$$

Substituting the values of $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial g}{\partial x}$ and $\frac{\partial g}{\partial y}$ in equation (1.14), we obtain a Jacobian matrix in the following form

$$A(x,y) = \begin{pmatrix} -1 - 2x - \frac{\alpha y^2}{(x+y)^2} & \frac{-\alpha x^2}{(x+y)^2} \\ \delta\beta\frac{y^2}{x^2} & \delta(1 - \frac{2\beta y}{x}) \end{pmatrix}. \tag{2.8}$$

Substituting the values of $x^* = 1 - \frac{\alpha}{1+\beta}$ and $y^* = \frac{1}{\beta}(1 - \frac{\alpha}{1+\beta})$ in (2.8), we derive

$$A(x^*,y^*) = \begin{pmatrix} -1 + \frac{\alpha(1+2\beta)}{(1+\beta)^2} & -\frac{\alpha\beta^2}{(1+\beta)^2} \\ \frac{\delta}{\beta} & -\delta \end{pmatrix}. \tag{2.9}$$

2.2 Stable (or unstable) nodes, focus, center and saddle near interior equilibrium

In this section we present some new results on the predator-prey model to determine the types of equilibrium points such as node, saddle, focus and centre. We also discuss the Hopf bifurcation and the limit cycles.

The following result shows that the determinant $|A(x^*, y^*)|$ is positive and provides a formula for $|A(x^*, y^*)|$ which will be used later.

Lemma 2.2.1. *Assume that* $\beta > 0$, $\delta > 0$ *and* $1+\beta > \alpha > 0$. *Then* $|A(x^*, y^*)| > 0$.

Proof. By (2.9), we have

$$\begin{aligned}|A(x^*, y^*)| &= -\delta[-1 + \frac{\alpha(1+2\beta)}{(1+\beta)^2}] + \frac{\delta}{\beta}\frac{\alpha\beta^2}{(1+\beta)^2}\\ &= \delta - \frac{\alpha\delta + 2\alpha\delta\beta}{(1+\beta)^2} + \frac{\alpha\delta\beta}{(1+\beta)^2} = \delta - \frac{\alpha\delta + 2\alpha\delta\beta - \alpha\delta\beta}{(1+\beta)^2}\\ &= \delta - \frac{\alpha\delta + \alpha\delta\beta}{(1+\beta)^2} = \delta - \frac{\alpha\delta(1+\beta)}{(1+\beta)^2} = \delta - \frac{\alpha\delta}{1+\beta} = \delta(1 - \frac{\alpha}{1+\beta}).\end{aligned}$$

Since $\delta > 0$ and $x^* = 1 - \dfrac{\alpha}{1+\beta} > 0$, then $|A(x^*, y^*)| > 0$. $\square$

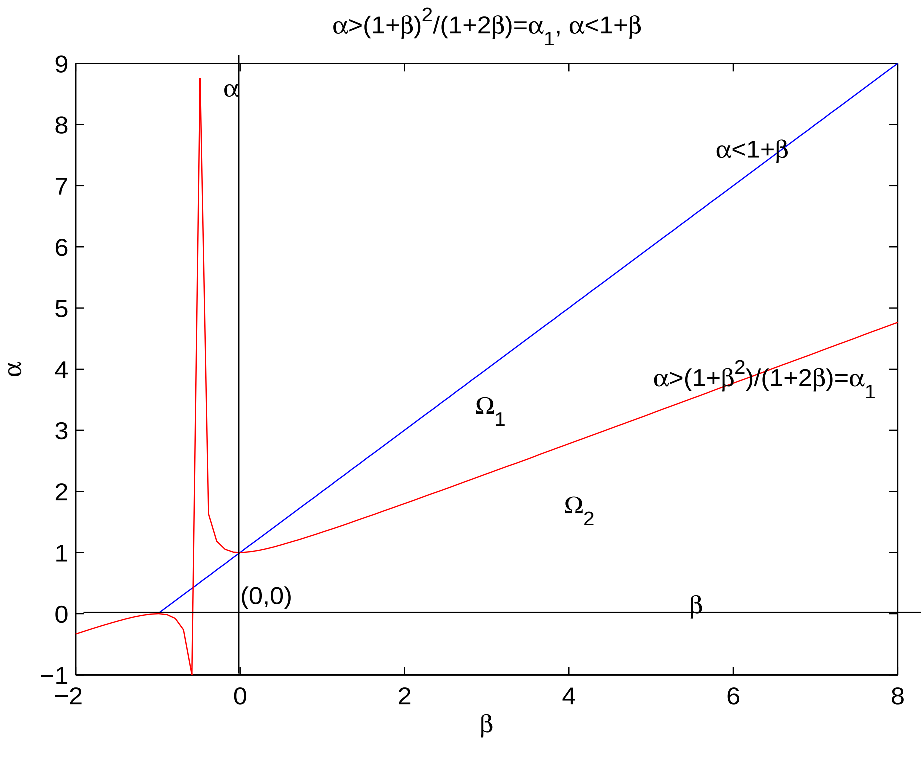

Figure 2.2: Solution of rgion of new results

2.3 Some new results

In this section, we analyze stable or unstable nodes, foci and centres.

Theorem 2.3.1. *If $\delta > 0$, $\beta > 0$, $1+\beta > \alpha > 0$ and $\alpha_1(\beta) = \dfrac{(1+\beta)^2}{1+2\beta}$. Then the following hold:*

(1) *If $\alpha > \alpha_1$ and $\delta > \dfrac{\alpha-\alpha_1}{\alpha_1}$, then $(x^*, y^*) \in \Omega_1$ is a stable node or focus of* (2.3).

(2) *If $\alpha > \alpha_1$ and $\delta < \dfrac{\alpha-\alpha_1}{\alpha_1}$, then $(x^*, y^*) \in \Omega_1$ is an unstable node or focus of* (2.3).

(3) *If $\alpha > \alpha_1$ and $\delta = \dfrac{\alpha-\alpha_1}{\alpha_1}$, then $(x^*, y^*) \in \Omega_1$ is a stable centre or focus of* (2.3).

(4) *If $0 < \alpha \leq \alpha_1$ and $\delta > 0$, then $(x^*, y^*) \in \Omega_2$ is a stable node or focus of* (2.3) .

Proof. We calculate the trace of the matrix from (2.9),

$$\begin{aligned} \operatorname{tr}(A(x^*, y^*)) &= -1 + \frac{\alpha(1+2\beta)}{(1+\beta)^2} - \delta = -[\delta + 1 - \frac{\alpha(1+2\beta)}{(1+\beta)^2}] \\ &= -[\delta + 1 - \frac{\alpha}{\alpha_1}] = -[\delta - (\frac{\alpha-\alpha_1}{\alpha_1})]. \end{aligned} \tag{2.10}$$

(1) Let $\alpha > \alpha_1$. Then from (2.10),
$\operatorname{tr}(A(x^*, y^*)) = -[\delta - (\frac{\alpha-\alpha_1}{\alpha_1})] < 0$, since $\alpha_1(\beta) = \dfrac{(1+\beta)^2}{(1+2\beta)}$. where we have $\delta - \dfrac{(\alpha-\alpha_1)}{\alpha_1} \geq 0$, and $0 < \alpha_1(\beta) = \dfrac{(1+\beta)^2}{1+2\beta} < \alpha$. So, (x^*, y^*) is a stable node or focus of (2.3) from lemma 1.4.1.

(2) Let $\alpha > \alpha_1$. Then from (2.10),
$\operatorname{tr}(A(x^*, y^*)) = -[\delta - (\frac{\alpha-\alpha_1}{\alpha_1})] > 0$, since $\alpha_1(\beta) = \dfrac{(1+\beta)^2}{(1+2\beta)}$. where $\delta < \dfrac{(\alpha-\alpha_1)}{\alpha_1}$, and $0 < \alpha_1(\beta) = \dfrac{(1+\beta)^2}{1+2\beta} < \alpha$. So, (x^*, y^*) is an unstable node or focus of (2.3) from lemma 1.4.1.

(3) Let $\alpha > \alpha_1$. Then from (2.10),
From (2.10), $\operatorname{tr}(A(x^*, y^*)) = -[\delta - (\frac{\alpha-\alpha_1}{\alpha_1})] = 0$, since $\alpha_1(\beta) = \dfrac{(1+\beta)^2}{(1+2\beta)}$. where

$\delta = \dfrac{(\alpha - \alpha_1)}{\alpha_1}$, and $0 < \alpha_1(\beta) = \dfrac{(1+\beta)^2}{1+2\beta} < \alpha$. So, (x^*, y^*) is a stable center or focus of (2.3) from lemma 1.4.1.

(4) Let $\alpha \leq \alpha_1$. Then from (2.10),

$\mathrm{tr}(A(x^*, y^*)) = -[\delta - (\dfrac{\alpha - \alpha_1}{\alpha_1})] < 0$, since $\alpha_1(\beta) = \dfrac{(1+\beta)^2}{(1+2\beta)}$. where $0 < \alpha \leq \alpha_1$ and $\delta > 0$. So, (x^*, y^*) is a stable node or focus of (2.3) from lemma 1.4.1.

□

In Figure 2.2, Ω_1 is the region of the plane which lies below the line $\alpha = 1 + \beta$ and above the curve $\alpha_1(\beta) = \dfrac{(1+\beta)^2}{1+2\beta}$ in the first quadrant. In addition, Ω_2 is the region of the plane which lies below the curve $\alpha_1(\beta) = \dfrac{(1+\beta)^2}{1+2\beta}$ and the above the β-axis, in the first quadrant.

Theorem 2.3.2. *Assume that $\beta > 0$, $1 + \beta > \alpha > 0$ and $\delta > \delta_1(\beta) = 1 + 2\beta$ for $\beta \in (0, \infty)$ and $\delta_1(\beta) \in [1, \infty)$. Then (x^*, y^*) is a stable or an unstable node of (2.3).*

Proof. Let $\rho = (\mathrm{tr}\, A(x^*, y^*))^2 - 4|A(x^*, y^*)|$.

Substituting the values of $\mathrm{tr}\, A(x^*, y^*)$ and $|A(x^*, y^*)|$ from (2.9) on the above equation, we derive

$$
\begin{aligned}
\rho &= [-1-\delta+\frac{\alpha(1+2\beta)}{(1+\beta)^2}]^2-4\delta(1-\frac{\alpha}{1+\beta}) \\
&= (-1-\delta)^2+2(-1-\delta)\frac{\alpha(1+2\beta)}{(1+\beta)^2}+\frac{\alpha^2(1+2\beta)^2}{(1+\beta)^4}-4\delta+4\frac{\delta\alpha}{1+\beta} \\
&= (1+\delta)^2-4\delta-2(1+\delta)\frac{\alpha(1+2\beta)}{(1+\beta)^2}+\frac{\alpha^2(1+2\beta)^2}{(1+\beta)^4}+4\frac{\delta\alpha}{1+\beta} \\
&= (1-\delta)^2+\frac{\alpha^2(1+2\beta)^2}{(1+\beta)^4}+4\frac{\delta\alpha}{1+\beta}-2(1+\delta)\frac{\alpha(1+2\beta)}{(1+\beta)^2} \\
&= (1-\delta)^2+\frac{\alpha^2(1+2\beta)^2}{(1+\beta)^4}+\frac{2\alpha}{(1+\beta)^2}[2\delta(1+\beta)-(1+\delta)(1+2\beta)] \\
&= (1-\delta)^2+\frac{\alpha^2(1+2\beta)^2}{(1+\beta)^4}+\frac{2\alpha}{(1+\beta)^2}[2\delta+2\delta\beta-1-\delta-2\beta-2\delta\beta] \\
&= (1-\delta)^2+\frac{\alpha^2(1+2\beta)^2}{(1+\beta)^4}+\frac{2\alpha}{(1+\beta)^2}[\delta-(1+2\beta)]>0.
\end{aligned}
$$

If $\delta > \delta_1(\beta) = 1+2\beta$ then (x^*, y^*) is a stable or an unstable node of (2.3). Since $|A(x^*, y^*)| > 0$ from Lemma 2.2.1. □

Corollary 2.3.3. *If* $\alpha = \beta = \delta > 0$, *then* $\mathrm{tr}(A(x^*, y^*)) < 0$. *Thus,* (x^*, y^*) *is a stable node or focus of the equation* (2.3).

Proof. Substituting $\delta = \alpha$ and $\beta = \alpha$ in equation (2.9), we get

$$
\begin{aligned}
\mathrm{tr}(A(x^*, y^*)) &= -1-\delta+\frac{\alpha(1+2\beta)}{(1+\beta)^2} = -1-\alpha+\frac{\alpha(1+2\alpha)}{(1+\alpha)^2} \\
&= \frac{-(1+\alpha)(1+\alpha)^2+\alpha(1+2\alpha)}{(1+\alpha)^2} = \frac{1}{(1+\alpha)^2}\{-\alpha^3-\alpha^2-2\alpha-1\} \\
&= -\{\frac{(\alpha^3+\alpha^2+2\alpha+1)}{(1+\alpha)^2}\} < 0
\end{aligned}
$$

Therefore, it is a stable node or focus of the equation (2.3). If $\alpha^3+\alpha^2+2\alpha+1 = 0$ then $\mathrm{tr}(A(x^*, y^*)) = 0$ at $\alpha = -0.2151+1.3071i$ or $\alpha = -0.2151-1.3071i$ or $\alpha = -0.5698$. So, (x^*, y^*) is a centre or focus of (2.3). □

Theorem 2.3.4. *If* $\alpha = \beta = \delta > 0$ *and* $\alpha > 3 + \sqrt{15}$, *then* $(\operatorname{tr} A(x^*, y^*))^2 - 4|A(x^*, y^*)| > 0$.

Proof. Let $\rho_1 = (\operatorname{tr} A(x^*, y^*))^2 - 4|A(x^*, y^*)|$. By substituting $\delta = \alpha$ and $\beta = \alpha$ in (2.9), we obtain

$$\begin{aligned}
\rho_1 &= \frac{(\alpha^3 + \alpha^2 + 2\alpha + 1)^2}{(1+\alpha)^4} - 4\alpha(1 - \frac{\alpha}{1+\alpha}) \\
&= \frac{[\alpha^3 + (\alpha+1)^2]^2}{(1+\alpha)^4} - 4\alpha\frac{(1+\alpha-\alpha)}{1+\alpha} = \frac{[\alpha^3 + (\alpha+1)^2]^2}{(1+\alpha)^4} - \frac{4\alpha}{1+\alpha} \\
&= \frac{1}{(1+\alpha)^4}[(\alpha^3)^2 + 2\alpha^3(\alpha+1)^2 + (\alpha+1)^4 - 4\alpha(1+\alpha)^3] \\
&= \frac{1}{(1+\alpha)^4}[\alpha^6 + (\alpha+1)^2\{2\alpha^3 + (\alpha+1)^2 - 4\alpha(1+\alpha)\}] \\
&= \frac{1}{(1+\alpha)^4}[\alpha^6 + (\alpha+1)^2(2\alpha^3 + \alpha^2 + 2\alpha + 1 - 4\alpha - 4\alpha^2)] \\
&= \frac{1}{(1+\alpha)^4}[\alpha^6 + 2\alpha^5 + \alpha^4 - 6\alpha^3 - 6\alpha^2 + 1] \\
&= \frac{1}{(1+\alpha)^4}[\alpha^6 + 2\alpha^5 + \alpha^2(\alpha^2 - 6\alpha - 6) + 1] \\
&= \frac{1}{(1+\alpha)^4}[\alpha^6 + 2\alpha^5 + \alpha^2\{(\alpha-3)^2 - 15\} + 1] > 0,
\end{aligned}$$

where $(\alpha - 3)^2 - 15 > 0$. This implies that $\alpha > 3 + \sqrt{15}$. Thus, (x^*, y^*) is a stable node of the equation (2.3). If $\alpha^6 + 2\alpha^5 + \alpha^4 - 6\alpha^3 - 6\alpha^2 + 1 = 0$ then $\rho_1 = 0$ at $\alpha = -1.2603 \pm 1.4982i$ or $\alpha = 1.5206$ or $\alpha = -0.6779 \pm 0.1509i$ or $\alpha = 0.3557$. So, (x^*, y^*) is a stable node of (2.3). $\square$

Theorem 2.3.5. *If* $\beta > 0$, $\delta > 0$, $1 + \beta > \alpha > 0$ *and* $\alpha \to 0$, *then* $|A(x^*, y^*)| > 0$, $\operatorname{tr} A(x^*, y^*) < 0$ *and* $(\operatorname{tr} A(x^*, y^*))^2 - 4|A(x^*, y^*)| > 0$.

Proof. We have $x^* = 1 - \dfrac{\alpha}{1+\beta}$ and $y^* = \dfrac{1}{\beta}(1 - \dfrac{\alpha}{1+\beta})$. If $\alpha \to 0$ then $x^* = 1$ and $y^* = \dfrac{1}{\beta}$.

From (2.9), we derive

$$A(x^*, y^*) = \begin{pmatrix} -1 + \dfrac{\alpha(1+2\beta)}{(1+\beta)^2} & -\dfrac{\alpha\beta^2}{(1+\beta)^2} \\ \dfrac{\delta}{\beta} & -\delta \end{pmatrix}, \tag{2.11}$$

$$= \begin{pmatrix} -1 & 0 \\ \dfrac{\delta}{\beta} & -\delta \end{pmatrix}. \tag{2.12}$$

Thus $|A(x^*, y^*)| = \delta > 0,\quad$ since $\quad\delta > 0$. $(trA(x^*, y^*)) = -1 - \delta = -(1+\delta) < 0$, and $(\operatorname{tr} A(x^*, y^*))^2 - 4|A(x^*, y^*)| = (-1-\delta)^2 - 4\delta = (1-\delta)^2 \geq 0$. Consequently, (x^*, y^*) is a stable node of the equation (2.3). $\square$

Corollary 2.3.6. *When $\beta > 0$, $\delta > 0$, $1 + \beta > \alpha > 0$ and $\beta \to \infty$, the point (x^*, y^*) is a stable node of the equation* (2.3).

Proof. From (2.9), we can write

$$B(x^*, y^*) := A(x^*, y^*)|_{\beta\to\infty} = \begin{pmatrix} -1 + \dfrac{\alpha(1+2\beta)}{(1+\beta)^2} & \dfrac{-\alpha\beta^2}{(1+\beta)^2} \\ \dfrac{\delta}{\beta} & -\delta \end{pmatrix}|_{\beta\to\infty}, \tag{2.13}$$

$$= \begin{pmatrix} -1 & -\alpha \\ 0 & -\delta \end{pmatrix}. \tag{2.14}$$

Thus, $|A(x^*, y^*)| = \delta > 0$, and $\quad\operatorname{tr} A(x^*, y^*) = -1 - \delta = -(1+\delta) < 0$.

In addition, $(\operatorname{tr} A(x^*, y^*))^2 - 4|A(x^*, y^*)| = [-(1+\delta)]^2 - 4\delta = (1-\delta)^2 \geq 0$.

This shows that (x^*, y^*) is a stable node of equation the (2.3). $\square$

Lemma 2.3.7. *When $\alpha = \beta > 0$, the following hold:*

(1) *If $\delta > \delta_2(\beta) = \dfrac{\beta^2 - \beta - 1}{(1+\beta)^2} > 0$ for $\beta \in (\dfrac{1+\sqrt{5}}{2}, \infty)$, then (x^*, y^*) is a stable node or focus of* (2.3).

(2) *If $\delta > \delta_2(\beta) = \dfrac{\beta^2 - \beta - 1}{(1+\beta)^2}$ for $\beta \in [0, \dfrac{1+\sqrt{5}}{2}]$, then (x^*, y^*) is a stable node*

or focus of (2.3).
(3) *If* $\delta < \delta_2(\beta) = \dfrac{\beta^2-\beta-1}{(1+\beta)^2}$ *for* $\beta \in (\dfrac{1+\sqrt{5}}{2}, \infty)$, *then* (x^*, y^*) *is an unstable node or focus of equation* (2.3).
(4) *If* $0 < \delta = \delta_2(\beta) = \dfrac{\beta^2-\beta-1}{(1+\beta)^2}$ *for* $\beta = \dfrac{1+\sqrt{5}}{2}$, *then* (x^*, y^*) *is a center or focus of equation* (2.3).

Proof. From (2.9), we obtain that

$$C(x^*, y^*) := A(x^*, y^*)|_{\beta=\alpha} = \left(\begin{array}{cc} -1 + \dfrac{\alpha(1+2\beta)}{(1+\beta)^2} & \dfrac{-\alpha\beta^2}{(1+\beta)^2} \\ \dfrac{\delta}{\beta} & -\delta \end{array} \right)|_{\beta=\alpha}, \tag{2.15}$$

$$= \left(\begin{array}{cc} \dfrac{-1-\beta+\beta^2}{(1+\beta)^2} & -\dfrac{\beta^3}{(1+\beta)^2} \\ \dfrac{\delta}{\beta} & -\delta \end{array} \right). \tag{2.16}$$

Therefore,

$$\begin{aligned} |C(x^*, y^*)| &= -\delta(\frac{-1-\beta+\beta^2}{(1+\beta)^2}) + \frac{\delta}{\beta}\frac{\beta^3}{(1+\beta)^2} \\ &= \frac{\delta}{(1+\beta)^2}[1+\beta-\beta^2+\beta^2] = \frac{\delta}{1+\beta} > 0 \end{aligned}$$

$$\mathrm{tr}(C(x^*, y^*)) = \frac{-1-\beta+\beta^2}{(1+\beta)^2} - \delta = -[\delta - \frac{\beta^2-\beta-1}{(1+\beta)^2}] \tag{2.17}$$

(1) Let $\delta > \delta_2$. Then from (2.17),
$\mathrm{tr}(C(x^*, y^*)) = -[\delta - \dfrac{\beta^2-\beta-1}{(1+\beta)^2}] < 0$,
when $\delta - \dfrac{\beta^2-\beta-1}{(1+\beta)^2} > 0$. Therefore, we have $\delta > \delta_2(\beta) = \dfrac{\beta^2-\beta-1}{(1+\beta)^2} > 0$ for $\beta \in (\dfrac{1+\sqrt{5}}{2}, \infty)$. Thus, (x^*, y^*) is a stable node or focus of (2.3).
(2) Let $\delta > \delta_2$. Then from (2.17),
$\mathrm{tr}(C(x^*, y^*)) = -[\delta - \dfrac{\beta^2-\beta-1}{(1+\beta)^2}] < 0$,

when $\delta - \frac{\beta^2-\beta-1}{(1+\beta)^2} > 0$. Therefore, $\delta > \delta_2(\beta) = \frac{\beta^2-\beta-1}{(1+\beta)^2}$ for $\beta \in [0, \frac{1+\sqrt{5}}{2}]$. Thus, (x^*, y^*) is a stable node or focus of (2.3).

(3) Let $\delta < \delta_2$. Then from (2.17),
$\mathrm{tr}(C(x^*, y^*)) = -[\delta - \frac{\beta^2-\beta-1}{(1+\beta)^2}] > 0$,
when $\delta - \frac{\beta^2-\beta-1}{(1+\beta)^2} < 0$. Therefore, $\delta < \delta_2(\beta) = \frac{\beta^2-\beta-1}{(1+\beta)^2}$ for $\beta \in (\frac{1+\sqrt{5}}{2}, \infty)$. Thus, (x^*, y^*) is an unstable node or focus of (2.3).

(4) Let $\delta = \delta_2$. Then from (2.17),
$\mathrm{tr}(C(x^*, y^*)) = \frac{-1-\beta+\beta^2}{(1+\beta)^2} - \delta = -[\delta - \frac{\beta^2-\beta-1}{(1+\beta)^2}] = 0$,
when $\delta - \frac{\beta^2-\beta-1}{(1+\beta)^2} = 0$. Therefore, $\delta = \delta_2(\beta) = \frac{\beta^2-\beta-1}{(1+\beta)^2} = 0$ for $\beta = \frac{1+\sqrt{5}}{2}$, and thus, (x^*, y^*) is a center or focus of (2.3). $\square$

Theorem 2.3.8. *If $\alpha = \beta > 0$ and $\delta_3(\beta) = \frac{1+\beta-2\beta^2}{2(1+\beta)^2} > \delta$ for $\beta \in (0, 1)$, then (x^*, y^*) is a stable node of the equation* (2.3).

Proof. Let us suppose that $\rho_2 = (\mathrm{tr}\, C(x^*, y^*))^2 - 4|C(x^*, y^*)|$. Then, from (2.16), we get

$$\begin{aligned}
\rho_2 &= [\delta + \frac{1}{1+\beta} - \frac{\beta^2}{(1+\beta)^2}]^2 - 4\frac{\delta}{1+\beta} \\
&= \delta^2 + \frac{1}{(1+\beta)^2} + \frac{\beta^4}{(1+\beta)^4} + \frac{2\delta}{1+\beta} - 2\frac{\delta\beta^2}{(1+\beta)^2} - \frac{2\beta^2}{(1+\beta)^3} - \frac{4\delta}{1+\beta} \\
&= \frac{\beta^4}{(1+\beta)^4} - 2\delta\frac{\beta^2}{(1+\beta)^2} + \delta^2 + \frac{1}{(1+\beta)^2} - \frac{2\delta}{1+\beta} - \frac{2\beta^2}{(1+\beta)^3} \\
&= \{\frac{\beta^2}{(1+\beta)^2} - \delta\}^2 + \frac{(1+\beta-2\beta^2)}{(1+\beta)^3} - \frac{2\delta}{1+\beta} \\
&= \{\frac{\beta^2}{(1+\beta)^2} - \delta\}^2 + \frac{2}{1+\beta}\{\frac{(1+\beta-2\beta^2)}{2(1+\beta)^2} - \delta\} \\
&= \{\frac{\beta^2}{(1+\beta)^2} - \delta\}^2 + \frac{2}{1+\beta}\{\delta_3 - \delta\} > 0,
\end{aligned}$$

where $\delta_3 = \frac{1+\beta-2\beta^2}{2(1+\beta)^2} > \delta$, and

$\delta_3 := \delta_3(\beta) = \dfrac{1}{2(1+\beta)^2}\{1+\beta-2\beta^2\} = \dfrac{1}{2(1+\beta)^2}\{-2[(\beta-\dfrac{1}{4})^2 - \dfrac{1}{16}]+1\}$
$= \dfrac{1}{2(1+\beta)^2}\{\dfrac{9}{8} - 2(\beta-\dfrac{1}{4})^2\} > 0.$
We can also write $\dfrac{9}{16} > (\beta-\dfrac{1}{4})^2$, thus $\dfrac{3}{4} > \beta - \dfrac{1}{4}$ which leads to $0 < \beta < 1$.
This shows that, (x^*, y^*) is a stable node of (2.3). □

Theorem 2.3.9. *When* $\beta > 0$, $\delta > 0$, $1+\beta > \alpha > 0$ *and* $\beta \to 0$, *the point* (x^*, y^*) *is a stable node of* (2.3).

Proof. From equation (2.9), we derive

$$A(x^*, y^*) = \begin{pmatrix} -1 + \dfrac{\alpha(1+2\beta)}{(1+\beta)^2} & -\dfrac{\alpha\beta^2}{(1+\beta)^2} \\ \dfrac{\delta}{\beta} & -\delta \end{pmatrix} \tag{2.18}$$

and

$$\begin{aligned} |A(x^*, y^*)| &= \delta\{1 - \frac{\alpha(1+2\beta)}{(1+\beta)^2}\} + \frac{\delta}{\beta}\frac{\alpha\beta^2}{(1+\beta)^2} = \delta\{1 - \frac{\alpha(1+2\beta)}{(1+\beta)^2} + \frac{\alpha\beta}{(1+\beta)^2}\} \\ &= \delta\{1 - \frac{\alpha(1+2\beta-\beta)}{(1+\beta)^2}\} = \delta\{1 - \frac{\alpha}{1+\beta}\} \end{aligned}$$

where

$$\lim_{\beta\to 0} |A(x^*, y^*)| = \delta(1-\alpha) > 0.$$

Since $x^* = 1 - \dfrac{\alpha}{1+\beta} \to (1-\alpha) > 0$ as $\beta \to 0$, we obtain

$$\begin{aligned} \lim_{\beta\to 0}\{\operatorname{tr} A(x^*, y^*))\} &= \{-1 + \frac{\alpha(1+2\beta)}{(1+\beta)^2} - \delta\} = -1 + \alpha - \delta \\ &= -(1-\alpha+\delta) < 0, \end{aligned}$$

since $1 - \alpha > 0$. Let $\rho_3 = (\operatorname{tr} A(x^*, y^*))^2 - 4|A(x^*, y^*)|$, then we derive

$$\begin{aligned} \rho_3 &= [1-\alpha+\delta]^2 - 4\delta(1-\alpha) = (1-\alpha)^2 + 2(1-\alpha)\delta + \delta^2 - 4\delta(1-\alpha) \\ &= (1-\alpha)^2 - 2\delta(1-\alpha) + \delta^2 = (1-\alpha-\delta)^2 \geq 0. \end{aligned}$$

Therefore, (x^*, y^*) is a stable node of (2.3). $\square$

Theorem 2.3.10. *If* $\delta > 0$, $\beta > 0$, $1+\beta > \alpha > 0$, $\delta \to 0_+$ *and* $\alpha > \alpha_1(\beta) = \dfrac{(1+\beta)^2}{1+2\beta}$,
then (x^*, y^*) *is an unstable node of* (2.3).

Proof. From equation (2.9), we get

$$\begin{aligned}\mathrm{tr}(A(x^*, y^*)) &= -1 + \frac{\alpha(1+2\beta)}{(1+\beta)^2} - \delta \\ &= -1 - \delta + \frac{\alpha(1+2\beta)}{(1+\beta)^2}.\end{aligned}$$

Moreover,

$$\lim_{\delta \to 0_+} \mathrm{tr}(A(x^*, y^*)) = -1 + \frac{\alpha(1+2\beta)}{(1+\beta)^2} = \frac{1+2\beta}{(1+\beta)^2}[\alpha - \frac{(1+\beta)^2}{1+2\beta}] > 0,$$

where $\alpha > \alpha_1(\beta) = \dfrac{(1+\beta)^2}{1+2\beta}$.
Let $\rho = (\mathrm{tr}(A(x^*, y^*)))^2 - 4|A(x^*, y^*)| = [-1-\delta+\dfrac{\alpha(1+2\beta)}{(1+\beta)^2}]^2 - 4\delta(1-\dfrac{\alpha}{1+\beta})$, then

$$\lim_{\delta \to 0_+} \rho = [-1 + \frac{\alpha(1+2\beta)}{(1+\beta)^2}]^2 = [\frac{\alpha(1+2\beta)}{(1+\beta)^2} - 1]^2 > 0.$$

We showed that $|A(x^*, y^*)| > 0$ as $\delta \to 0_+$, $\mathrm{tr}(A(x^*, y^*)) > 0$ and $\rho > 0$ as $\delta \to 0_+$.
So, (x^*, y^*) is an unstable node of (2.3). $\square$

2.4 Hopf bifurcations and limit cycles of (2.3)

In this section, we study the Hopf bifurcation and the limit cycles of equation (2.3) at the interior equilibrium (x^*, y^*) defined in (2.7). It is well-known that the necessary condition for the existence of the Hopf bifurcation at (x^*, y^*) is that $|B(x^*, y^*)| > 0$ and $\operatorname{tr} B(x^*, y^*) = 0$ for $\alpha = \beta > 0$. By Lemma 2.3.7(3), we see that $(x^*, y^*) = (x^*, \frac{x^*}{\alpha})$ satisfies the necessary condition when $\alpha \in (\frac{1+\sqrt{5}}{2}, \infty)$. In fact, we prove below that the Hopf bifurcation of (2.3) occurs at $(x^*, \frac{x^*}{\alpha})$ for each $\alpha \in (\frac{1+\sqrt{5}}{2}, \infty)$. We need a well-known result which provides the sufficient conditions for a system to have Hopf bifurcation and limit cycles at the equilibrium $(0,0)$. To state the result, we consider the following system:

$$\begin{cases} \dot{u} = au + bv + p(u,v) := f_1(u,v), \\ \dot{v} = cu + dv + q(u,v) := g_1(u,v), \end{cases} \tag{2.19}$$

where $p(u,v) = \sum_{i+j=2}^{\infty} a_{ij} u^i v^j$ and $q(u,v) = \sum_{i+j=2}^{\infty} b_{ij} u^i v^j$. The Jacobian matrix of f_1 and g_1 at the equilibrium $(0,0)$ is

$$A(0,0) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{2.20}$$

If $\Delta = |A(0,0)| = ad - bc > 0$, then it follows from the formula $(3')$ of Section 4.4 in [31] that the Liapunov number, denoted by σ_2, of (2.19), is given by

$$\sigma_2 = \frac{-3\pi}{2b\Delta^{3/2}} \sum_{i=1}^{8} \xi_i, \tag{2.21}$$

where

$$\xi_1 = ac(a_{11}^2 + a_{11}b_{02} + a_{02}b_{11}),\ \xi_2 = ab(b_{11}^2 + a_{20}b_{11} + a_{11}b_{02}),$$
$$\xi_3 = c^2(a_{11}a_{02} + 2a_{02}b_{02}),\ \xi_4 = -2ac(b_{02}^2 - a_{20}a_{02}),\ \xi_5 = -2ab(a_{20}^2 - b_{20}b_{02}),$$
$$\xi_6 = -b^2(2a_{20}b_{20} + b_{11}b_{20}),\ \xi_7 = (bc - 2a^2)(b_{11}b_{02} - a_{11}a_{20}),$$
$$\xi_8 = -(a^2 + bc)[3(cb_{03} - ba_{30}) + 2a(a_{21} + b_{12}) + (ca_{12} - bb_{21})].$$

The following result is a special case of Theorem 1 and Remark 1 in the section 4.4 of [31] (also see pages 253, and 261-264 in [6]).

Lemma 2.4.1. *Assume that $\Delta > 0$, $\tau = a+d = 0$ and $\sigma_2 < 0$. Then the following assertions hold:*
(1) *The equilibrium* $(0,0)$ *is a stable center or a stable focus with multiplicity one.*
(2) *A Hopf bifurcation of* (2.19) *occurs at* $(0,0)$ *at the bifurcation value $\tau = a+d = 0$.*
(3) *A unique stable limit cycle of* (2.19) *bifurcates from* $(0,0)$ *of as the bifurcation value $\tau = a+d$ increases from zero.*

Below, we give our main result in this section.

Theorem 2.4.2. *Assume that $\alpha \in (\frac{1+\sqrt{5}}{2}, \infty)$. Then the following assertions hold:*

1. $(x^*, \frac{x^*}{\alpha})$ *is a stable center or a stable focus with multiplicity one of* (2.3).
2. *A Hopf bifurcation occurs at* $(x^*, \frac{x^*}{\alpha})$ *of* (2.3) *at the bifurcation value $\tau = a + d = 0$.*
3. *A unique stable limit cycle bifurcates from* $(x^*, \frac{x^*}{\alpha})$ *of* (2.3), *as the bifurcation value $\tau = a + d$ increases from zero.*

Proof. Let $u = x - x^*$, then $x = x^* + u$ and $v = y - x^* \Rightarrow y = y^* + v = \dfrac{x^*}{\alpha} + v$. Substituting the values of x and y in equation (2.19), the equations become

$$\dot{u} = (x^* + u)(1 - x^* - u) - \alpha \frac{(x^* + u)(\frac{x^*}{\alpha} + v)}{(x^* + u) + (\frac{x^*}{\alpha} + v)} := f_1(u, v) \tag{2.22}$$

$$\dot{v} = \delta(\frac{x^*}{\alpha} + v)\{1 - \beta \frac{(\frac{x^*}{\alpha} + v)}{x^* + u}\} := g_1(u, v) \tag{2.23}$$

Let $c = x^* + \dfrac{x^*}{\alpha} = \dfrac{1}{1+\alpha} + \dfrac{1}{\alpha(1+\alpha)} = \dfrac{1}{\alpha}$ where $x^* = \dfrac{1}{1+\alpha}$.
Simplifying (2.22), indeed we get,

$$\begin{aligned} f_1(u, v) &= (x^* + u)(1 - x^* - u) - \alpha[\frac{(x^* + u)(\frac{x^*}{\alpha} + v)}{(x^* + u) + (\frac{x^*}{\alpha} + v)}] \\ &= (x^* + u)(1 - x^* - u) - \alpha[\frac{1}{c}(x^* + u)(\frac{x^*}{\alpha} + v)(1 + \frac{u + v}{c})^{-1}] \end{aligned} \tag{2.24}$$

We use the Taylor's series expansion from (2.24).

Let $f_1 = f_1(u, v)$ then

$$\begin{aligned} f_1 &= (x^* + u)(1 - x^* - u) - \frac{\alpha}{c}(x^* + u)(\frac{x^*}{\alpha} + v)[\sum_{n=0}^{3}(-1)^n(\frac{u + v}{c})^n \\ &\quad + \sum_{n=4}^{\infty}(-1)^n(\frac{u + v}{c})^n] \\ &= (x^* + u)(1 - x^* - u) - \frac{\alpha}{c}(x^* + u)(\frac{x^*}{\alpha} + v)[1 - \frac{u + v}{c} + (\frac{u + v}{c})^2 \\ &\quad -(\frac{u + v}{c})^3 + \sum_{n=4}^{\infty}(-1)^n(\frac{u + v}{c})^n] \\ &= (x^* + u)(1 - x^* - u) - \frac{\alpha}{c}(\frac{x^{*2}}{\alpha} + x^*v + \frac{x^*}{\alpha}u + uv)[1 - \frac{u}{c} - \frac{v}{c} + \frac{u^2}{c^2} + \frac{2}{c^2}uv \\ &\quad + \frac{v^2}{c^2} - \frac{u^3}{c^3} - \frac{3}{c^3}u^2v - \frac{3}{c^3}uv^2 - \frac{v^3}{c^3} + \sum_{n=4}^{\infty}(-1)^n(\frac{u + v}{c})^n] \end{aligned}$$

From the above expression, rearranging the constant and the coefficients of u, v, u^2, u v, v^2, u^3, u^2v, uv^2 and v^3, we derive

$$\begin{aligned}f_1 &= (x^* - x^{*2} - \frac{x^{*2}}{c}) + (1 - 2x^* - \frac{x^*}{c} + \frac{x^{*2}}{c^2})u + (-\frac{\alpha x^*}{c} + \frac{x^{*2}}{c^2})v \\ &+ (-1 - \frac{x^{*2}}{c^3} + \frac{x^*}{c^2})u^2 + (-\frac{2x^{*2}}{c^3} + \frac{\alpha x^*}{c^2} + \frac{x^*}{c^2} - \frac{\alpha}{c})uv + (-\frac{x^{*2}}{c^3} + \frac{\alpha x^*}{c^2})v^2 \\ &+ (-\frac{x^*}{c^3} + \frac{x^{*2}}{c^4})u^3 + (3\frac{x^{*2}}{c^4} - \alpha\frac{x^*}{c^3} - 2\frac{x^*}{c^3} + \frac{\alpha}{c^2})u^2v \\ &+ (3\frac{x^{*2}}{c^4} - 2\alpha\frac{x^*}{c^3} - \frac{x^*}{c^3} + \frac{\alpha}{c^2})uv^2 + (-\alpha\frac{x^*}{c^3} + \frac{x^{*2}}{c^4})v^3 + O_4(u,v). \end{aligned} \tag{2.25}$$

From (2.19), we obtain that this expression is equivalent to the following expression

$$\begin{aligned} f_1 &= \eta_0 + au + bv + a_{20}u^2 + a_{11}uv + a_{02}v^2 + a_{30}u^3 + a_{21}u^2v \\ &+ a_{12}uv^2 + a_{03}v^3 + O_4(u,v) \\ &= au + bv + a_{20}u^2 + a_{11}uv + a_{02}v^2 + a_{30}u^3 + a_{21}u^2v \\ &+ a_{12}uv^2 + a_{03}v^3 + O_4(u,v). \end{aligned} \tag{2.26}$$

We compare the coefficients of u, u^2, uv, v^2, u^3, u^2v, uv^2, v^3 and the constant from expressions (2.25) and (2.26).

First of all, we get the following relations in terms of x^* and c, then substituting values $x^* = \dfrac{1}{1+\alpha}$ and $c = \dfrac{1}{\alpha}$, we derive

$$a = 1 - 2x^* - \frac{x^*}{c} + \frac{x^{*2}}{c^2} = 1 - \frac{2}{1+\alpha} - \frac{\alpha}{1+\alpha} + \frac{\alpha^2}{(1+\alpha)^2} = \frac{\alpha^2 - \alpha - 1}{(1+\alpha)^2},$$

$$b = -\frac{\alpha x^*}{c} + \frac{x^{*2}}{c^2} = -\frac{\alpha^2}{1+\alpha} + \frac{\alpha^2}{(1+\alpha)^2} = \frac{-\alpha^2 - \alpha^3 + \alpha^2}{(1+\alpha)^2} = -\frac{\alpha^3}{(1+\alpha)^2} \tag{2.27}$$

$$a_{20} = -1 - \frac{x^{*2}}{c^3} + \frac{x^*}{c^2} = -1 - \frac{\alpha^3}{(1+\alpha)^2} + \frac{\alpha^2}{1+\alpha}$$
$$= \frac{-1 - 2\alpha - \alpha^2 - \alpha^3 + \alpha^2 + \alpha^3}{(1+\alpha)^2} = -\frac{1+2\alpha}{(1+\alpha)^2},$$
$$a_{11} = \frac{-2x^{*2}}{c^3} + \alpha\frac{x^*}{c^2} + \frac{x^*}{c^2} - \frac{\alpha}{c} = \frac{-2\alpha^{*3}}{(1+\alpha)^2} + \frac{\alpha^3}{1+\alpha} + \frac{\alpha^{*2}}{1+\alpha} - \alpha^{*2}$$

$$= \frac{-2\alpha^3 + \alpha^3 + \alpha^4 + \alpha^2 + \alpha^3 - \alpha^2 - 2\alpha^3 - \alpha^4}{(1+\alpha)^2} = \frac{-2\alpha^3}{(1+\alpha)^2},$$

$$a_{02} = \frac{-x^{*2}}{c^3} + \frac{\alpha x^*}{c^2} = \frac{-\alpha^3}{(1+\alpha)^2} + \frac{\alpha^3}{1+\alpha} = \frac{-\alpha^3 - \alpha^4 + \alpha^4}{(1+\alpha)^2} = \frac{\alpha^4}{(1+\alpha)^2},$$

$$a_{30} = \frac{-x^*}{c^3} + \frac{x^{*2}}{c^4} = \frac{-\alpha^3}{1+\alpha} + \frac{\alpha^4}{(1+\alpha)^2} = \frac{-\alpha^3 - \alpha^4 + \alpha^4}{(1+\alpha)^2} = \frac{-\alpha^3}{(1+\alpha)^2},$$

$$a_{21} = \frac{3x^{*2}}{c^4} - \frac{\alpha x^*}{c^3} - \frac{2x^*}{c^3} + \frac{\alpha}{c^2} = \frac{3\alpha^4}{(1+\alpha)^2} - \frac{\alpha^4}{1+\alpha} - \frac{2\alpha^3}{1+\alpha} + \alpha^3$$

$$= \frac{3\alpha^4 - \alpha^4 - \alpha^5 - 2\alpha^3 - 2\alpha^4 + \alpha^3 + 2\alpha^4 + \alpha^5}{(1+\alpha)^2} = \frac{2\alpha^4 - \alpha^3}{(1+\alpha)^2},$$

$$a_{12} = \frac{3x^{*2}}{c^4} - \frac{2\alpha x^*}{c^3} - \frac{x^*}{c^3} + \frac{\alpha}{c^2} = \frac{3\alpha^4}{(1+\alpha)^2} - \frac{2\alpha^4}{1+\alpha} - \frac{\alpha^3}{1+\alpha} + \alpha^3$$

$$= \frac{3\alpha^4 - 2\alpha^4 - 2\alpha^5 - \alpha^3 - \alpha^4 + \alpha^3 + 2\alpha^4 + \alpha^5}{(1+\alpha)^2} = \frac{2\alpha^4 - \alpha^5}{(1+\alpha)^2},$$

$$a_{03} = \frac{-\alpha x^*}{c^3} + \frac{x^{*2}}{c^4} = \frac{-\alpha^4}{1+\alpha} + \frac{\alpha^4}{(1+\alpha)^2} = \frac{-\alpha^4 - \alpha^5 + \alpha^4}{(1+\alpha)^2} = \frac{-\alpha^5}{(1+\alpha)^2}$$

and $\eta_0 = x^* - x^{*2} - \dfrac{x^{*2}}{c} = \dfrac{1}{1+\alpha} - \dfrac{1}{(1+\alpha)^2} - \dfrac{\alpha}{(1+\alpha)^2} = \dfrac{1+\alpha-1-\alpha}{(1+\alpha)^2} = 0.$

Here $\eta_0 = x^* - x^{*2} - \dfrac{x^{*2}}{c}$ is independent of u and v, thus it is a constant term which is equivalent to zero.

Now, we expand g_1 into the Taylor's series. Simplifying (2.23), we get

$$\begin{aligned} g_1(u,v) &= \delta(\frac{x^*}{\alpha} + v)[1 - \alpha\frac{(\frac{x^*}{\alpha} + v)}{x^* + u}] \\ &= \delta(\frac{x^*}{\alpha} + v)[1 - \alpha(\frac{(\frac{x^*}{\alpha} + v)}{x^*(1 + \frac{u}{x^*})}]. \end{aligned} \tag{2.28}$$

Using the Taylor series expansion of (2.28), we derive

$$\begin{aligned}
g_1(u,v) &= \delta(\frac{x^*}{\alpha}+v)[1-\alpha(\frac{1}{\alpha}+\frac{v}{x^*})(1+\frac{u}{x^*})^{-1}] \\
&= \delta(\frac{x^*}{\alpha}+v)[1-(1+\frac{\alpha v}{x^*})\{\sum_{n=0}^{3}(-1)^n(\frac{u}{x^*})^n+\sum_{n=4}^{\infty}(-1)^n(\frac{u}{x^*})^n\}] \\
&= \delta(\frac{x^*}{\alpha}+v)[1-(1+\frac{\alpha v}{x^*})\{1-\frac{u}{x^*}+\frac{u^2}{x^{*2}}-\frac{u^3}{x^{*3}}+\sum_{n=4}^{\infty}(-1)^n(\frac{u}{x^*})^n\}] \\
&= \delta(\frac{x^*}{\alpha}+v)[1-\{1-\frac{u}{x^*}+\frac{u^2}{x^{*2}}-\frac{u^3}{x^{*3}}+\frac{\alpha}{x^*}v-\frac{\alpha}{x^{*2}}uv \\
&\quad +\frac{\alpha}{x^{*3}}u^2v+P_4(u,v)\}]
\end{aligned}$$

Rearranging the coefficients of u, v, u^2, uv, v^2, u^3, u^2v and uv^2 from the above expression, we obtain

$$\begin{aligned}
g_1(u,v) &= \delta(\frac{x^*}{\alpha}+v)[1-1+\frac{u}{x^*}-\frac{u^2}{x^{*2}}+\frac{u^3}{x^{*3}}-\frac{\alpha}{x^*}v \\
&\quad +\frac{\alpha}{x^{*2}}uv-\frac{\alpha}{x^{*3}}u^2v+P_4(u,v)] \\
&= \delta[\frac{1}{\alpha}u-v-\frac{1}{\alpha x^*}u^2+\frac{1}{x^*}uv+\frac{1}{x^*}uv-\frac{\alpha}{x^*}v^2+\frac{1}{\alpha x^{*2}}u^3 \\
&\quad -\frac{1}{x^{*2}}u^2v-\frac{1}{x^{*2}}u^2v+\frac{\alpha}{x^{*2}}uv^2+P_4(u,v)] \\
&= \delta[\frac{1}{\alpha}u-v-\frac{1}{\alpha x^*}u^2+\frac{2}{x^*}uv-\frac{\alpha}{x^*}v^2+\frac{1}{\alpha x^{*2}}u^3 \\
&\quad -\frac{2}{x^{*2}}u^2v+\frac{\alpha}{x^{*2}}uv^2+P_4(u,v)].
\end{aligned} \tag{2.29}$$

The expression (2.29) is equivalent to (2.30)

$$\begin{aligned}
g_1(u,v) &= cu+dv+b_{20}u^2+b_{11}uv+b_{02}v^2+b_{30}u^3 \\
&\quad +b_{21}u^2v+b_{12}uv^2+P_4(u,v).
\end{aligned} \tag{2.30}$$

Comparing the coefficients of u, v, u^2, uv, v^2, u^3, u^2v and uv^2 from (2.29) and (2.30), we get $c=\frac{\delta}{\alpha}=\frac{\alpha^2-\alpha-1}{\alpha(1+\alpha)^2}$, $d=-\delta=-\frac{(\alpha^2-\alpha-1)}{(1+\alpha)^2}$,
$b_{20}=-\frac{\delta}{\alpha x^*}=-\frac{(\alpha^2-\alpha-1)(1+\alpha)}{\alpha(1+\alpha)^2}=-\frac{(\alpha^2-\alpha-1)}{\alpha(1+\alpha)}$,

$$b_{11} = \frac{2\delta}{x^*} = \frac{2(\alpha^2-\alpha-1)(1+\alpha)}{(1+\alpha)^2} = \frac{2(\alpha^2-\alpha-1)}{1+\alpha},$$
$$b_{02} = -\frac{\alpha\delta}{x^*} = \frac{-\alpha(\alpha^2-\alpha-1)(1+\alpha)}{(1+\alpha)^2} = \frac{-\alpha(\alpha^2-\alpha-1)}{1+\alpha},$$
$$b_{30} = \frac{\delta}{\alpha x^{*2}} = \frac{(\alpha^2-\alpha-1)(1+\alpha)^2}{\alpha(1+\alpha)^2} = \frac{\alpha^2-\alpha-1}{\alpha},$$
$$b_{21} = \frac{-2\delta}{x^{*2}} = -\frac{2(\alpha^2-\alpha-1)(1+\alpha)^2}{(1+\alpha)^2} = -2(\alpha^2-\alpha-1),$$
$$b_{12} = \frac{\alpha\delta}{x^{*2}} = \frac{\alpha(\alpha^2-\alpha-1)(1+\alpha)^2}{(1+\alpha)^2} = \alpha(\alpha^2-\alpha-1),\ b_{03} = 0.$$

Hence, equations (2.22) and (2.23) can be transformed into the following system

$$\begin{cases} \dot{u} & = au + bv + \sum_{i+j=2}^{3} a_{ij}u^iv^j + \sum_{i+j\geq 4}^{\infty} a_{ij}u^iv^j \\ \dot{v} & = cu + dv + \sum_{i+j=2}^{3} b_{ij}u^iv^j + \sum_{i+j\geq 4}^{\infty} b_{ij}u^iv^j \end{cases}. \tag{2.31}$$

It is easy to verify that under the translation $u = x - x^*$ and $v = y - x^*$, the Jacobian determinant remains unchanged. Hence, by Lemma 2.2.1, we have

$$\Delta = ad - bc = |A(x^*, y^*)| > 0.$$

This can be written as

$$\Delta = ad - bc = -\frac{(\alpha^2-\alpha-1)^2}{(1+\alpha)^2} - \frac{(-\alpha^3)}{(1+\alpha)^2}\frac{(\alpha^2-\alpha-1)}{\alpha(1+\alpha)^2}$$
$$= \frac{(\alpha^2-\alpha-1)(-\alpha^2+\alpha+1+\alpha^2)}{(1+\alpha)^4}$$
$$= \frac{\alpha^2-\alpha-1}{(1+\alpha)^3},$$ where $\alpha \in (\frac{1+\sqrt{5}}{2}, \infty)$. It is easy to see that

$$\operatorname{tr}(A) = a + d = \frac{-(1+\alpha-\alpha^2)}{(1+\alpha)^2} + \frac{1+\alpha-\alpha^2}{(1+\alpha)^2} = \frac{-1-\alpha+\alpha^2+1+\alpha-\alpha^2}{(1+\alpha)^2} = 0.$$

We determine the sign of the Liapunov number σ_2 from (2.21), substitute the values of a, c, a_{11}, b_{o2} a_{02},and b_{11}, into ξ_1, and get

$$\begin{aligned}
\xi_1 &= ac(a_{11}^2 + a_{11}b_{02} + a_{02}b_{11}) \\
&= -\frac{(1+\alpha-\alpha^2)}{(1+\alpha)^2}\frac{(\alpha^2-\alpha-1)}{\alpha(1+\alpha)^2}[(\frac{-2\alpha^3}{(1+\alpha)^2})^2 \\
&\quad + \frac{(-2\alpha^3)\alpha(-\alpha^2+\alpha+1)}{(1+\alpha)^2(1+\alpha)} + \frac{\alpha^4}{(1+\alpha)^2}\frac{2(\alpha^2-\alpha-1)}{1+\alpha}] \\
&= \frac{(\alpha^2-\alpha-1)^2}{\alpha(1+\alpha)^4}[\frac{4\alpha^6}{(1+\alpha)^4} + \frac{2\alpha^4(\alpha^2-\alpha-1)}{(1+\alpha)^3} + \frac{2\alpha^4(\alpha^2-\alpha-1)}{(1+\alpha)^3}] \\
&= \frac{(\alpha^2-\alpha-1)^2}{\alpha(1+\alpha)^4}[\frac{4\alpha^6}{(1+\alpha)^4} + \frac{4\alpha^4(\alpha^2-\alpha-1)}{(1+\alpha)^3}] \\
&= \frac{4\alpha^4(\alpha^2-\alpha-1)^2}{\alpha(1+\alpha)^4}[\frac{\alpha^2}{(1+\alpha)^4} + \frac{(\alpha^2-\alpha-1)}{(1+\alpha)^3}] \\
&= \frac{4\alpha^3(\alpha^2-\alpha-1)^2}{(1+\alpha)^4}[\frac{\alpha^2+\alpha^2-\alpha-1+\alpha^3-\alpha^2-\alpha}{(1+\alpha)^4}] \\
&= \frac{4\alpha^3(\alpha^2-\alpha-1)^2}{(1+\alpha)^8}(\alpha^3+\alpha^2-2\alpha-1).
\end{aligned}$$

Substituting the values of a, b, b_{11}, a_{20}, a_{11} and b_{02} in equation (2.21) for ξ_2, we derive

$$\begin{aligned}
\xi_2 &= ab(b_{11}^2 + a_{20}b_{11} + a_{11}b_{02}) \\
&= \frac{-(1+\alpha-\alpha^2)}{(1+\alpha)^2}\frac{(-\alpha^3)}{(1+\alpha)^2}[\frac{4(\alpha^2-\alpha-1)^2}{(1+\alpha)^2} \\
&\quad + (\frac{-(1+2\alpha)}{(1+\alpha)^2})(\frac{2(\alpha^2-\alpha-1)}{1+\alpha}) + \frac{2\alpha^4(\alpha^2-\alpha-1)}{(1+\alpha)^3}] \\
&= 2\frac{\alpha^3(1+\alpha-\alpha^2)}{(1+\alpha)^4}[\frac{2(\alpha^2-\alpha-1)^2}{(1+\alpha)^2} \\
&\quad - \frac{(\alpha^2-\alpha-1)(1+2\alpha)}{(1+\alpha)^3} + \frac{\alpha^4(\alpha^2-\alpha-1)}{(1+\alpha)^3}] \\
&= \frac{2\alpha^3(\alpha^2-\alpha-1)(1+\alpha-\alpha^2)}{(1+\alpha)^2(1+\alpha)^4}[2(\alpha^2-\alpha-1) - \frac{(1+2\alpha)}{1+\alpha} + \frac{\alpha^4}{1+\alpha}] \\
&= \frac{-2\alpha^3(\alpha^2-\alpha-1)^2}{(1+\alpha)^7}(\alpha^4+2\alpha^3-6\alpha-3).
\end{aligned}$$

Also, substituting the values of c, a_{11}, a_{02} and b_{02} in equation (2.21) for ξ_3, we note that

$$\begin{aligned}
\xi_3 &= c^2(a_{11}a_{02} + 2a_{02}b_{02}) \\
&= \frac{(\alpha^2-\alpha-1)^2}{\alpha^2(1+\alpha)^4}[(\frac{-2\alpha^3}{(1+\alpha)^2})(\frac{\alpha^4}{(1+\alpha)^2}) \\
&\quad +2(\frac{\alpha^4}{(1+\alpha)^2})(\frac{-\alpha(\alpha^2-\alpha-1)}{1+\alpha})] \\
&= \frac{(\alpha^2-\alpha-1)^2}{\alpha^2(1+\alpha)^4}[\frac{-2\alpha^7}{(1+\alpha)^4} - 2\frac{\alpha^5(\alpha^2-\alpha-1)}{(1+\alpha)^3}] \\
&= \frac{-2\alpha^5(\alpha^2-\alpha-1)^2}{\alpha^2(1+\alpha)^8}[\alpha^2 + (\alpha^2-\alpha-1)(1+\alpha)] \\
&= \frac{-2\alpha^3(\alpha^2-\alpha-1)^2}{(1+\alpha)^8}[\alpha^2+\alpha^2-\alpha-1+\alpha^3-\alpha^2-\alpha] \\
&= \frac{-2\alpha^3(\alpha^2-\alpha-1)^2(\alpha^3+\alpha^2-2\alpha-1)}{(1+\alpha)^8}.
\end{aligned}$$

Moreover, substituting the values of a, c b_{02}, a_{20} and a_{02} in equation (2.21) into ξ_4, we see that

$$\begin{aligned}
\xi_4 &= -2ac(b_{02}^2 - a_{20}a_{02}) \\
&= (\frac{-2(1+\alpha-\alpha^2)}{(1+\alpha)^2})(\frac{\alpha^2-\alpha-1}{\alpha(1+\alpha)^2}) \\
&\quad [\frac{\alpha^2(\alpha^2-\alpha-1)^2}{(1+\alpha)^2} + (\frac{-(1+2\alpha)}{(1+\alpha)^2}\frac{\alpha^4}{(1+\alpha)^2})] \\
&= \frac{-2(\alpha^2-\alpha-1)^2}{\alpha(1+\alpha)^4}\frac{\alpha^2}{(1+\alpha)^4}[(1+\alpha)^2(\alpha^2-\alpha-1)^2 + \alpha^2(1+2\alpha)] \\
&= \frac{-2\alpha(\alpha^2-\alpha-1)^2}{(1+\alpha)^8}[\alpha^6 - 4\alpha^4 - 2\alpha^3 + 4\alpha^2 + 4\alpha + 1 + \alpha^2 + 2\alpha^3] \\
&= \frac{-2\alpha(\alpha^2-\alpha-1)^2}{(1+\alpha)^8}(\alpha^6 - 4\alpha^4 + 5\alpha^2 + 4\alpha + 1).
\end{aligned}$$

In addition, substituting the values of a, b, a_{20}, b_{20} and b_{02} in equation (2.21) into

ξ_5, we remark that

$$\begin{aligned}
\xi_5 &= -2ab(a_{20}^2 - b_{20}b_{02}) \\
&= -2\{\frac{-(1+\alpha-\alpha^2)}{(1+\alpha)^2}\frac{(-\alpha^3)}{(1+\alpha)^2}\}[\frac{(1+2\alpha)^2}{(1+\alpha)^4} \\
&\quad -(\frac{-(\alpha^2-\alpha-1)}{\alpha(1+\alpha)})(\frac{-\alpha(\alpha^2-\alpha-1)}{1+\alpha})] \\
&= \frac{2\alpha^3(\alpha^2-\alpha-1)}{(1+\alpha)^8}[(1+2\alpha)^2-(1+\alpha)^2(\alpha^2-\alpha-1)^2] \\
&= \frac{2\alpha^3(\alpha^2-\alpha-1)}{(1+\alpha)^8}[-\alpha^3(\alpha^3-4\alpha-2)].
\end{aligned}$$

Also, substituting the values of b, a_{20}, b_{20} and b_{11} in equation (2.21) into ξ_6, we write

$$\begin{aligned}
\xi_6 &= -b^2(2a_{20}b_{20} + b_{11}b_{20}) \\
&= \frac{(-\alpha^6)}{(1+\alpha)^4}[2(\frac{-(1+2\alpha)}{(1+\alpha)^2})(\frac{-(\alpha^2-\alpha-1)}{\alpha(1+\alpha)}) \\
&\quad +2(\frac{(\alpha^2-\alpha-1)}{1+\alpha})(\frac{-(\alpha^2-\alpha-1)}{\alpha(1+\alpha)})] \\
&= \frac{-\alpha^6}{(1+\alpha)^4}\frac{2(\alpha^2-\alpha-1)}{\alpha(1+\alpha)^3}[(1+2\alpha)-(1+\alpha)(\alpha^2-\alpha-1)] \\
&= \frac{-2\alpha^5(\alpha^2-\alpha-1)}{(1+\alpha)^7}[1+2\alpha-\alpha^2+\alpha+1-\alpha^3+\alpha^2+\alpha] \\
&= \frac{-2\alpha^5(\alpha^2-\alpha-1)}{(1+\alpha)^7}(2+4\alpha-\alpha^3) = \frac{2\alpha^5(\alpha^2-\alpha-1)(\alpha^3-4\alpha-2)}{(1+\alpha)^7}.
\end{aligned}$$

Similarly, substituting the values of b, c, a, b_{11}, b_{02},

a_{11} and a_{20} in equation (2.21) into ξ_7, we derive

$$\begin{aligned}
\xi_7 &= (bc - 2a^2)(b_{11}b_{02} - a_{11}a_{20}) \\
&= [\frac{-\alpha^3(\alpha^2-\alpha-1)}{\alpha(1+\alpha)^4} - 2\frac{(\alpha^2-\alpha-1)^2}{(1+\alpha)^4}] \\
&\quad [\frac{-2\alpha(\alpha^2-\alpha-1)^2}{(1+\alpha)^2} - (\frac{-2\alpha^3}{(1+\alpha)^2})(\frac{-(1+2\alpha)}{(1+\alpha)^2})] \\
&= \frac{2\alpha(\alpha^2-\alpha-1)}{(1+\alpha)^8}[\alpha^2+2\alpha^2-2\alpha-2][(1+\alpha)^2(\alpha^2-\alpha-1)^2+\alpha^2(1+2\alpha)] \\
&= \frac{2\alpha(\alpha^2-\alpha-1)}{(1+\alpha)^8}(3\alpha^2-2\alpha-2)(\alpha^6-4\alpha^4+5\alpha^2+4\alpha+1).
\end{aligned}$$

Finally, substituting the values of a, b, c, b_{03}, a_{30}, a_{21}, b_{12}, a_{12} and b_{21} in equation (2.21) into ξ_8, we get

$$\begin{aligned}
\xi_8 &= -(a^2+bc)[3(cb_{03}-ba_{30}) + 2a(a_{21}+b_{12}) + (ca_{12}-bb_{21}) \\
&= -[\frac{(\alpha^2-\alpha-1)^2}{(1+\alpha)^4} + (\frac{-\alpha^3}{(1+\alpha)^2})(\frac{\alpha^2-\alpha-1}{\alpha(1+\alpha)^2})][0-(\frac{-\alpha^3}{(1+\alpha)^2}) \\
&\quad (\frac{-\alpha^3}{(1+\alpha)^2}) + 2\frac{\alpha^2-\alpha-1}{(1+\alpha)^2}\{\frac{2\alpha^4-\alpha^3}{(1+\alpha)^2} + \alpha(\alpha^2-\alpha-1)\} \\
&\quad +(\frac{\alpha^2-\alpha-1}{\alpha(1+\alpha)^2})(\frac{2\alpha^4-\alpha^5}{(1+\alpha)^2}) + \frac{\alpha^3}{(1+\alpha)^2}\{-2(\alpha^2-\alpha-1)\}] \\
&= -[\frac{(\alpha^2-\alpha-1)^2}{(1+\alpha)^4} - \frac{\alpha^2(\alpha^2-\alpha-1)}{(1+\alpha)^4}][0 - \frac{-3\alpha^3}{(1+\alpha)^4} \\
&\quad +2\frac{\alpha^2-\alpha-1}{(1+\alpha)^4}\{(2\alpha^4-\alpha^3)+(\alpha^2+\alpha+1)(\alpha^3-\alpha^2-\alpha)\} \\
&\quad \{\frac{(\alpha^2-\alpha-1)(2\alpha^3-\alpha^4)}{(1+\alpha)^4} - \frac{2\alpha^3(\alpha^2-\alpha-1)}{(1+\alpha)^2}\}].
\end{aligned}$$

Thus,

$$\begin{aligned}
\xi_8 &= -\frac{1}{(1+\alpha)^4}(\alpha^2-\alpha-1)(\alpha^2-\alpha-1-\alpha^2)[\frac{-3\alpha^6}{(1+\alpha)^4}(2\alpha^4-\alpha^3+\alpha^5 \\
&\quad +\alpha^4-2\alpha^3-3\alpha^2-\alpha)+\frac{\alpha^2-\alpha-1}{(1+\alpha)^2}\{2\alpha^3-\alpha^4-2\alpha^3(\alpha^2+2\alpha+1)\}] \\
&= \frac{(\alpha^2-\alpha-1)(\alpha+1)}{(1+\alpha)^4}[\frac{-3\alpha^6}{(1+\alpha)^4}+\frac{2(\alpha^2-\alpha-1)(\alpha^5+3\alpha^4-3\alpha^3-3\alpha^2-\alpha)}{(1+\alpha)^4} \\
&\quad +\frac{(\alpha^2-\alpha-1)}{(1+\alpha)^4}(2\alpha^3-\alpha^4-2\alpha^5-4\alpha^4-2\alpha^3)].
\end{aligned}$$

Taking the common factor for the simplification from the expression above, we can write

$$\begin{aligned}
\xi_8 &= \frac{(\alpha^2-\alpha-1)}{(1+\alpha)^3}[\frac{-3\alpha^6}{(1+\alpha)^4}+\frac{(\alpha^2-\alpha-1)}{(1+\alpha)^4} \\
&\quad (2\alpha^5+6\alpha^4-6\alpha^3-6\alpha^2-2\alpha-2\alpha^5-5\alpha^4)] \\
&= \frac{(\alpha^2-\alpha-1)}{(1+\alpha)^3}[\frac{-3\alpha^6}{(1+\alpha)^4}+\frac{(\alpha^2-\alpha-1)}{(1+\alpha)^4}(\alpha^4-6\alpha^3-6\alpha^2-2\alpha)] \\
&= \frac{(\alpha^2-\alpha-1)}{(1+\alpha)^7}(-3\alpha^6+\alpha^6-7\alpha^5-\alpha^4+10\alpha^3+8\alpha^2+2\alpha) \\
&= \frac{-\alpha(\alpha^2-\alpha-1)}{(1+\alpha)^7}(2\alpha^5+7\alpha^4+\alpha^3-10\alpha^2-8\alpha-2).
\end{aligned}$$

By adding ξ_1 to ξ_6 from the above expressions, we get

$$\begin{aligned}
\sum_{i=1}^{6}\xi_i &= \xi_1+\xi_2+\xi_3+\xi_4+\xi_5+\xi_6 \\
&= \frac{-2\alpha(\alpha^2-\alpha-1)}{(1+\alpha)^8}(\alpha^9+3\alpha^8-5\alpha^7-16\alpha^6 \\
&\quad +7\alpha^5+23\alpha^4+8\alpha^3-6\alpha^2-5\alpha-1).
\end{aligned}$$

where adding ξ_7 and ξ_8, we can write

$$\begin{aligned}
\xi_7+\xi_8 &= \frac{-\alpha(\alpha^2-\alpha-1)}{(1+\alpha)^8}(-6\alpha^8+4\alpha^7+30\alpha^6 \\
&\quad -7\alpha^5-38\alpha^4-13\alpha^3+12\alpha^2+10\alpha+2).
\end{aligned}$$

Now, substituting the values of ξ_1, ξ_2, ξ_3, ξ_4, ξ_5, ξ_6, ξ_7 and ξ_8 in equation (2.21), we obtain

$$\begin{aligned}
\sum_{i=1}^{8}\xi_i &= \xi_1+\xi_2+\xi_3+\xi_4+\xi_5+\xi_6+\xi_7+\xi_8 \\
&= \frac{-2\alpha(\alpha^2-\alpha-1)}{(1+\alpha)^8}(\alpha^9+3\alpha^8-5\alpha^7-16\alpha^6+7\alpha^5+23\alpha^4 \\
&\quad +8\alpha^3-6\alpha^2-5\alpha-1)+\frac{-\alpha(\alpha^2-\alpha-1)}{(1+\alpha)^8(-6\alpha^8+4\alpha^7} \\
&\quad +30\alpha^6-7\alpha^5-38\alpha^4-13\alpha^3+12\alpha^2+10\alpha+2) \\
&= \frac{-\alpha(\alpha^2-\alpha-1)}{(1+\alpha)^8}(2\alpha^9-6\alpha^7-2\alpha^6+7\alpha^5+8\alpha^4+3\alpha^3) \\
&= \frac{-\alpha^4(\alpha^2-\alpha-1)}{(1+\alpha)^8}(2\alpha^6-6\alpha^4-2\alpha^3+7\alpha^2+8\alpha+3).
\end{aligned}$$

Substituting $w(\alpha)=\alpha^2-\alpha-1$ in the above expression, we remark that

$$\begin{aligned}
\sum_{i=1}^{8}\xi_i &= \frac{-\alpha^4 w(\alpha)}{(1+\alpha)^8}[(\alpha^2-\alpha-1)(2\alpha^4+2\alpha^3-2\alpha^2-2\alpha+3)+(9\alpha+6)] \\
&= \frac{-\alpha^4 w(\alpha)}{(1+\alpha)^8}[w(\alpha)(2\alpha^4+2\alpha^3-2\alpha^2-2\alpha+3)+(9\alpha+6)] \\
&= \frac{-\alpha^4 w(\alpha)}{(1+\alpha)^8}[w(\alpha)\{2\alpha^4+2\alpha(\alpha^2-\alpha-1)+3\}+(9\alpha+6)] \\
&= \frac{-\alpha^4 w(\alpha)}{(1+\alpha)^8}[w(\alpha)\{2\alpha^4+2\alpha w(\alpha)+3\}+(9\alpha+6)],
\end{aligned}$$

and thus

$$\sum_{i=1}^{8}\xi_i = \frac{-\alpha^4}{(1+\alpha)^8}[w^2(\alpha)\{2\alpha^4+2\alpha w(\alpha)+3\}+w(\alpha)(9\alpha+6)]<0,$$

if $w(\alpha)=\alpha^2-\alpha-1>0$ for $\alpha\in(\frac{1+\sqrt{5}}{2},\infty)$. □

From equation (2.21), we derive $\sigma_2=\frac{-3\pi}{2b\Delta^{3/2}}\sum_{i=1}^{8}\xi_i<0$, since $\sum_{i=1}^{8}\xi_i<0$ and noting that $b=-\frac{\alpha^3}{(1+\alpha^2)}$ from equation (2.27). The result follows from Lemma 2.4.1.

Chapter 3

Stochastic Model

3.1 A stochastic mathematical model of the predator-prey interaction

Deterministic models are stable with a cyclic behaviour in the common period for the sizes of species populations. However, in practice, stochastic variations will occur in the values of x and y, which may produce a qualitatively different behaviour. These variations may lead to an extinction of the predator as a result of a possible extinction of the prey. Deterministic models may be inadequate for capturing the exact variability in nature. Then, stochastic models are required for an accurate approximation of the dynamics of such interactions. The random fluctuations result in changing some degree of parameters in the deterministic environment. In nature, real environments are stochastic.

In fact, biological systems are inherently random in nature and the noise plays a vital role in the structure and function of such systems [28]. Stochastic differential equation models were introduced by May in [28] to investigate limits to the niche overlap in randomly fluctuating environments . This was a quantum leap in the development of mathematically sophisticated ecological modelling. An important concept in stochastic modelling is that of a Wiener process.

A Wiener process is a continuous time stochastic process which is named in honour of Nobert Wiener . It is also known as the standard Brownian motion [17], after Robert Brown.

Definition 3.1.1. A *Wiener process* W(t) depends continuously on $t \in [0, T]$ and satisfies the following conditions:

1. $W(0) = 0$ (with probability 1).

2. For $0 \leq s < t$, the increment $W(t) - W(s)$ is a random variable normally distributed with mean zero and variance $t - s$. It is equivalent to $W(t) - W(s) \sim \sqrt{t-s}N(0,1) = N(0, t-s)$, where N(0,1) denotes a normally distributed random variable with zero mean and unit variance.

3. For $0 \leq s < t < u < v \leq T$, the increments $W(t) - W(s)$ and $W(v) - W(u)$ are independent random variables.

Definition 3.1.2. The *normal distribution* is defined as a continuous probability distribution with a bell-shaped probability density function, which is known as the Gaussian function

$$f(x, \mu, \sigma_1^2) = \frac{1}{\sigma_1\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma_1}\right)^2}, \tag{3.1}$$

where μ is the mean or expectation and σ_1^2 is the variance. If the distribution has $\mu = 0$ and $\sigma_1^2 = 1$, then it is called the unit normal distribution.

In general, $N(\mu, \sigma_1^2)$ denotes a normal distribution with expected value μ and variance σ_1^2. If the random variable X is distributed normally with expected value μ and variance ${\sigma_1}^2$, then we can write $x \sim N(\mu, {\sigma_1}^2) \sim \mu + \sigma_1 N(0, 1)$.

3.2 Itô and Stratonovich integrals

In order to describe the stochastic model we investigate, we need to define the two different forms of a stochastic differential equations: the Itô form and the Stratonovich form [3].

Definition 3.2.1. Let $[0, T] \subset \Re$ and $0 = t_0 < t_1 < .... < t_n = T$ be a discretisation of the time-interval $[0, T]$ with the largest step being
$\Delta t = \max\limits_{k\in\{0,1,\ldots,n-1\}} (t_{k+1} - t_k)$, then

$$\int_0^T f(t)dW_t = \lim_{\Delta t \to 0} \sum_{i=0}^{n-1} f(t_i)[W(t_{i+1}) - W(t_i)] \tag{3.2}$$

is called the *Itô integral* of the function f.

Definition 3.2.2. Let $[0, T] \subset \Re$ and $0 = t_0 < t_1 < .... < t_n = T$ be a discretisation of the time-interval $[0, T]$ with the largest step being
$\Delta t = \max\limits_{k\in\{0,1,\ldots,n-1\}} (t_{k+1} - t_k)$ then

$$\int_0^T f(t) \circ dW_t = \lim_{\Delta t \to 0} \sum_{i=0}^{N-1} f(t_{i+1/2})[W(t_{i+1}) - W(t_i)] \tag{3.3}$$

is called the *Stratonovich integral* of the function of f, where $t_{i+1/2} = \dfrac{t_i + t_{i+1}}{2}$.
Thus there is an important difference between the Itô and the Stratonovich integrals, since f is evaluated at the left-end point of the interval in the Itô case and at the midpoint in the Stratonovich case [3].

The Itô form of a stochastic differential equation can be written in the differential form

$$dx(t) = f(t, x(t))dt + g(t, x(t))dW(t) \tag{3.4}$$

for $0 \leq t \leq T$, where W(t) is a Wiener process.

Stochastic differential equation model for the predator-prey interaction

We can write the stochastic differential equation model for the predator-prey interaction by adding a stochastic term to the model (2.3) as

$$\begin{cases} dx = [x(1-x) - \alpha \dfrac{xy}{x+y}]dt + \sigma x dW_t \\ dy = \delta y[1 - \beta \dfrac{y}{x}]dt + \sigma y dW_t \end{cases} \tag{3.5}$$

where $W(t) = W_t$ is a standard Wiener process, or a Brownian motion, on the time interval $[0, T]$ and σ is a parameter representing the strength of noise.

Stochastic models are more accurate models for studying important biological processes. An example of such a biological application in ecology is the study of the predator-prey interaction.

The deterministic model fails to describe a basic phenomenon of a natural system in the changing environment which may cause random variations in the predator-prey growth rate and death rate. The Gaussian white noise, which is a useful concept to model rapidly fluctuating phenomena and the main source of noise is that there are inherent uncertainties in an ecological system such as varying seasons or nutrient inputs. On the other hand, there are considerable human disturbances that exacerbate the uncertainty in the way an ecosystem responds such as the global warming which is caused by human activities.

Chapter 4

Numerical results

4.1 Numerical methods for stochastic continuous models

Many areas of science and engineering are relying on quantitative analysis, as more complex mathematical models of the real world phenomena become available. Since most of these models don't have a closed form exact solution, numerical approximations are the only tools available for analysing them.

In particular, the Euler-Maruyama method (see for example Higham [17]) is widely used to approximate the solution of stochastic differential equation models arising in applications.

Let us consider an Itô stochastic differential equation in the general form:

$$dx(t) = f(x(t))dt + g(x(t))dW(t), \tag{4.1}$$

for $0 \leq t \leq T$. Here f and g are vector functions , and W is a single Wiener process. The solution is subjected to the initial condition $x(0) = x_0$, where x_0 may be a random variable. The interval $[0, T]$ is discretized as $0 = t_0 < t_1 < ...... < t_n = T$

where $t_j = j\Delta t$ and $\Delta t = \dfrac{T}{n}$ for some integer n. The Euler-Maruyama (EM) method applied to equation (4.1) can be written in the following form

$$x_j = x_{j-1} + f(x_{j-1})\Delta t + g(x_{j-1})(W(t_j) - W(t_{j-1})) \tag{4.2}$$

for $j = 1, 2, ..., n$.

Depending on the desired properties of the numerical solution, the exact solution of the SDE may be approximated by strong or weak numerical methods (see, for example, Higham [17]).

Definition 4.1.1. If x_k is the numerical approximation of $x(t_k)$ on a grid $0 = t_0 < t_1 < ...... < t_n = T$ with the largest step $\Delta t = \max_{k\in\{0,1,...,n-1\}}(t_{k+1} - t_k)$ then the numerical approximation $(x_k)_k$ is said to converge to $x(t)$ with *strong global order* $\gamma > 0$ if there exists a constant $c > 0$ which does not depend on Δt or on the grid and $\Delta t_0 > 0$ such that

$$E|x_n - x(t_n)| \leq c\Delta t^{\gamma}$$

for any grid with $\Delta t < \Delta t_0$.

Note that γ can be fractional, since the root mean-square order of a Wiener increment $\Delta W = W(t + \Delta t) - W(t)$ is $\Delta t^{1/2}$. The strong approximations are needed when the numerical solution is required to follow closely the exact solution on each path. However, when we are interested in obtaining an accurate approximation of the moments of the exact solution , then weak numerical methods will be employed.

Definition 4.1.2. If x_k is a numerical approximation of $x(t_k)$ on a grid $0 = t_0 < t_1 < .... < t_n = T$ with the largest step $\Delta t = \max_{k\in\{0,1,......,n-1\}}(t_{k+1} - t_k)$ then the numerical approximation $(x_k)_k$ is said to converge to $x(t)$ with *weak global order* $\gamma > 0$ if , for any fixed $T > 0$ and any polynomial p, the following holds:

$$|E(p(x_n)) - E(p(x(T)))| \leq c\Delta t^{\gamma},$$

for any $\Delta t < \Delta t_0$ for some $\Delta t_0 > 0$ where c is a constant which is independent on Δt and on the grid.

We remark that the Euler-Maruyama method is of strong order of convergence $\frac{1}{2}$ and weak order of convergence 1. We recall that the Euler method in the deterministic framework was of order of convergence 1. The $\frac{1}{2}$ strong order of convergence is due to the presence of the Wiener increments which behave like $W(t+\Delta) - W(t) \sim \sqrt{\Delta t}N(0,1)$.

Maiti and Pathak [27] proposed a prey-predator model represented as an Itô stochastic differential equation driven by one Wiener process. The stochastic prey-predator systems can be written as:

$$\begin{cases} dx = [x(1-x) - \alpha \dfrac{xy}{x+y}]dt + \sigma x dW_t \\ dy = \delta y[1 - \beta \dfrac{y}{x}]dt + \sigma y dW_t \end{cases} \tag{4.3}$$

where W_t is a Wiener process. We consider the following parameters $\alpha = 0.05$, $\delta = 0.2$ and $\beta = 0.3$. The initial conditions are $x(0) = 0.1$ and $y(0) = 0.1$.

Euler-Maruyama for the stochastic predator-prey model

We apply the Euler-Maruyama method to the system (4.3) and obtain:

$$\begin{cases} x(i+1) = x(i) + (x(i)(1-x(i)) - \alpha \dfrac{x(i)y(i)}{(x(i)+y(i)})\Delta t + \sigma x(i)\sqrt{\Delta t}N(0,1) \\ y(i+1) = y(i) + (\delta y(i)(1 - \beta \dfrac{y(i)}{x(i)}))\Delta t + \sigma y(i)\sqrt{\Delta t}N(0,1) \end{cases} \tag{4.4}$$

for $i = 0, 1, ..., n$ where $x(0) = 0.9$, $y(0) = 0.7$, $\delta = 0.22$, $\beta = 0.9$, $\alpha = 1.6$ The number of steps in the mesh is $n = 1000$, $T = 80$; $\Delta t = T/n$; The value of the parameter σ is varied from $\sigma = 0$ to $\sigma = 0.3$. When the strength of noise is increased then the fluctuations of the curve are increased

Figure 4.1 represents the phase-portrait of the stochastic model of the predator-prey model corresponding to the strength of noise value $\sigma = 0.07$.

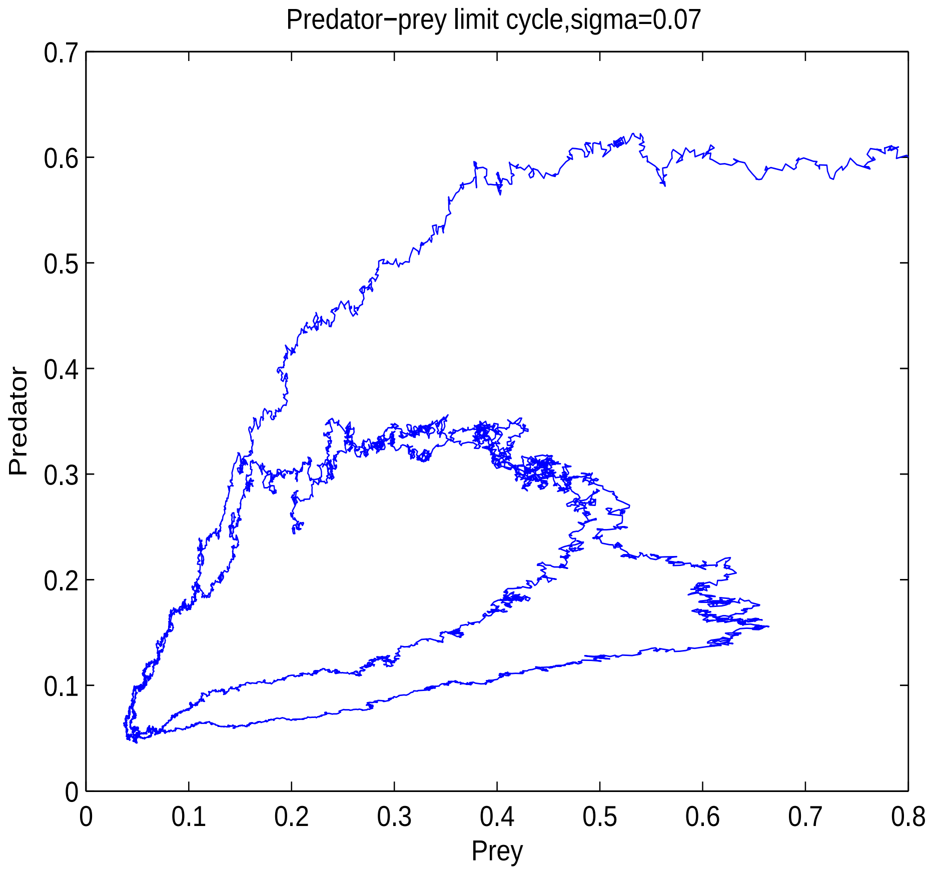

Figure 4.1: The phase-portrait of the predator-prey model with low noise($\sigma = 0.07$).

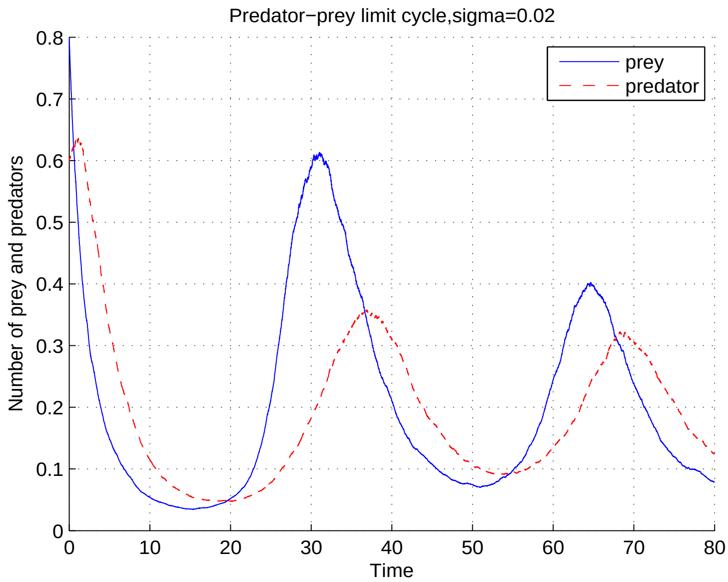

Figure 4.2: The evolution in time of the predator and of the prey with low noise($\sigma = 0.02$).

4.2 Characteristics of noise

We study below the following plots for the prey-predator system:

(i) The evolution in time of the species x and y, respectively.

(ii) The phase portrait of species (x, y).

(iii) Different qualitative behaviour can be studied for a variation of parameters.

Figure 4.1 represents the phase portrait of the prey-predator system. It is helpful to study the qualitative behaviour of the dynamical systems near the equilibria. It does not only predict the extinction of species but it also provides insight on the optimum management of resources existing in nature.

Figure 4.10 is a good example of a sink or an attractor fixed point where an attractor is a set towards which a dynamical system evolves over time. As t

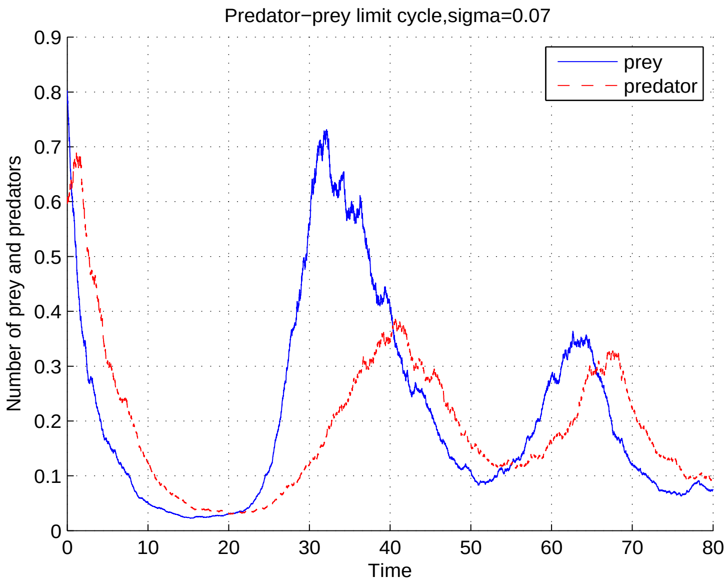

Figure 4.3: The evolution in time of the predator and of the prey with medium noise($\sigma = 0.07$).

increases, the trajectories will spiral around a fixed point and, decreasing will return in an anticlockwise direction.

In a dynamical system , different types of fluctuations can arise depending on the intensity of noise. The Gaussian white noise is employed when continuous random perturbations are present. For our model we observe that (i) When $\sigma = 0.02$, the curve follows closely the deterministic curve, for this small intensity of the noise. (ii) When $\sigma = 0.07$, medium intensity noise, the system exhibits oscillations in the presence of the Gaussian white noise, in a similar manner as for the deterministic system. (iii) When $\sigma = 0.2$, there are more fluctuations than in the case of the medium intensity noise with intensity $\sigma = 0.07$. Sometimes, the noise leads to an extremely "spiky" looking function. (iv) When $\sigma = 1.6$, the oscillations of the system almost decay after this intensity of noise .

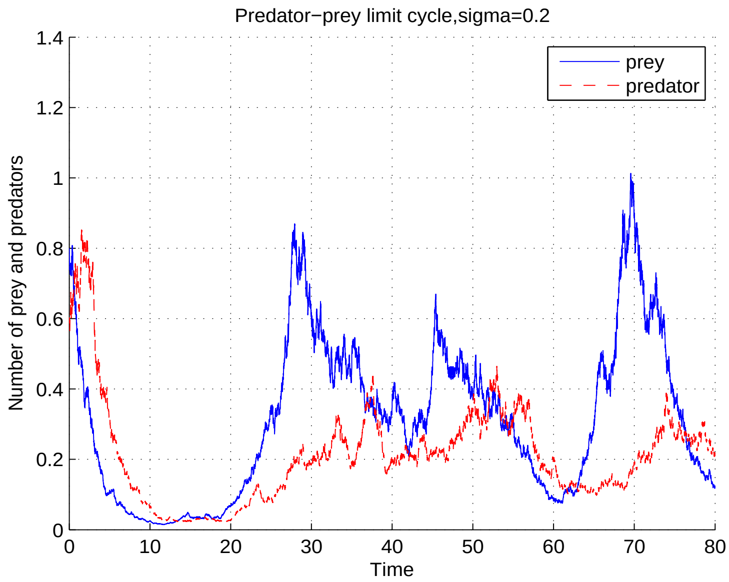

Figure 4.4: The evolution in time of the predator and of the prey with high noise($\sigma = 0.2$).

We observed that, for a range of strengths of noise between $0.02 \leq \sigma \leq 0.2$ the system preserved its oscillatory dynamics .

Figure 4.8 which contains ten trajectories, shows the solution of the prey and of the predator species of the stochastic prey-predator system. These trajectories exhibit a similar pattern with independent fluctuations.

Figure 4.9 represents the mean and the standard deviation over 1000 trajectories for these species. The standard deviation is more consistent than the mean since it reduces fluctuations of the number of observations.

Figure 4.1 indicates the predator-prey limit cycle on the equilibrium point. It provides information about the stable or the unstable limit cycles. If the limit cycle is unstable then the species may become extinct.

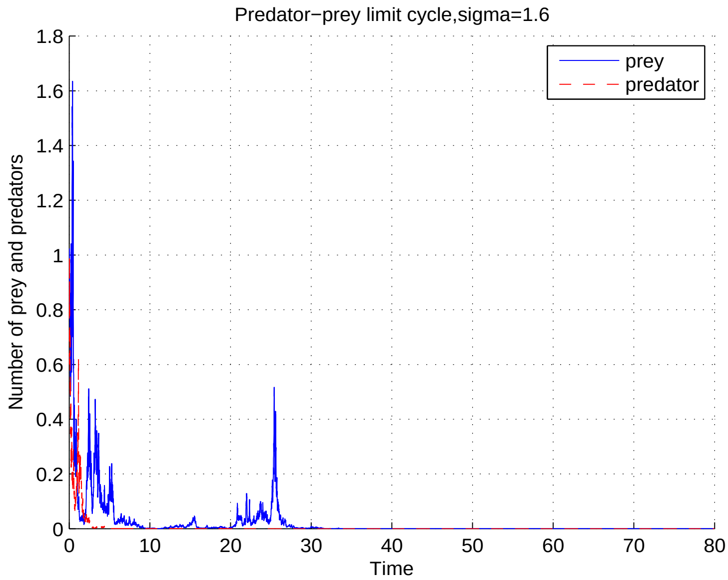

Figure 4.5: The evolution in time of the predator and of the prey with high noise($\sigma = 1.6$).

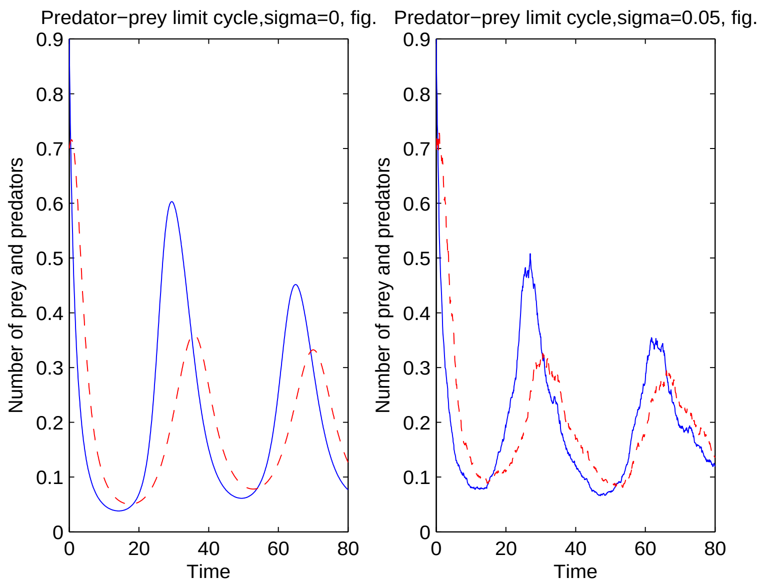

Figure 4.6: The evolution in time of the predator and of the prey with an increasing sequence of noises (left $\sigma = 0$ and right $\sigma = 0.05$).

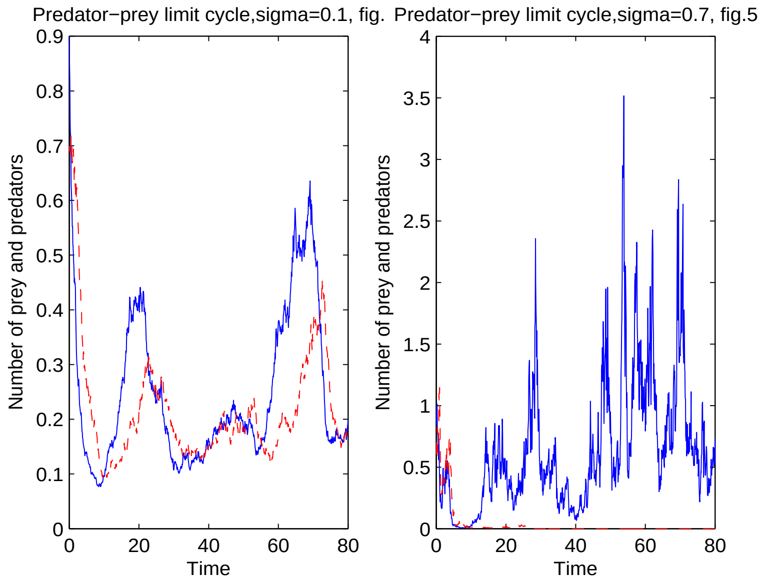

Figure 4.7: The evolution in time of the predator and of the prey with an increasing sequence of noises (left $\sigma = 0.1$ and right $\sigma = 0.7$).

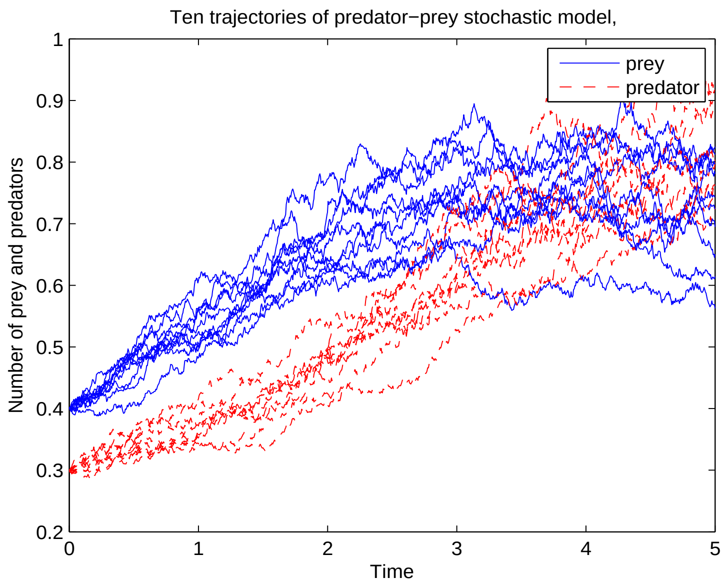

Figure 4.8: Ten trajectories representing the evolution in time of the predator and of the prey for $\sigma = 0.09$.

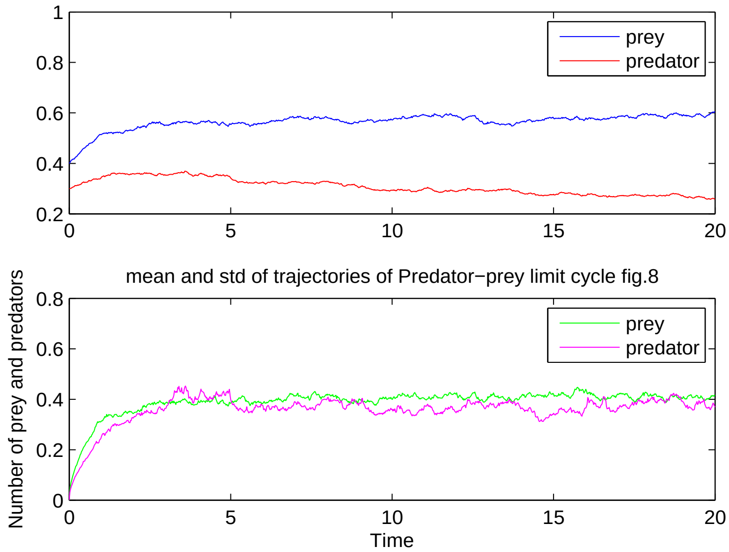

Figure 4.9: The evolution in time of the of mean and of the standard deviation of the predator and of the prey respectively model over 5,000 trajectories.

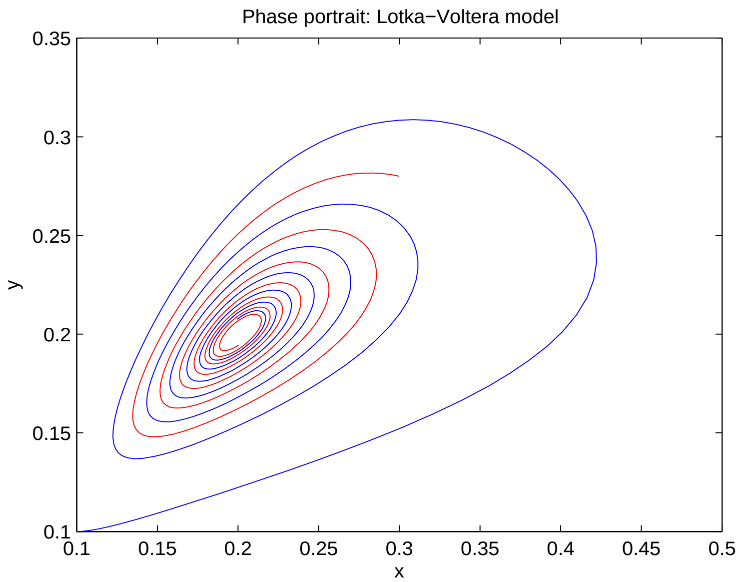

Figure 4.10: The phase portrait of the predator-prey model without noise.

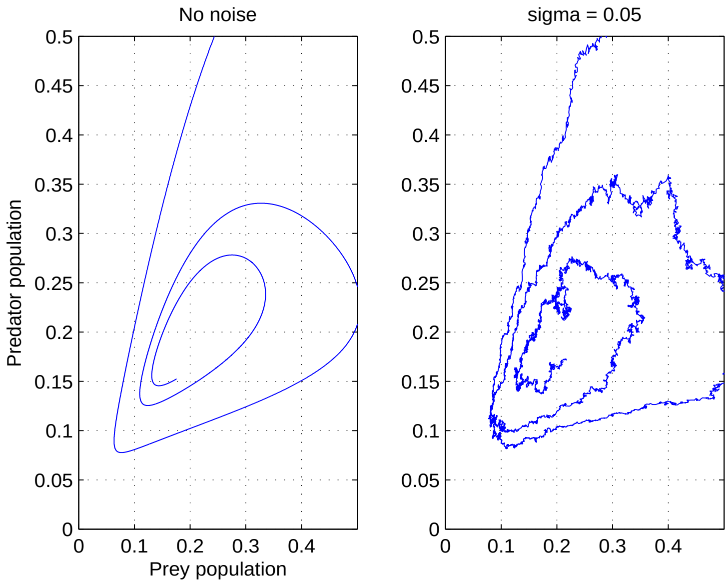

Figure 4.11: Sequence of phase portraits of the stochastic predator-prey model(left $\sigma = 0$, right $\sigma = 0.05$).

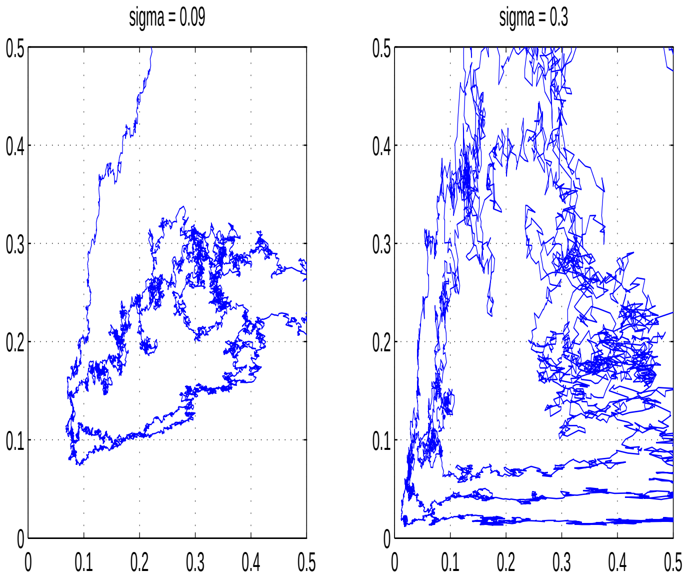

Figure 4.12: Sequence of phase portraits of the stochastic predator-prey model (left $\sigma = 0.09$, right $\sigma = 0.3$).

Chapter 5

Conclusion

5.1 Summary

The main focus of this thesis was to introduce mathematical models of biological systems and techniques for their analysis. The Lotka-Volterra predator-prey equation is a critical model in ecology. The study of the dynamics in the Lotka-Volterra model and its generalisation is a key problem in ecology.

In this thesis, we established some new results by using lemma 1.4.1 such as the existence of stable or unstable equilibrium points under suitable values of parameters in the models. Two species can coexist in the case of stable condition, otherwise they might be extinct in the case of unstable condition.

The Gaussian white noise was employed to observe oscillations in stochastic environment. The prediction of coexistence of the populations of two species can be made with the help of observation of fluctuations in the stochastic model.

5.2 Limitation of research

The region of our new results lies on $(\beta - \alpha)$ plane below the line $\alpha = 1 + \beta$ and above the $\beta - axis$, where Ω_1 and Ω_2 are exact regions of new results. We also restrict the parameters $\alpha > 0$, $\beta > 0$, $\delta > 0$ and $1 + \beta > \alpha$. Therefore, our new results do not validate out of these regions Ω_1 and Ω_2.

The Holling-Tanner model is a nonlinear differential equation, so we analyse the local stability near the equilibrium after linearisation. Furthermore, we can analyse the global stability after developing new techniques.

5.3 Future research

Our thesis is useful for future research work regarding the predator-prey model. We focused on establishing results on certain regions of parameter values. In the future, we plan to get some results outside these regions.

We need further some techniques and research to analyse different types of equilibria on the outside of Ω_1 and Ω_2. For example, the Holling-Tanner model can be modified in the form of Leslie-Gower predator-prey systems with harvesting rates. In addition, we can establish the global stability near the equilibrium according to (Jordan and Smith, 1999) a Lyapunov function and Poincaré-Bendixon theorem.

Appendix

Matlab code used for the project:

Appendix one

(1) This MATLAB code is used to draw the figure of phase portrait in Brownian motion.

```
clear all; close all;
N=10000;              % number of steps to take
  T=80;                 % maximum time
  h=T/N;                % time step
  t=(0:h:T);            % t is the vector [0 1h 2h 3h ... Nh]
  %axis([0 2 0 1]);     % set axis limits
  x=zeros(size(t));     % place to store predator population size
  y=zeros(size(t));     % place to store prey population size
  tt=zeros(size(t));
  x(1)=0.8;             % initial prey population
  y(1)=0.6;             % initial predator population
  delta=0.22; beta=0.9; alpha=1.6;
  %sigma=0.02;          %low strength of noise
  sigma=0.07;           %medium strength of noise with oscillations
  %high strenth of noise
  %sigma= 1.5;         %highest strength of noise
  for i=1:N            %start taking steps
    x(i+1)=x(i)+(x(i)*(1-x(i))-alpha*x(i)*y(i)/(x(i)+y(i)))*h
```

```
  + sigma*x(i)*randn*sqrt(h);
  y(i+1)=y(i)+(delta*y(i)*(1-beta*y(i)/x(i)))*h
  + sigma*y(i)*randn*sqrt(h);
  tt(i+1) =tt(i) +h;
  end;
hlegl=legend('prey','predator');
plot(x,y) %phase portrait
xlabel('Prey');
ylabel('Predator');
title('Predator-prey limit cycle,sigma=0.07 ');
```

Appendix two

(2) In these figures, oscillations appear from $\sigma = 0.02$ to $\sigma = 0.2$ and ceases oscillations after $\sigma = 1.6$

```
N=10000;               % number of steps to take
T=80;                  % maximum time
h=T/N;                 % time step
t=(0:h:T);             % t is the vector [0 1h 2h 3h ... Nh]
%axis([0 2 0 1]);      % set axis limits

x=zeros(size(t));      % place to store predator population size
y=zeros(size(t));      % place to store prey population size
tt=zeros(size(t));

x(1)=0.8;              % initial prey population
```

```
y(1)=0.6;            % initial predator population
delta=0.22; beta=0.9; alpha=1.6;
%sigma=0.02;         %low strength of noise
sigma=0.07;          %medium strength of noise with oscillations
%sigma=0.2;          %higer medium strenth of noise
%sigma=1.6;          %high strenth of noise

for i=1:N            % start taking steps
  x(i+1)=x(i)+(x(i)*(1-x(i))-alpha*x(i)*y(i)/(x(i)+y(i)))*h
  + sigma*x(i)*randn*sqrt(h);
  y(i+1)=y(i)+(delta*y(i)*(1-beta*y(i)/x(i)))*h
  + sigma*y(i)*randn*sqrt(h);
  tt(i+1) =tt(i) +h;
end;
plot(tt,x, '-b', tt, y,'--r');  % plot of evolution of x (red) and y(blue)
hlegl=legend('prey','predator');
xlabel('Prey');
ylabel('Predator');
xlabel('Time');
ylabel('Number of prey and predators');
title('Predator-prey limit cycle,sigma=0.07 ');
```

Appendix three

(3) The MATLAB code indicate that oscillations change according to strength of noise.

```
subplot(2,2,1);
N=1000;              % number of steps to take
T=80;                % maximum time
h=T/N;               % time step
t=(0:h:T);           % t is the vector [0 1h 2h 3h ... Nh]

x=zeros(size(t));    % place to store pred population size
y=zeros(size(t));    % place to store prey population size
tt=zeros(size(t));
x(1)=0.9;            % initial prey population
y(1)=0.7;            % initial pred population
delta=0.22; beta=0.9; alpha=1.6;
sigma=0;              % strength of noise

for i=1:N             % start taking steps
  x(i+1)=x(i)+(x(i)*(1-x(i))-alpha*x(i)*y(i)/(x(i)+y(i)))*h
  + sigma*x(i)*randn*sqrt(h);
  y(i+1)=y(i)+(delta*y(i)*(1-beta*y(i)/x(i)))*h
  + sigma*y(i)*randn*sqrt(h);
  tt(i+1) =tt(i) +h;
end;
plot(tt,x, '-b', tt, y,'--r');
```

```
xlabel('Time');
ylabel('Number of prey and predators');
title('Predator-prey limit cycle,sigma=0, fig.');
subplot(2,2,2);
N=1000;                 % number of steps to take
T=80;                   % maximum time
h=T/N;                  % time step
t=(0:h:T);              % t is the vector [0 1h 2h 3h ... Nh]
x=zeros(size(t));       % place to store pred population size
y=zeros(size(t));       % place to store prey population size
tt=zeros(size(t));
x(1)=0.9;               % initial prey population
y(1)=0.7;               % initial pred population
delta=0.22; beta=0.99; alpha=1.6;
sigma=0.05;             % strength of noise

for i=1:N               % start taking steps
  x(i+1)=x(i)+(x(i)*(1-x(i))-alpha*x(i)*y(i)/(x(i)+y(i)))*h
  + sigma*x(i)*randn*sqrt(h);
  y(i+1)=y(i)+(delta*y(i)*(1-beta*y(i)/x(i)))*h
  + sigma*y(i)*randn*sqrt(h);
  tt(i+1)=tt(i)+h;
end
plot(tt,x,'b-',tt,y,'r--');
xlabel('Time');
```

```
 ylabel('Number of prey and predators');
 title('Predator-prey limit cycle,sigma=0.05, fig.');
subplot(2,2,3);
 N=1000;               % number of steps to take
 T=80;                 % maximum time
 h=T/N;                % time step
 t=(0:h:T);            % t is the vector [0 1h 2h 3h ... Nh]

 x=zeros(size(t));     % place to store pred population size
 y=zeros(size(t));     % place to store prey population size
 tt=zeros(size(t));
 x(1)=0.9;             % initial prey population
 y(1)=0.7;             % initial pred population
 delta=0.22; beta=0.99; alpha=1.6;
 sigma=0.09;           % strength of noise

 for i=1:N             % start taking steps
   x(i+1)=x(i)+(x(i)*(1-x(i))-alpha*x(i)*y(i)/(x(i)+y(i)))*h
   + sigma*x(i)*randn*sqrt(h);
   y(i+1)=y(i)+(delta*y(i)*(1-beta*y(i)/x(i)))*h
   + sigma*y(i)*randn*sqrt(h);
   tt(i+1)=tt(i)+h;
 end

 plot(tt,x,'b-',tt,y,'r--');
```

```
xlabel('Time');
ylabel('Number of prey and predators');
title('Predator-prey limit cycle,sigma=0.09, fig.');
subplot(2,2,4);
N=1000;               % number of steps to take
T=80;                 % maximum time
h=T/N;                % time step
t=(0:h:T);            % t is the vector [0 1h 2h 3h ... Nh]

x=zeros(size(t));     % place to store pred population size
y=zeros(size(t));     % place to store prey population size
tt=zeros(size(t));
x(1)=0.9;             % initial prey population
y(1)=0.7;             % initial pred population
delta=0.22; beta=0.99; alpha=1.6;
sigma=0.3;            % strength of noise

for i=1:N             % start taking steps
  x(i+1)=x(i)+(x(i)*(1-x(i))-alpha*x(i)*y(i)/(x(i)+y(i)))*h
  + sigma*x(i)*randn*sqrt(h);
  y(i+1)=y(i)+(delta*y(i)*(1-beta*y(i)/x(i)))*h
  + sigma*y(i)*randn*sqrt(h);
  tt(i+1)=tt(i)+h;
end;
plot(tt,x,'b-',tt,y,'r--');
```

```
  xlabel('Time');
  ylabel('Number of prey and predators');
  title('Predator-prey limit cycle,sigma=0.3, fig.5');
```

Appendix four

(4) This MATLAB code represents different trajectories of predator-prey limit cycle.

```
clear all;close all;
  N=1000;              % number of steps to take
  T=5;                 % maximum time
  h=T/N;               % time step
  tt=(0:h:T);          % t is the vector [0 1h 2h 3h ... Nh]
  axis([0 2 0 1]);     % set axis limits
  M=5 ; %number of different trajectories
  x =zeros(size(tt));
  y =zeros(size(tt));
  x0 = 0.4; y0 = 0.3; tt(1)=0;
  for j=1:M
      x(j,1)= x0;        % initial prey population
      y(j,1) = y0;       % initial pred population
  end;
  delta=0.4; beta=0.6; alpha=0.5;
  sigma= 0.09;          % strength of noise
  meanx=mean(x);
  meany=mean(y);
```

```
  stdx=std(x);
  stdy=std(y);
  for j=1:M               %index of trajectory
  for i=1:N               %index of time (start taking steps)
     x(j,i+1)=x(j,i)+(x(j,i)*(1-x(j,i))
     -alpha*x(j,i)*y(j,i)/(x(j,i)+y(j,i)))*h
     + sigma*x(j,i)*randn*sqrt(h);
     y(j,i+1)=y(j,i)+delta*(y(j,i)*(1-beta*y(j,i)/x(j,i)))*h
      + sigma*y(j,i)*randn*sqrt(h);
     tt(i+1) =tt(i) +h;
    %set(drawprey,'xdata',[x(i),x(i+1)],'ydata',[y(i),y(i+1)]);
    %drawnow;
  end;
  end;
  for i=1:(N+1)
      meanx(i)=mean(x(:, i));
      meany(i)=mean(y(:, i));
      stdx(i)=std(x(:, i));
      stdy(i)=std(y(:, i));
  end
for j=1:M
  plot(tt,x(j,:), '-b', tt, y(j,:),'--r');  % plot of evolution of x
  hlegl=legend('prey','predator');
  hold on
end
```

```
  xlabel('Time');
  ylabel('Number of prey and predators');
  title('Three trajectories of predator-prey stochastic model');
```

Appendix five

(5) This MATLAB code is for mean and standard deviation of trajectories of Predator-prey limit cycle.

```
clear all;close all;
  N=1000;              % number of steps to take
  T=20;                % maximum time
  h=T/N;               % time step
  tt=(0:h:T);          % t is the vector [0 1h 2h 3h ... Nh]
  M=1000 ; %number of different trajactories
  x=zeros(size(tt));
  y=zeros(size(tt));
  tt=zeros(size(tt));
  x0 = 0.4;
  y0 = 0.3;
  tt(1)=0;
  for j=1:M
      x(j,1)= x0;           % initial prey population
      y(j,1) = y0;          % initial pred population
  end;
  delta=0.4; beta=0.6; alpha=0.5;
  %sigma= 0.1;   % strength of noise
```

```
sigma=0.7;
meanx=mean(x);
meany=mean(y);
stdx=std(x);
stdy=std(y);
for j=1:M          %index of trajectory
for i=1:N          %index of time (start taking steps)
x(j,i+1)=x(j,i)+(x(j,i)*(1-x(j,i))
-alpha*x(j,i)*y(j,i)/(x(j,i)+y(j,i)))*h
+ sigma*x(j,i)*randn*sqrt(h);
y(j,i+1)=y(j,i)+delta*(y(j,i)*(1-beta*y(j,i)/x(j,i)))*h
+ sigma*y(j,i)*randn*sqrt(h);
tt(i+1)=tt(i)+h;
end;
end;
for i=1:(N+1)
    meanx(i)=mean(x(:, i));
    meany(i)=mean(y(:, i));
    stdx(i)=std(x(:, i));
    stdy(i)=std(y(:, i));
end
 subplot(2,1,1),plot(tt, meanx, 'b', tt, meany, 'r')
 hlegl=legend('prey','predator');
 subplot(2,1,2),plot(tt, stdx, 'g', tt, stdy, 'm')
 hlegl=legend('prey','predator');
```

```
  xlabel('Time');
  ylabel('Number of prey and predators');
  title('mean and std of trajectories of Predator-prey limit cycle fig.8');
```

Appendix six

(6) This MATLAB code is for the phase portrait: Lotka-Volterra model.

```
function yderivative = Project(t,y)
 % the right hand side of the system of differential equations which model
 % the Holling-Tanner system
 delta = 0.2;
 beta = 0.3 ;
 alpha = 0.05;
 yderivative = [y(1).*(1-y(1))-y(1).*y(2)./(alpha+y(1));
  y(2).*(delta-beta*y(2)./y(1))];
```

Appendix seven

```
clear all; close all;
opts=odeset('RelTol',10^(-8));
y0=[0.1;0.1];
[t,y]=ode45(@Project,[0,200],y0,opts);
z0=[0.3;0.28];
[tau,z]=ode45(@Project,[0,200],z0,opts);
plot (y(:,1),y(:,2),'b',z(:,1),z(:,2),'r')
xlabel('x')
```

```
ylabel('y')
title('phase-space: Holling-Tanner model')
```

Appendix eight

(8) This is the mat lab code for phase portrait with different strength of noise.

```
subplot(2,2,1);
  N=10000;              % number of steps to take
  T=80;                 % maximum time
  h=T/N;                % time step
  t=(0:h:T);            % t is the vector [0 1h 2h 3h ... Nh]
  x=zeros(size(t));     % place to store pred population size
  y=zeros(size(t));     % place to store prey population size
  %tt=zeros(size(t));
  x(1)=0.9;             % initial prey population
  y(1)=0.7;             % initial pred population
  delta=0.22; beta=0.99; alpha=1.6;
  sigma=0;             % strength of noise
  for i=1:N            % start taking steps
    x(i+1)=x(i)+(x(i)*(1-x(i))-alpha*x(i)*y(i)/(x(i)+y(i)))*h
    + sigma*x(i)*randn*sqrt(h);
    y(i+1)=y(i)+(delta*y(i)*(1-beta*y(i)/x(i)))*h
    + sigma*y(i)*randn*sqrt(h);
    %tt(i+1) =tt(i) +h;
  end;
  plot(x,y);
```

```
  axis([0 1 0 1]);     % set axis limits
  grid on;
  title('No noise');
  xlabel('Prey population');
  ylabel('Predator population');
subplot(2,2,2);
  N=10000;             % number of steps to take
  T=80;                % maximum time
  h=T/N;               % time step
  t=(0:h:T);           % t is the vector [0 1h 2h 3h ... Nh]

  x=zeros(size(t));    % place to store pred population size
  y=zeros(size(t));    % place to store prey population size
  x(1)=0.9;            % initial prey population
  y(1)=0.7;            % initial pred population
  delta=0.22; beta=0.99; alpha=1.6;
  sigma=0.05;          % strength of noise
  for i=1:N            % start taking steps
    x(i+1)=x(i)+(x(i)*(1-x(i))-alpha*x(i)*y(i)/(x(i)+y(i)))*h
    + sigma*x(i)*randn*sqrt(h);
    y(i+1)=y(i)+(delta*y(i)*(1-beta*y(i)/x(i)))*h
     + sigma*y(i)*randn*sqrt(h);
    %tt(i+1)=tt(i)+h;
  end
  plot(x,y);
```

```
axis([0 1 0 1]);     % set axis limits
grid on;
title('sigma = 0.05');
subplot(2,2,3);
N=10000;             % number of steps to take
T=80;                % maximum time
h=T/N;               % time step
t=(0:h:T);           % t is the vector [0 1h 2h 3h ... Nh]

x=zeros(size(t));    % place to store pred population size
y=zeros(size(t));    % place to store prey population size
%tt=zeros(size(t));
x(1)=0.9;            % initial prey population
y(1)=0.7;            % initial pred population
delta=0.22; beta=0.99; alpha=1.6;
sigma=0.09;          % strength of noise
for i=1:N            % start taking steps
  x(i+1)=x(i)+(x(i)*(1-x(i))-alpha*x(i)*y(i)/(x(i)+y(i)))*h
   + sigma*x(i)*randn*sqrt(h);
  y(i+1)=y(i)+(delta*y(i)*(1-beta*y(i)/x(i)))*h
  + sigma*y(i)*randn*sqrt(h);
  %tt(i+1)=tt(i)+h;
end
plot(x,y);
axis([0 1 0 1]);     % set axis limits
```

```
grid on;
title('sigma = 0.09');
subplot(2,2,4);
N=10000;             % number of steps to take
T=80;                % maximum time
h=T/N;               % time step
t=(0:h:T);           % t is the vector [0 1h 2h 3h ... Nh]
x=zeros(size(t));    % place to store pred population size
y=zeros(size(t));    % place to store prey population size
x(1)=0.9;            % initial prey population
y(1)=0.7;            % initial pred population
delta=0.22;  beta=0.99; alpha=1.6;
sigma=0.3;           % strength of noise
for i=1:N            % start taking steps
  x(i+1)=x(i)+(x(i)*(1-x(i))-alpha*x(i)*y(i)/(x(i)+y(i)))*h
  + sigma*x(i)*randn*sqrt(h);
  y(i+1)=y(i)+(delta*y(i)*(1-beta*y(i)/x(i)))*h
  + sigma*y(i)*randn*sqrt(h);
end;
plot(x,y);
axis([0 1 0 1]);     % set axis limits
grid on; title('sigma = 0.3');
```

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