# Comparison Between Heidler Function And The Pulse Function For Modeling The Lightning Return-Stroke Current 

Khaled Elrodesly<br>Ryerson University

Follow this and additional works at: http:// digitalcommons.ryerson.ca/dissertations
Part of the Electrical and Computer Engineering Commons

## Recommended Citation

Elrodesly, Khaled, "Comparison Between Heidler Function And The Pulse Function For Modeling The Lightning Return-Stroke Current" (2010). Theses and dissertations. Paper 1323.

# Comparison between Heidler Function and the Pulse Function for Modeling the Lightning Return-Stroke Current 

## By

## Khaled Elrodesly

Bachelor of Science Electronics and Electrical Communications Engineering Ain Shams University, Cairo, Egypt, 2008

A thesis<br>presented to Ryerson University<br>in partial fulfillment of the requirements for the degree of<br>Master of Applied Science in the Program of Electrical and Computer Engineering

Toronto, Ontario, Canada, 2010
©Khaled Elrodesly, 2010

## AUTHOR DECLARATION

I hereby declare that I am the sole author of this thesis or dissertation.
I authorize Ryerson University to lend this thesis or dissertation to other institutions or individuals for the purpose of scholarly research.

Khaled Elrodesly

I further authorize Ryerson University to reproduce this thesis or dissertation by photocopying or by other means, in total or in part, at the request of other institutions or individuals for the purpose of scholarly research.

Khaled Elrodesly

ABSTRACT<br>Comparison between Heidler Function and the Pulse Function for Modeling Lightning Return-Stroke Current<br>Master of Applied Science, 2010<br>Khaled Elrodesly<br>Electrical and Computer Engineering<br>Ryerson University

In the past, many functions were considered for simulating the lightning returnstroke current. Some of these functions were found to have problems related to their discontinuities or the discontinuities of their derivatives at onset time. Such problems appear in the double exponential function and its modifications. However, other functions like the Pulse function and Heidler function do not suffer from such problems.

One of the main objectives of this work is to simulate the lightning return-stroke current full wave, including the decay part, using either Heidler function or the Pulse function. This work is not only necessary for the evaluation and development of lightning return-stroke modeling, but also for the calculation of the lightning current waveform parameters.

Although the lightning return-stroke current, measured at the CN Tower, is simulated using the Pulse function and Heidler function, the simulation of the CN Tower lightning current derivative signal is considered using the derivative of the Pulse and Heidler functions.

First, we build a modeling environment for each function, which can be described as parameter estimation system. This system, which represents an automated approach for estimating the analytical parameters of a given function, is capable to best fit the function with the measured data. Using these analytical parameters transforms the discrete data into a continuous signal, from which the current waveform parameters can be estimated.

This analytical parameters estimation system is recognized as a curve fitting system. For curve fitting technique, the initial value of each analytical parameter and its feasible region, where the optimal value of this analytical parameter is located, must be specified. The more accurate the initial point is the easier and faster the optimal value can be estimated. On choosing the best approach of the initial condition, which gives the nearest location to the optimal point, applying the estimation system and achieving the analytical model that fits the CN Tower measured current derivative, the current waveform parameters can be easily studied.

In order to be sure that the analytical parameter extraction system gives the best fit of a function, it needs to be evaluated. Instead of going through the measured data, we first use artificial digital data as a productive way to evaluate the system. Also, a comparison between both the Pulse and Heidler functions is performed.

The described fitting process is applied on 15 flashes, containing 31 return strokes. The calculated current waveform parameters were used to form statistics to determine the probability distribution of the value of each parameter, including the range and the $50 \%$ probability level, which is fundamental in building lightning protection systems.

## ACKNOWLEDGEMENTS

I feel deeply indebted to my supervisor, Professor Dr. Ali Hussein, for his patience, guidance, constant supervision, creative advising and personal involvement throughout the progression of this work. He has been, for me, not only a source of inspiration and encouragement but also a model of ideal academic relations. Much of what I have learned during the progress of this thesis I owe, in fact, to Professor Hussein. Working under the supervision of a very great investigator of the highest academic and human caliber like Professor Hussein has formed me and will continue to be an unforgettable and a unique experience of my life.

I would like to express my great appreciation and gratitude to Dr. Kaamran Raahemifar for his guidance, advice and moral support. My thanks go to all my Professors at Ryerson University who taught me during my graduate study at the University.

I am thankful to the Department of Electrical and Computer Engineering, Ryerson University for offering all facilities, financial resources and the academic environments that made this work possible

I would like to express my deepest appreciation to my parents and brother for their moral support, patience and unconditional love.

## TABLE OF CONTENTS

1. CHAPTER 1: INTRODUCTION ..... 1
2. CHAPTER 2: LIGHTNING PHYSICS ..... 5
2.1 ELECTRIFICATION OF THUNDERSTORMS ..... 7
2.1.1 Precipitation Thesis .....  8
2.1.2 Convection Thesis ..... 9
2.1.3 Tri-pole Modification ..... 9
2.2 LIGHTNING CATEGORIES ..... 11
2.2.1 Cloud-to-cloud ..... 11
2.2.2 Cloud to earth ..... 11
2.3 LIGHTNING FORMATION ..... 13
2.4 LIGHTNING TO TALL STRUCTURES ..... 14
2.5 CN TOWER AND ITS MEASUREMENT SYSTEM ..... 16
2.6 CURRENT WAVEFORM PARAMETERS ..... 21
3. CHAPTER 3: SIMULATING FUNCTIONS ..... 23
3.1 DOUBLE EXPONENTIAL FUNCTION ..... 24
3.2 JONES MODIFICATION ..... 27
3.3 HEIDLER FUNCTION ..... 32
3.4 PULSE FUNCTION ..... 44
4. CHAPTER 4: CURRENT FITTING TECHNIQUE ..... 51
4.1 CURVE FITTING ..... 51
4.1.1 Least Square Regression ..... 52
4.1.2 Non Linear Regression ..... 54
4.1.3 Goodness of the Fit ..... 57
4.2 MODELING ENVIRONMENT ..... 58
4.2.1 Artificial Signal Concept ..... 58
4.2.2 Modeling Using Artificial Signal ..... 60
4.3 INITIAL CONDITION ..... 68
4.3.1 Initial Condition for the Pulse Function ..... 68
4.3.2 Initial Condition for Heidler Function ..... 75
5. CHAPTER 5:RESULTS AMD DISCUSSIONS ..... 82
5.1 ESTIMATING $\tau_{2}$ ..... 86
5.2 ESTIMATING $\mathrm{I}_{\mathrm{MAX}}$ ..... 91
5.3 APPLYING CONSTRAINTS ..... 92
5.3.1 Condition of Current Derivative Zero Crossing ..... 93
5.3.2 Time of Maximum Steepness Condition ..... 96
5.4 CURRENT WAVEFORM PARAMETERS ..... 99
5.5 Cumulative Distripution ..... 100
5.5.1 Current Peak ..... 100
5.5.2 Maximum Steepness ..... 101
5.5.3 Risetime to the Current Peak ..... 102
5.5.4 Decay time from the Current Peak ..... 103
5.5.5 Current Pulse Width ..... 104
5.5.6 Charge ..... 105
6. CHAPTER 6: CONCLUSIONS AND RECOMMENDATIONS ..... 107
REFERENCES ..... 109

## List of Figures

Figure 2.1: Earth-Atmosphere charge ..... 6
Figure 2.2: World map showing frequency of Lightning ..... 7
Figure 2.3: Dipole Models ..... 8
Figure 2.4: Precipitation Thesis ..... 8
Figure 2.5: Convection Thesis ..... 9
Figure 2.6: Charge Reversal Temperature ..... 10
Figure 2.7: Different types of lightning ..... 11
Figure 2.8: Lighting Formation Process ..... 14
Figure 2.9: Branching of upward-initiated lightning ..... 15
Figure 2.10: CN Tower and its surrounding ..... 17
Figure 2.11: CN Tower and the lightning measurement location ..... 18
Figure 2.12: Schematic of old Rogowski coil connection ..... 19
Figure 2.13: Schematic of new Rogowski coil connection ..... 20
Figure 2.14: Demonstration of current maximum steepness ..... 22
Figure 2.15 Demonstration of current peak and risetime of current ..... 22
Figure 3.1: Current simulation using double exponential function with $\mathrm{I}=10 \mathrm{kA}, \tau_{1}=5$
$\mu \mathrm{s}$ and $\tau_{2}=0.5 \mu \mathrm{~s}$ ..... 24
Figure 3.2: Current Derivative simulation using double exponential derivative function
with $\mathrm{I}=10 \mathrm{kA}, \tau_{1}=5 \mu \mathrm{~s}$ and $\tau_{2}=0.5 \mu \mathrm{~s}$ ..... 26
Figure 3.3: Current simulation using Jones modification for double exponential functionwith I $=10 \mathrm{kA}, \tau_{1}=5 \mu \mathrm{~s}$ and $\tau_{2}=0.5 \mu \mathrm{~s}$.......................................................................... 28
Figure 3.4: Current Derivative simulation using double exponential derivative function
with $\mathrm{I}=10 \mathrm{kA}, \tau_{1}=5 \mu \mathrm{~s}$ and $\tau_{2}=0.5 \mu \mathrm{~s}$ ..... 30

Figure 3.5: Current simulation using double exponential and Jone modification for double exponential derivative function with $\mathrm{I}=10 \mathrm{kA}, \tau_{1}=5 \mu \mathrm{~s}$ and $\tau_{2}=0.5 \mu \mathrm{~s}$

Figure 3.6: Current Derivative simulation using double exponential and Jone modification for double exponential derivative function with $\mathrm{I}=10 \mathrm{kA}, \tau_{1}=5 \mu \mathrm{~s}$ and $\tau_{2}=$ $0.5 \mu \mathrm{~s}$ .31

Figure 3.7: Current simulation using Heidler function with $\mathrm{I}=10 \mathrm{kA}, \tau_{1}=0.3 \mu \mathrm{~s}, \tau 2=3$ us and $n=7$ .37

Figure 3.8: Current derivative simulation using Heidler derivative function with $\mathrm{I}=10$ $\mathrm{kA}, \tau_{1}=0.3 \mu \mathrm{~s}, \tau_{2}=3 \mu \mathrm{~s}$ and $\mathrm{n}=7$ .38

Figure 3.9: Current simulation using Pulse function with $\mathrm{I}=10 \mathrm{kA}, \tau_{1}=0.5 \mu \mathrm{~s}, \tau_{2}=10$ $\mu \mathrm{s}$ and $\mathrm{n}=5$ .45

Figure 3.10: Current derivative simulation using Pulse function derivative with $\mathrm{I}=10 \mathrm{kA}$, $\tau_{1}=0.5 \mu \mathrm{~s}, \tau_{2}=10 \mu \mathrm{~s}$ and $\mathrm{n}=5$ .47

Figure 4.1: Example of the criteria for "best fit" based on minimizing the sum of the residual .52

Figure 4.2: Example of the criteria for "best fit" based on minimizing the sum of the absolute value of the residual .53

Figure 4.3: Example of the criteria for "best fit" based on minimizing the maximum error of any individual point .54

Figure 4.4: Artificial signal production simulation for current and current derivative with rate 10 ns .60

Figure 4.5: Curve fitting results using Pulse function artificial signal with $\mathrm{I}=10 \mathrm{kA}, \tau_{1}=$


Figure 4.6: Curve fitting results using Heidler function artificial signal with $\mathrm{I}=10 \mathrm{kA}, \tau_{1}$ $=0.1 \mu \mathrm{~s}, \tau_{2}=2 \mu \mathrm{~s}$ and $\mathrm{n}=5$

Figure 4.7: Current fitting results using Pulse current function and the corresponding current derivative simulation with Heidler artificial signal of parameters $\mathrm{I}=10 \mathrm{kA}, \tau_{1}=$ $0.1 \mu \mathrm{~s}, \tau_{2}=2 \mu \mathrm{~s}$ and $\mathrm{n}=5$

Figure 4.8: Current derivative fitting results using Pulse current derivative function and the corresponding current simulation with Heidler artificial signal of parameters $I=10$ $\mathrm{kA}, \tau_{1}=0.1 \mu \mathrm{~s}, \tau_{2}=2 \mu \mathrm{~s}$ and $\mathrm{n}=5$

Figure 4.9: Current derivative fitting results using Pulse current derivative function and the corresponding current simulation with Heidler artificial signal of parameters $\mathrm{I}=10$ $\mathrm{kA}, \tau_{1}=0.1 \mu \mathrm{~s}, \tau_{2}=2 \mu \mathrm{~s}$ and $\mathrm{n}=5$

Figure 4.10: Current derivative fitting results using Heidler current derivative function and the corresponding current simulation with Pulse artificial signal of parameters $I=10$ $\mathrm{kA}, \tau_{1}=0.07 \mu \mathrm{~s}, \tau_{2}=1 \mu \mathrm{~s}$ and $\mathrm{n}=6$ .66

Figure 4.11: Current fitting results using Heidler current function and the corresponding current simulation with Pulse artificial signal of parameters $\mathrm{I}=10 \mathrm{kA}, \tau_{1}=0.07 \mu \mathrm{~s}, \tau_{2}=1$ $\mu \mathrm{s}$ and $\mathrm{n}=6$

Figure 4.12: Relative error value for the estimated values of $\tau_{1}, \tau_{2}$ and $n$ with the change of its value individually using the Pulse function initial condition estimation .72

Figure 4.13: Relative error value for the estimated values of $n$ with the change of $\tau_{1}$ and $\tau_{2}$ using the Pulse function initial condition estimation .73

Figure 4.14: Relative error value for the estimated values of $\tau_{2}$ with the change of $\tau_{1}$ and n using the Pulse function initial condition estimation .73

Figure 4.15: Relative error value for the estimated values of $\tau_{1}$ with the change of $\tau_{2}$ and n using the Pulse function initial condition estimation .74

Figure 4.16: Relative error value for the estimated values of $\tau_{1}, \tau_{2}$ and $n$ with the change of its value individually using Heidler function initial condition estimation

Figure 4.17: Relative error value for the estimated values of $n$ with the change of $\tau_{1}$ and $\tau_{2}$ using Heidler function initial condition estimation

Figure 4.18: Relative error value for the estimated values of $\tau_{2}$ with the change of $\tau_{1}$ and n using Heidler function initial condition estimation .80

Figure 4.19: Relative error value for the estimated values of $\tau_{1}$ with the change of $\tau_{2}$ and n using Heidler function initial condition estimation

Figure 5.1: First impulse of the measured signal .............................................................. 83
Figure 5.2: Fitting result of the Pulse and Heidler functions for the measured current derivative signal and the corresponding current signal ..................................................... 84

Figure 5.3: Fitting result of the Pulse and Heidler functions for the measured current derivative signal and the corresponding current signal ..................................................... 85

Figure 5.4: Fitting the rising portion of the Pulse and Heidler functions ......................... 86
Figure 5.5: Classifications of the measured current signal parts ...................................... 87
Figure 5.6: Fitting result of the Pulse and Heidler functions on the integration of the measured signal decay part (current decay part)............................................................... 88

Figure 5.7: Fitting result of the Pulse and Heidler functions for the measured current derivative signal with fixed $\tau_{2}$ and the corresponding current ........................................... 89

Figure 5.8: Fitting result of the Pulse and Heidler functions for the measured current derivative signal with fixed $\tau_{2}$ and the corresponding current .90

Figure 5.9:Fitting result of the Pulse and Heidler functions for the measured current derivative signal with fixed $\tau_{2}$ and $\mathrm{I}_{\max }$ and the current.................................................... 92

Figure 5.10: Fitting result of the Pulse and Heidler functions for the measured current derivative signal with fixed $\tau_{2}, \mathrm{I}_{\text {max }}$ and with forcing time constraints of the zero crossing of the current derivative signal and the corresponding current ......................................... 95

Figure 5.11: Fitting result of the Pulse and Heidler functions for the measured current derivative signal with fixed $\tau_{2}, \mathrm{I}_{\max }$ and with forcing time constraints of the maximum steepness of current signal and the corresponding current .............................................. 98

Figure 5.12: Cumulative distribution of the current peak................................................ 100
Figure 5.13: Cumulative distribution of the maximum steepness .................................. 101
Figure 5.14: Cumulative distribution of the risetime to the current peak........................ 102
Figure 5.15: Cumulative distribution of the decay time from the current peak............... 103
Figure 5.16: Cumulative distribution of the current pulse width..................................... 104
Figure 5.17: Cumulative distribution of the charge ......................................................... 105

## List of Tables

Table 2.1: Ground Lightning Categories ............................................................................ 12
Table 4.1: Simulation results of the current, current derivative and current derivative with adjusted initial value of the parameters using the Pulse function

Table 4.2: Simulation results of the current and current derivative using Heidler function

Table 5.1: Parameters' initial and final values for the Pulse and Heidler functions fitting process and the associated $\mathrm{R}^{2}$ .85

Table 5.2: Parameters' initial and final values for the Pulse and Heidler functions fitting process with fixed $\tau_{2}$ and the associated $\mathrm{R}^{2}$ .90

Table 5.3: Parameters' initial and final values for the Pulse and Heidler functions fitting


Table 5.4: Parameters' initial and final values for the Pulse and Heidler functions fitting process with fixed $\tau_{2}, \mathrm{I}_{\text {max }}$ and with forcing time constraints of the zero crossing of the current derivative signal and the associated $\mathrm{R}^{2}$.

Table 5.5: Parameters' initial and final values for the Pulse and Heidler functions fitting process with fixed $\tau_{2}, \mathrm{I}_{\text {max }}$ and with forcing time constraints of the maximum steepness of current signal and the associated $\mathrm{R}^{2}$ .98

Table 5.6: Current waveform parameters value calculated from the Pulse function .99

Table 5.7: Summary of the current waveform parameters cumulative statistics distribution 106

## Chapter 1

## Introduction

Lightning is one of the most common and most spectacular natural phenomena. Although beautiful, destructive causing loss of life and properties, frequent blackouts, communication network interruption, etc. Recent systematic studies, however, show that lightning is necessary to maintain fine weather electric field.

Atmospheric electricity has fascinated scientists for centuries. Much has been learned about lightning since Benjamin Franklin's 1752 famous experiment as if a kite and a key to proof lightning as a giant electric discharge. In the 1930`s, the lightning research was motivated by the need to reduce the damage of lightning on power systems and to understand its physical process. In the 1960 's, the lightning research was primarily related to the unexpected vulnerability of solid-state electronics due to the induced voltage and current from lightning, which results in hazards to both ground base and airborne systems. The most exposed of these are power lines, telecommunications systems, aircrafts and spacecrafts.

The present studies of lightning emphasizes the design of lightning protection circuits, which necessitate the estimation of the lightning return-stroke current waveform parameters as the peak, the risetime to the peak, maximum rate of rise (maximum steepness) and the charge. The lightning-generated electromagnetic pulse (LEMP) is also of great importance. The returnstroke model represents a relation between the current and the generated fields. The accuracy of the model will be affected by the characteristics of the structure being modeled e.g. discontinuities, conductivity, grounding, elevation etc... Modeling the return-stroke current represents a step towards calculating lightning return-stroke generated electric and magnetic fields and return-stroke model evaluation.

Due to the random nature of lightning, the lightning return-stroke current has been studied using the direct measurement of the current at elevated structures. The 553 m Canadian National (CN) Tower located in Toronto represents a hub for capturing lightning. Lightning strikes to the CN Tower have been observed since 1978. Since 1991, the tower has been equipped with a lightning current derivative measurement system. The measured return stroke current derivative signal is modeled in this thesis using different simulating functions.

As a first step towards the evaluation and development of a return-stroke model, Heidler function and its modified form have been widely used to simulate the lightning return-stroke current [1-3]. More recently, the derivative of Heidler function and the derivative of its modified form have been successfully used to simulate the lightning return-stroke current derivative, measured at the CN Tower $[4,5]$.

An in-depth investigation of Heidler function and its derivative is carried out in order to evaluate its suitability to simulate the lightning return-stroke current and its derivative, respectively, especially for tall-structures where reflections from tall-structure discontinuities present some difficulty in the simulation. As a part of the evaluation of Heidler function, it is compared with other relevant functions, such as the Pulse function.

In the past, many functions were considered for simulating the lightning return-stroke current [6-10]. Some of these functions were found to have problems related to their discontinuities or the discontinuities of their first and second derivatives at the onset time [6-10]. Such problems appear in the double exponential function and its modifications (Jones [6], Gardner [7], etc). However, other functions like the Pulse function and Heidler function do not suffer from such problems [8].

Therefore, one of the objectives of this work is to simulate the current and the current derivative measured at the tower using either Heidler function or the Pulse function. Although this work is fundamental for the evaluation and development of CN Tower lightning returnstroke modeling, it is also necessary for the determination of at least some of the waveform parameters of the current measured at the CN Tower, including the charge.

For high signal-to-noise ratio and fast rate of rise currents, measured at the CN Tower, it has been possible to determine the current wavefront parameters (peak, maximum rate of rise and the $10 \%$ to $90 \%$ risetime) [9]. However, other important current parameters, such as the pulse width, decay time and the return-stroke charge, have been difficult to determine because of current reflections from CN Towers structure discontinuities.

All current waveform parameters (peak, maximum rate of rise, risetime ...) will be obtained from Heidler or the Pulse functions by matching these functions to the measured current derivative and current before the arrival of current reflections and when current reflections are minimized at much later time.

Both Heidler, and the Pulse functions satisfy the requirements of the analytical representation of the lightning return-stroke current. In order to compare the two functions in terms of their representations of the measured signal, the first step is to build individually a modeling environment for each function which can be described as lightning return-stroke current parameters extraction system. This system, which represents an automated approach for extracting the analytical parameters of the functions, is capable to best fit the function with the measured data. Using these analytical parameters transforms the discrete data into a continuous signal, from which parameters can be determined.

This analytical parameters extraction system is recognized as a curve fitting system. For curve fitting technique, the initial value of each analytical parameter and its feasible region, where the optimal value of this analytical parameter is located, must be specified. The more accurate the initial point is the easier and faster the optimal value can be reached. Since the whole system performance is affected by the chosen initial point, the choice is considered a key point in reaching optimal fitting results. The initial point search process represents by itself a similar curve fitting but on a smaller scale with some approximation

On choosing the best approach to find an initial condition, which gives the nearest location to the optimal point, we apply the extraction system achieving the analytical model -that fits the CN Tower measured current derivative-, the current waveform parameters can be easily studied.

Although the current signal is used here for the simulation, the current derivative measured signal is also considered for simulation. The current derivative signal represents the current change rate with time. Any error in the current usually leads to lager error in the current derivative.

In order to be sure that the parameter extraction system gives the best fit, we need to develop a method for evaluating this system. Therefore, instead of using the measured data, we first use an artificial data produced by known function in order to evaluate the parameter extraction system.

The artificial data will be free from any reflections and noise. It can be produced by having a signal source represented by an analytical function that resembles the original lightning signal. This function forms a continuous signal. This signal is made to pass through a digitizer having the same rate of the measured signal, which produces the so called artificial signal.

After applying the parameter extraction system on the artificial signal, a comparison can be carried out between the simulating functions and the function used for producing the artificial function. This comparison will be a good test on the ability of simulating function to fit the measured signal.

In Chapter 2, a description of lightning physics, categories and the return-stroke associated with its formation process is given. Also lightning triggering and lightning quantities being measured are considered. The CN Tower and its measuring system are described. Finally, the lightning return-stoke current waveform parameters that are important in the establishment of protection systems are defined.

In Chapter 3, the requirements for the analytical function representation of lightning return-stroke current are introduced. Different functions that are used for lightning return-stoke current modeling is introduced. Analytical analysis will be carried out for each function through defining its parameters, showing its effect on the function and investigating the function behavior using its first and second derivatives. Finally, comparison among them is made based on the fulfillment of the requirements. Each function is analyzed by deriving its first and second derivatives.

In Chapter 4, the analytical parameter extraction system for Heilder and the Pulse functions is developed for each function individually as a first step in the comparison between both functions. Since initial value for each analytical parameter that represents the guide for reaching its optimal point is very important, the mechanism of generating initial values will be investigated. The use of artificial signals is introduced and is used to evaluate the analytical parameter extraction system.

In Chapter 5, the developed analytical parameter extraction system is used to fit the measured current derivative signal through which some constraints are imposed to overcome the problems of the measured signal. Also, all steps of fitting up to the final stage will be shown. Comparison between Heidler and the Pulse function is achieved. Finally, the current waveform parameters are calculated for several signals and cumulative statistics are carried out.

In Chapter 6, the conclusion and the recommendations are provided.

## Chapter 2

## Lightning Physics

Lightning is one of the most common and most spectacular natural phenomena. In ancient cultures people were terrified from such a scary phenomenon and some may have believed that lightning is gods' weapons, which are used to punish humans [11, 12]. Systematic studies of thunderstorm electricity began in 1752 when an experiment, carried out by Benjamin Franklin, proved that lightning is a giant electric discharge [13]. Lightning and thunderstorms have been subject of numerous scientific investigations. Yet the exact mechanism by which a cloud is electrified has not been fully understood.

## Definition of Lightning:

"Transient high current electric discharge, which occurs when some region of the atmosphere gain such a large charge that electric field associated with it can cause an electric break down of air."

A total discharge is termed a flash and has time duration of about half a second. A flash is made up of various discharge components, among which are typically three or four high current pulses called return strokes. Each return stroke last for up to milliseconds, the separation time between strokes being typically several tens of milliseconds. Lightning appears to "Flicker" because the human eye cannot resolve the individual light pulse associated with each stroke [11, 12].

Although lightning has been known as one of the most dangerous environmental phenomena, it was found to be a source for generating the necessary molecules from which life could evolve at the period of the time during which life evolved on earth. Since it is a source of forest ignition, it plays an important role in determining the composition of trees and plants in the world forests taking into consideration that these fires are small and did not damage the trees.

It also produces a chemical around its hot discharge channel that does not exist in the atmosphere, the so called fixed nitrogen, which in is necessary for plantation [11-13].

Moreover, lightning plays important role in maintaining the fine-weather electric field ( $100 \mathrm{~V} / \mathrm{m}$ ) pointing downward keeping 300 kV voltage between the earth and the electro sphere as shown in Figure 2.1. On average, a negative charge of $10^{6}$ Coulombs is distributed on the surface of the earth and an equivalent positive charge is distributed throughout the atmosphere. Total atmospheric current of the order of 1 kA is continuously depleting these charges, which is replaced by the effect of thunderstorms and lightning [11, 12].


Figure 2.1: Earth-Atmosphere charge [12]

It is also very important to mention that lightning transient high currents reaching the earth can be devastating to modern society's infrastructures. They frequently cause blackouts, outage of power and they can destroy or interrupt the operations of communication networks, aircrafts, spacecrafts and electric and electronic devices. They also results in the over voltages induced in the electrical networks (by the indirect effects of lighting strokes). So the necessity of protection from lightning hazards has made the lightning phenomenon an important area of research since the seventies [11, 12]. Figure 2.2 shows world map and how frequent it hit by lightning.


Figure 2.2: World map showing frequency of Lightning

### 2.1 Electrification of Thunderstorms

Benjamin Franklin identified one of the basic difficulty that the cloud of the thunder guest are most commonly in negative state of electricity but sometimes it is positive. Since then, it has been accepted that lightning is the transfer of either positive or negative charge. Much microphysical process might cause the charge separation, with result that whole object is neutral but one region has more positive or negative charge than other. Charge separation is measured in volts. Typical lightning bolt represents a potential difference of hundred million volts [13].

After Franklin's observation the charge distribution of rainy cloud configured as dipole in which positive charge in one region of cloud and negative in the other region as shown in

Figure 2.3. Investigators have invocated two different models for the emergence of the dipole structure, namely the Precipitation and Convection model.


Figure 2.3: Dipole Models

### 2.1.1 Precipitation Model

It assumes that the raindrops, hailstones and graupels (particles which are ice pallets of millimeter to centimeter size) representing large particles are pulled by gravity downwards through air. They pass through smaller water droplets and ice crystals leading to collusions between large particles and small ones causing the transfer of negative charge to the large particles. The lower region of the cloud then accumulates negative charged particles and the upper region is positive charged particles leading to the formation of the electric dipole [13].


Figure 2.4: Precipitation Model [13]

### 2.1.2 Convection Model

It assumes that the charge in the cloud is supplied by two external sources. The first source is the cosmic rays which impinge on the air molecules above the cloud causing its ionization. The other source is the strong electric filed at the earth surface around sharp edges producing 'Corona Discharge' positive ions. These positive ions are carried by warm air to the cloud [13].

After reaching the cloud it attracts the negative ions to enter the cloud and quickly attach them self to water droplet and ice crystal forming layer called "Screening Layer." By the hypothesis of the down drift at the cloud periphery then carry the negative charge downwards forming the dipole [13].


Figure 2.5: Convection Model [13]

### 2.1.3 Tri-Pole Model

Wilson [11], one of the researchers for the electrification of the thunderclouds, made an observation of several thunderstorms from a distance and concluded that the basic structure of a thundercloud cab be represented by a positive dipole. While Simpson [11] another researcher working in the same area, measured the charge of the falling rain and concluded the opposite (Negative dipole) [13].

Electric field measurements are used to determine the charge in thundercloud. Having more than one charged body, electric fields can be so complicated and different charge configurations may lead to the same measurements.

Many observations have established that thunderclouds can be modeled as tri-poles, in which the main region of negative charge is centered between of two positive charges regions [13].

Although the Convection theory leads more naturally to the tri-pole model, the corona discharges from sharp edges that produce flux of positive charge toward the base of cloud is not enough for forming the tri-pole. The perception model can only account for simple dipole. Several modifications are needed to account for the lower positive charge region and the fact that rain generally carries positive charge.

Several modifications were proposed, but the final observation that the collision between ice crystals and grauples particles is dependent on the temperature called charge reversal temperature. Below this temperature negative charge is transferred to the small particles and vice versa [13].


Figure 2.6: Charge Reversal Temperature [13]

### 2.2 Lightning Categories

Lightning can be classified into several categories, but in our work we will mention two main categories which are the dominant categories:

### 2.2.1 Cloud-to-cloud

Lightning discharges may occur between areas of cloud having different potentials without contacting the ground. These are most common between the anvil and lower reaches of a given thunderstorm. This lightning can sometimes be observed at great distances at night as socalled "heat lightning". In such instances, the observer may see only a flash of light without thunder. The "heat" portion of the term is a folk association between locally-experienced warmth and the distant lightning flashes [11, 12].


Figure 2.7: Different types of lightning

### 2.2.2 Cloud to earth

Lighting between cloud and earth can be categorized in terms of the direction of motion, upward or downwards, and the sign of the charge, positive or negative, of the leader that initiate the discharge Table 2.1 [12].

Table 2.1: Ground Lighting Categories [12]
Negative Cloud to Downward Initiated

Negative cloud to ground is the most common lightning; it accounts for over $90 \%$ of the world cloud to ground flashes [11, 12].

### 2.3 Lightning Formation

As a thundercloud having a certain amount of charge moves over an area, an equal but opposite charge is induced on the earth's surface below it, as this charges accumulates more and more, the air path between the cloud and earth is ionized forming Leader extending downwards. The negative charged leaders called a "stepped leader", the stepped leader may branch into a number of paths as it continues to descend. When a stepped leader approaches the ground, the presence of opposite charges on the ground enhances the electric field at the ground level, especially at relative tall structures. When the electric field becomes strong enough at certain points, a conductive discharge (called a streamer) extends in the upward direction. Then the lightning strike begins as an extremely large negative electric current (return stroke current) along the path defined by the return stroke, which has speed of about one third of the speed of light at the attachment point, but in the opposite direction from ground to cloud [11, 12]. The whole process is shown in Figure 2.8.

During the strike, successive portions of air become a conductive discharge channel as the electrons and positive ions of air molecules are pulled away from each other and forced to flow in opposite directions. The electrical discharge rapidly superheats the discharge channel, causing the air to expand rapidly and produce a shock wave heard as thunder. The rolling and gradually dissipating rumble of thunder is caused by the time delay of sound coming from different portions of a long stroke [11, 12].

A single stroke flash happens when the first return stroke ceases to flow and no more charge is available to sustain the flash. High speed videos (examined frame-by frame) show that most lightning strikes are made up of multiple individual strokes. A typical strike is made of 3 to 4 strokes. Successive stokes are usually separated by a relatively large amount of time, typically 40 to 50 milliseconds. Subsequent strokes can cause a noticeable "strobe light" effect. Each subsequent stroke is preceded by intermediate dart leader, which is usually but weaker than the
initial stepped leader. A subsequent stroke usually re-uses the discharge channel taken by the previous stroke [11, 12].


Figure 2.8: Lighting Formation Process [12]
The variations in successive discharges are the result of smaller regions of charge within the cloud being depleted by successive strokes. The sound of thunder from lightning is prolonged by successive strokes.

### 2.4 Lightning to Tall Structures

The upward-initiated lightning can occur at mountain tops and at manmade tall structures, such as towers and sky scrapers. Upward-initiated lightning can be triggered artificially using rockets trailing conducting wires.

Tall manmade structures are struck by both downward-initiated and upward-initiated flashes. Upward initiated flashes are branched in the upward direction as shown in Figure 2.9
[14], while downward-initiated flashes are branched downward but are rare to occur at elevated objects.

The frequency of occurrence of upward-initiated lightning increases with the height of the structure. For structures smaller or equal 50 meters in height $90 \%$ of flashes are downwardinitiated, while for structures higher than 400 meters only 5\% of flashes are downward-initiated [5, 6].


Figure 2.9: Branching of upward-initiated lightning

Many research projects have been conducted with tall man-made structures and the discovery of upward-initiated lightning is generally attributed to McEachron (1939), who studied lightning flashes to the Empire State Building in New York City [12]. He observed that most of the flashes to the Empire State Building are upward-initiated and start at the top of the structure [17, 18].

After McEachron, many other researchers studied lightning to tall structures including Berger in Lugano Switzerland on two towers located on Mount San Salvadore [11], Hagenguth and Anderson at the Empire State Building [12], Garbafnaty at the two 40 m television towers, one located at Monte Sasso Di Pale and the second located at Monte Rosa [14-18].

In all these studies, it was observed that lightning to tall structures, in the most cases, was initiated by a leader travelling from the top of the structure towards the cloud. According to Berger and his measurements, about $85 \%$ of lightning flashes were initiated by upward moving stepped leader. In 1991, studies conducted on CN Tower showed that only $2.8 \%$ of the flashes in that year were downward initiated flashes. Also, studies at the 530 meters high Ostankino Television tower is Moscow showed that more than $90 \%$ of flashes were upward initiated [12].

### 2.5 CN Tower and Its Lightning Measurement System

The Toronto CN Tower, shown in Figure 2.10 (located at 43.64 N and 79.40 W ), was formerly the world's tallest manmade freestanding structure. It was surpassed in height by the rising Burj Khalifa, Dubai on 12 September 2007. The work on the tower began in 1973 and after 40 month of construction, the tower was opened to the public on June 26, 1976 .With a height of 553 meter and large number of lightning strikes per year, the CN Tower represents an ideal location for studying the lightning phenomenon. While the flash density in Toronto is about 2 flashes per square kilometer per year, the tower receives tens of strikes each year, for example in 1991 the tower was hit by 72 flashes [19-21]. Simultaneous measurements of parameters CN Tower lightning strikes began in 1991. The measurements system can be divided into video monitoring system, current measurement system and field measurement. There location is shown in Figure 2.11.


Figure 2.10: CN Tower and its surrounding
The video monitoring system consists of two VHS cameras with 30 frames / sec and connected to VCRs for recording, one is placed at the Rosebrugh building of the University of Toronto and the other one at the Ontario Hydro Technology Building. The location of the cameras provides almost perpendicular views with an angel of 82.5 between them [20]. The system is used for lighting flash trajectory tracking and also to determine the numbers of strokes
per flash. In some cases the strokes are shorter than the camera resolution that why in 1996 a high speed camera with 500 frames / sec was installed.


Figure 2.11: CN Tower and the lightning measurement location

The current measurement system located at the CN Tower consists of two Rogowski coils. A Rogowski coil has the property that its induced voltage is proportional to the time variation of the net current which represents the current derivative.

The first (old) coil placed at 474 meters AGL (Above Ground Level). It encircles $1 / 5^{\text {th }}$ of the CN Tower's steel structure. Assuming a uniform distribution of the current in the azimuth direction along the CN Tower's steels structure, the measured current is estimated to be $20 \%$ of the total current. The 3 meters long coil consists of two 1.5 meters segment, which are terminated with a resistor at one end (resistor are used to absorb reflections and to damp oscillations in the coil), and connected to the impedance matching box at the other end. The impedance matching box is connected to a Sony-Tektronics RTD-710A digitizer through 165 meters of a $50 \Omega$ tri-axial cable. The impedance of the coil seen through the matching box is made to be $50 \Omega$, which is the same as the impedance of the tri-axial cable. A schematic of the coil connection and coil placement at the CN Tower are shown in Figure 2.12.


Figure 2.12: Schematic of old Rogowski coil connection

The second (new) Rogowski coil is placed at 509 meters AGL. It encircles $100 \%$ of the total lightning current. It consists of four, 1.5 meters long segments for a total length of 6 meters. It is connected to a Sony-Tektronics RTD-710A digitizer through an optical fiber. Two segments of the coil are connected to the matching box 1 and the other two segments are connected to the
matching box 2 . Both boxes are connected to a third matching box to ensure that the impedance looking into the third box is $50 \Omega$. A 30 dB attenuator is inserted between the third matching box and the fiber optical transmitter to limit the output of the new coil to below saturation level. A schematic of the new Rogowski coil is shown in Figure 2.13.


Figure 2.13: Schematic of new Rogowski coil connection

The data obtained with the new Rogowski coil has considerably better signal-to-noise ratio (SNR) as compared to data obtained with the old Rogowski coil because the new coil measure $100 \%$ of the total current and it is connected to the digitizer through an optical link.

The field measurement system is allocated at the Rosebrugh building of the University of Toronto ( 2.0 km north of the tower). It consists of two field sensors to measure the azimuth component of the magnetic field and the vertical component of the electric field. The two sensors are connected to two different channels of the digitizer through coaxial cables. The magnetic
field sensor and the electric field sensor not only measure the fields resulting from the direct strikes to the tower but also any fields created by the lightning strikes in the vicinity if the tower.

Since the measurement systems are placed at different locations and there is no time synchronization between them, a Global Positioning System (GPS) has been used. The GPS represents a master clock to time synchronize all measurements.

### 2.6 Current Waveform Parameters

Researchers are interested in five major current waveform parameters for lightning return-stroke studies, namely the current peak, current maximum steepness, risetime to the current peak ( $10 \%$ level to $90 \%$ level of the peak), current pulse width half peak and charge transferred.

Since the radiated electromagnetic field directly depends on the derivative of the lightning return-stroke current, the current maximum steepness represents an important parameter. The current derivative also influences overvoltages caused by lightning striking power lines and therefore knowledge of maximum steepness is important for protection of tall structures, power lines as well as static sensitive equipments.

The current peak is taken to be the first peak of the current due to the presence of another higher peak called the absolute peak. The absolute peak appears due to reflections from the CN Tower structure discontinuities.

The risetime to the current peak is taken from $10 \%$ to $90 \%$ of the peak level. This indicates how fast the signal is. Signals with slow risetime may not have a first peak. This occurs when reflections arrive before the arrival of the first current peak. In case of currents with low peaks, it is usually hard to determine the peak due to large SNR.

Figure 2.14 and 2.15 show a CN Tower measured current derivative signal and its current obtained by numerical integration. The absolute current is also indicated so it can be compared with the current peak. Other current parameters such as the current pulse width half peak and charge will be indicated in the later model.


Figure 2.14 Demonstration of current maximum steepness


Figure 2.15: Demonstration of current peak and risetime to the current peak

## Chapter 3

## Simulating Functions

Different simulating functions have been proposed to model the lightning return-stroke current using direct measurements preformed at tall structures, such as the CN tower. A suitable function should fulfill the requirements of the analytical representation of the lightning returnstroke current [22, 23]:

1. A good approximation to the observed waveshape of the current at the base of the return stroke channel.
2. Enable the determination of the lightning current waveform parameters:
a. Maximum current
b. Maximum current steepness
c. Risetime to the current peak
d. Current decay time
e. Current pulse width half peak
f. The charge transfer through the channel
3. No discontinuity should appear in first and second derivative especially at the onset time ( $\mathrm{t}=0$ ).
4. The current function should be differentiable in order to compute the lightninggenerated fields.
5. It should allow the variation of the location of the maximum current steepness.
6. It should be as simple as possible.

In this chapter, four functions will be investigated, namely, the Double Exponential function, Jones modified Double Exponential function, the Pulse function and Heidler function.

Each of these functions is individually analyzed to:

- Check the discontinuity for the current and current derivative at onset
- Give initial indication which function best describe the base current of the return stroke


### 3.1 Double Exponential Function

The Double Exponential function used to be the most famous simulating function for the modeling the lightning return-stroke current. This is due to its simplicity and ease for integration and differentiation [22]. It mathematically expressed by:

$$
\begin{equation*}
I(t)=\frac{I_{\max }}{\eta} *\left(e^{-\frac{t}{\tau_{1}}}-e^{-\frac{t}{\tau_{2}}}\right) \tag{3-1}
\end{equation*}
$$

Where:
$I_{\max }$ : Peak of the Current.
$\eta$ : Correction Factor
$\tau_{1}, \tau_{2}$ : Time Constants for determination of the current risetime and decay time


Figure 3.1: Current simulation using double exponential function with $\mathrm{I}=10 \mathrm{kA}, \tau_{1}=5 \mu \mathrm{~s}$ and $\tau_{2}=0.5 \mu \mathrm{~s}$

The correction factor is used to adjust the current peak value because at that $\mathrm{t}_{\text {max }}$ instant the value of the exponential part is less than 1.The expression of the $\eta$ can be found by differentiate the current function with respect to time and equating that to zero to get an expression for the instant $t_{\text {max }}$, then substitute it in the current function to get an expression for $\eta$ :

$$
\begin{gather*}
\frac{d I(t)}{d t}=\frac{I_{\max }}{\eta}\left(-\frac{1}{\tau_{1}} * e^{-\frac{t}{\tau_{1}}}+\frac{1}{\tau_{2}} * e^{-\frac{t}{\tau_{2}}}\right)  \tag{3-2}\\
\frac{d I\left(t_{\max }\right)}{d t}=0 \\
\frac{I_{\max }}{\eta}\left(-\frac{1}{\tau_{1}} * e^{-\frac{t_{\max }}{\tau_{1}}}+\frac{1}{\tau_{2}} * e^{-\frac{t_{\max }}{\tau_{2}}}\right)=0 \\
-\frac{1}{\tau_{1}} * e^{-\frac{t_{\max }}{\tau_{1}}}+\frac{1}{\tau_{2}} * e^{-\frac{t_{\max }}{\tau_{2}}}=0 \\
\frac{1}{\tau_{1}} * e^{-\frac{t_{\max }}{\tau_{1}}}=\frac{1}{\tau_{2}} * e^{-\frac{t_{\max }}{\tau_{2}}} \\
\frac{\tau_{2}}{\tau_{1}}=e^{-\frac{t_{\max }}{\tau_{2}}+\frac{t_{\max }}{\tau_{1}}} \\
\ln \left(\frac{\tau_{2}}{\tau_{1}}\right)=-\frac{t_{\max }}{\tau_{2}}+\frac{t_{\max }}{\tau_{1}} \\
t_{\max }=\frac{\tau_{2} * \tau_{1}}{\tau_{1}-\tau_{2}} * \ln \left(\frac{\tau_{1}}{\tau_{2}}\right)  \tag{3-3}\\
I\left(t_{\max }\right)=\frac{I_{\max }}{\eta} *\left(e^{-\frac{t_{\max }}{\tau_{1}}}+e^{-\frac{t_{\max }}{\tau_{2}}}\right) \\
I\left(t_{\max }\right)=I_{\max } \\
\eta=e^{-\frac{t_{\max }}{\tau_{1}}-e^{-\frac{t_{\max }}{\tau_{2}}}} \tag{3-4}
\end{gather*}
$$

By substituting $\mathrm{t}_{\text {max }}$ in the previous equation we will get the final expression of the correction factor:

$$
\begin{equation*}
\eta=\frac{\tau_{1}}{\tau_{2}}\left(e^{-\frac{\tau_{2}}{\tau_{1}-\tau_{2}}}-e^{-\frac{\tau_{2}}{\tau_{1}-\tau_{2}}}\right) \tag{3-5}
\end{equation*}
$$



Figure 3.2: Current Derivative simulation using Double Exponential Function derivative function with $\mathrm{I}=10 \mathrm{kA}, \tau_{1}$

$$
=5 \mu \mathrm{~s} \text { and } \tau_{2}=0.5 \mu \mathrm{~s}
$$

The current derivative simulation in Figure 3.2 shows the discontinuity at the onset time $t$ $=0$. The discontinuity value can be calculated as follows:

$$
\begin{gather*}
\frac{d I(t=0)}{d t}=\frac{I_{\max }}{\eta}\left(-\frac{1}{\tau_{1}} * e^{-\frac{0}{\tau_{1}}}+\frac{1}{\tau_{2}} * e^{-\frac{0}{\tau_{2}}}\right) \\
\frac{d I(t=0)}{d t}=\frac{I_{\max }}{\eta}\left(\frac{1}{\tau_{2}}-\frac{1}{\tau_{1}}\right) \neq 0 \tag{3-6}
\end{gather*}
$$

As shown in Figure 3.2, it is clear to conclude that the maximum steepness value occurs at the onset time $t=0$. In order to derive an expression for the time of occurrence, through which the current derivative value equals to zero, lets differentiae the Double Exponential function two times with respect to time and equating it to zero as follows:

$$
\begin{gather*}
\frac{d^{2} I(t)}{d t^{2}}=\frac{I_{\max }}{\eta}\left(\frac{1}{\tau_{1}{ }^{2}} * e^{-\frac{t}{\tau_{1}}}-\frac{1}{\tau_{2}{ }^{2}} * e^{-\frac{t}{\tau_{2}}}\right)  \tag{3-7}\\
\frac{d^{2} I\left(t_{\text {zero crossing double derivative }} \stackrel{\text { def }}{=} t_{z c d d}\right)}{d t^{2}}=0
\end{gather*}
$$

$$
\begin{gather*}
\frac{I_{\max }}{\eta}\left(\frac{1}{\tau_{1}{ }^{2}} * e^{-\frac{t_{z c d d}}{\tau_{1}}}-\frac{1}{\tau_{2}{ }^{2}} * e^{-\frac{t_{z c d d}}{\tau_{2}}}\right)=0 \\
\frac{1}{\tau_{1}{ }^{2}} * e^{-\frac{t_{z c c d d}}{\tau_{1}}}-\frac{1}{\tau_{2}{ }^{2}} * e^{-\frac{t_{z c d d}}{\tau_{2}}}=0 \\
\frac{1}{\tau_{1}{ }^{2}} * e^{-\frac{t_{z c d d}}{\tau_{1}}}=\frac{1}{\tau_{2}{ }^{2}} * e^{-\frac{t_{z c d d}}{\tau_{2}}} \\
e^{-\frac{t_{z c d d}}{\tau_{1}}+\frac{t_{z c d d}}{\tau_{2}}}=\frac{\tau_{1}{ }^{2}}{\tau_{2}{ }^{2}} \\
-\frac{t_{z c d d}}{\tau_{1}}+\frac{t_{z c d d}}{\tau_{2}}=\ln \left(\frac{\tau_{1}{ }^{2}}{\tau_{2}{ }^{2}}\right) \\
t_{z c d d}=2 * \frac{\tau_{2} * \tau_{1}}{\tau_{1}-\tau_{2}} * \ln \left(\frac{\tau_{1}}{\tau_{2}}\right) \\
t_{t_{z c d d}} \stackrel{\text { def }}{=} t_{\text {zero crossing double derivative }}=2 * \frac{\tau_{2} * \tau_{1}}{\tau_{1}-\tau_{2}} * \ln \left(\frac{\tau_{1}}{\tau_{2}}\right)=2 * t_{\max } \tag{3-8}
\end{gather*}
$$

The time of occurrence of the maximum steepness should occur during the rising edge and before the occurrence of the peak of the current but in the Double Exponential Function that condition is not satisfied and it occur at the onset time $t=0$, in which the time of occurrence of the second derivative equal to zero at two times of the location of the maximum current which means the theoretical maximum steepness occurs after the peak of the current.

### 3.2 Jones Modification

Several modifications have been imposed for the Double Exponential function using different functions and numerical techniques to eliminate the discontinuity. Time shifting is one of the imposed modifications. It was introduced by Jones [6]. Figure 3.3 shows the current simulation. Its mathematically expressed by:

$$
\begin{gather*}
I(t)=\frac{I_{\max }}{\eta} *\left(e^{-\frac{t^{*}}{\tau_{1}}}-e^{-\left(\frac{t^{*}}{\tau_{2}}\right)^{2}}\right)  \tag{3-9}\\
t^{*}=\frac{\left(\tau_{2}\right)^{2}}{\tau_{1}}+t
\end{gather*}
$$



Figure 3.3: Current simulation using Jones modification for Double Exponential function with $\mathrm{I}=10 \mathrm{kA}, \tau_{1}=5 \mu \mathrm{~s}$

$$
\text { and } \tau_{2}=0.5 \mu \mathrm{~s}
$$

In order to show the effect of the time shifting on the Double Exponential function, let's first find the expression of the correction factor for the modified function. Using the same method as explained in the Double Exponential function in which time of occurrence of the maximum current $\mathrm{t}_{\text {max }}$ is obtained using the first derivative of the function and then equates it with zero as follows:

$$
\begin{gather*}
\frac{d I}{d t}=\frac{d I}{d t^{*}} * \frac{d t^{*}}{d t} \\
\frac{d t^{*}}{d t}=1 \\
\frac{d I}{d t}=\frac{d I}{d t^{*}} \\
\frac{d I}{d t^{*}}=\frac{I_{\max }}{\eta} *\left(\frac{-1}{\tau_{1}} * e^{-\frac{t^{*}}{\tau_{1}}}+\frac{2 t^{*}}{\tau_{2}^{2}} e^{-\left(\frac{t^{*}}{\tau_{2}}\right)^{2}}\right)  \tag{3-10}\\
t_{\max }^{*}=\frac{\left(\tau_{2}\right)^{2}}{\tau_{1}}+t_{\max }
\end{gather*}
$$

$$
\begin{gather*}
\frac{d I\left(t_{\max }\right)}{d t^{*}}= \\
\frac{I_{\max }}{\eta} *\left(\frac{-1}{\tau_{1}} * e^{-\frac{t^{*} \max }{\tau_{1}}}+\frac{2 t^{*}{ }_{\max }}{\tau_{2}{ }^{2}} e^{-\left(\frac{t^{*} \max ^{2}}{\tau_{2}}\right)^{2}}\right)=0 \\
\frac{-1}{\tau_{1}} * e^{-\frac{t^{*} \max }{\tau_{1}}}+\frac{2 t^{*}{ }_{\text {max }}}{\tau_{2}{ }^{2}} e^{-\left(\frac{t^{*} \max }{\tau_{2}}\right)^{2}}=0 \\
\frac{2 t^{*}{ }_{\text {max }}}{\tau_{2}{ }^{2}} e^{-\left(\frac{t^{*} \max ^{2}}{\tau_{2}}\right)^{2}}=\frac{1}{\tau_{1}} * e^{-\frac{t^{*} \max ^{\tau_{1}}}{}}  \tag{3-11}\\
t^{*}{ }_{\text {max }}=\frac{\tau_{2}{ }^{2}}{2 \tau_{1}} * e^{-\frac{t^{*} \max ^{\tau_{1}}}{\tau_{1}}\left(\frac{t^{*} \max ^{2}}{\tau_{2}}\right)^{2}}
\end{gather*}
$$

The previous equation does not have general solution to get an expression for $t^{*}{ }_{\text {max }}$ numerical method will be used instead to get the value of $t^{*}{ }_{\max }$. Consequently $t_{\max }$ could be deduced. Now let's investigate the correction factor by substituting $t^{*}{ }_{\text {max }}$ in the current expression (3-9) through which the expression of the correction factor can be obtained:

$$
\begin{gather*}
I\left(t_{\max }\right)=\frac{I_{\max }}{\eta} *\left(e^{-\frac{t^{*} \max }{\tau_{1}}}-e^{-\left(\frac{t^{*} \max }{\tau_{2}}\right)^{2}}\right)=I_{\max } \\
\eta=e^{-\frac{t^{*} \max }{\tau_{1}}}-e^{-\left(\frac{t^{*} \max }{\tau_{2}}\right)^{2}} \tag{3-12}
\end{gather*}
$$

Since we are concerned with the onset time, then let's study the first derivative at $\mathrm{t}=0$ :

$$
\begin{gather*}
t^{*}=\frac{\left(\tau_{2}\right)^{2}}{\tau_{1}} \\
\frac{d I(t=0)}{d t}=\frac{I_{\max }}{\eta} *\left(\frac{-1}{\tau_{1}} * e^{-\frac{\tau_{2}{ }^{2}}{\tau_{1}{ }^{2}}}+\frac{2 \tau_{2}{ }^{2}}{\tau_{2}{ }^{2} * \tau_{1}} e^{-\left(\frac{t^{*}}{\tau_{2}}\right)^{2}}\right) \\
\frac{d I(t=0)}{d t}=\frac{I_{\max }}{\eta} *\left(\frac{-1}{\tau_{1}} * e^{-\frac{\tau_{2}{ }^{2}}{\tau_{1}{ }^{2}}}+\frac{2}{\tau_{1}} * e^{-\left(\frac{\tau_{2}}{\tau_{1}}\right)^{2}}\right) \\
\frac{d I(t=0)}{d t}=-\frac{I_{\max }}{\eta} * \frac{1}{\tau_{1}} * e^{-\frac{\tau_{2}{ }^{2}}{\tau_{1}{ }^{2}}} \tag{3-13}
\end{gather*}
$$



Figure 3.4: Current Derivative simulation using Double Exponential function derivative function with $\mathrm{I}=10 \mathrm{kA}, \tau_{1}$

$$
=5 \mu \mathrm{~s} \text { and } \tau_{2}=0.5 \mu \mathrm{~s}
$$

Reduction in the value of the discontinuity at the onset time of the current derivative is achieved in Jones modification compared with the value of the discontinuity at the onset time of the current derivative appeared in the Double Exponential function using the same values of the analytical parameters as shown in Figure 3.5 and 3.6. Also, in Jones's modification the maximum steepness appears during the rising edge and before the occurrence of the maximum current.


Figure 3.5: Current simulation using Double Exponential function and Jone modification for Double Exponential function derivative with $\mathrm{I}=10 \mathrm{kA}, \tau_{1}=5 \mu \mathrm{~s}$ and $\tau_{2}=0.5 \mu \mathrm{~s}$


Figure 3.6: Current Derivative simulation using Double Exponential function and Jone modification for Double Exponential function derivative with $\mathrm{I}=10 \mathrm{kA}, \tau_{1}=5 \mu \mathrm{~s}$ and $\tau_{2}=0.5 \mu \mathrm{~s}$

### 3.3 Heidler Function

Another set of functions have been proposed to solve the discontinuity It use the idea of decoupling between the current rise function $\mathrm{x}(\mathrm{t})$ and the current decay function $\mathrm{y}(\mathrm{t})$, in which at the current rising function $\mathrm{y}(\mathrm{t})=1$ and at the current decay function $\mathrm{x}(\mathrm{t})=1$. The current function can be written as :

$$
\begin{equation*}
I(t)=I_{\max } * x(t) * y(t) \tag{3-14}
\end{equation*}
$$

All of the those functions having same current decay function $y(t)$ :

$$
\begin{equation*}
y(t)=e^{-\frac{t}{\tau}} \tag{3-15}
\end{equation*}
$$

The current rise $x(t)$ can be modeled as:

$$
\begin{equation*}
x(y)=\frac{f(t)+\left(\frac{t}{T}\right)^{n}}{g(t)+\left(\frac{t}{T}\right)^{n}} \tag{3-16}
\end{equation*}
$$

For $t=0$ the current rise will depend on $f(t)$ and $g(t)$ which can be any function, trigonometric or polynomial, but taking into consideration the elimination of the onset discontinuity. The exponent $n$ play very important role in decoupling between the rising and decay current functions when it is value is large, while having low value leading to smooth change between the both functions. By substituting of both functions in the current function we get:

$$
\begin{gather*}
I(t)=\frac{I_{\max }}{\eta} * \frac{f(t)+\left(\frac{t}{T}\right)^{n}}{g(t)+\left(\frac{t}{T}\right)^{n}} * y(t)  \tag{3-17}\\
I(t)=\frac{I_{\max }}{\eta} *\left(\frac{f(t)}{g(t)+\left(\frac{t}{T}\right)^{n}} * y(t)+\frac{\left(\frac{t}{T}\right)^{n}}{g(t)+\left(\frac{t}{T}\right)^{n}} * y(t)\right)
\end{gather*}
$$

$$
\begin{equation*}
I(t)=\frac{I_{\max }}{\eta} *(A(t) * f(t)+B(t) * y(t)) \tag{3-18}
\end{equation*}
$$

Where: $\quad A(t)=\frac{y(t)}{g(t)+\left(\frac{t}{T}\right)^{n}}$

$$
B(t)=\frac{\left(\frac{t}{T}\right)^{n}}{g(t)+\left(\frac{t}{T}\right)^{n}}
$$

For high exponent value n and $\mathrm{t}<\mathrm{T}$ then $\mathrm{A}=1$ and $\mathrm{B}=0$, on the other hand if $\mathrm{t}>\mathrm{T}$ then $\mathrm{A}=0$ and $\mathrm{B}=1$, so at time $\mathrm{t}=\mathrm{TA}(\mathrm{t})$ and $\mathrm{B}(\mathrm{t})$ behaves like switch or by another word unit step function. Only a condition should be applied that $f(T)=y(T)$ for smooth changing of the curve at the switch time.

Although the discontinuity problem could be eliminated by choosing suitable $f(t)$ and $g(t)$ as stated before but the problem of maximum steepness location cannot vary between $0 \%$ to $90 \%$ of the current level, so modification should be applied to overcome such problem or by choosing different function putting into consideration that no discontinuity should appear. One of those functions is the power function having the form of:

$$
\begin{equation*}
I(t)=\frac{I_{\max }}{\eta} * \frac{a_{1}\left(\frac{t}{T}\right)^{k_{1}}+a_{2}\left(\frac{t}{T}\right)^{k_{2}}+\ldots \ldots \ldots+\left(\frac{t}{T}\right)^{n}}{b_{0}+b_{1}\left(\frac{t}{T}\right)^{m_{1}}+b_{2}\left(\frac{t}{T}\right)^{m_{2}}+\ldots \ldots \ldots+\left(\frac{t}{T}\right)^{n}} * e^{-\frac{t}{\tau}} \tag{3-19}
\end{equation*}
$$

Where: $\mathrm{k}_{\mathrm{i}}, \mathrm{m}_{\mathrm{i}}<\mathrm{n}$

In order to avoid discontinuities at $\mathrm{t}=0 \mathrm{k}_{\mathrm{i}}, \mathrm{m}_{\mathrm{i}}>1$. Using such function can lead to the concave shape of current having maximum steepness location approaching $90 \%$ of current level. Considering simple case as following:

$$
\begin{equation*}
I(t)=\frac{I_{\max }}{\eta} * \frac{a\left(\frac{t}{T}\right)^{n}+\left(\frac{t}{T}\right)^{n}}{1+\left(\frac{t}{T}\right)^{n}} * e^{-\frac{t}{\tau}} \tag{3-20}
\end{equation*}
$$

It can be shown that the slow rise of current is associated with the low value of $n$ and the maximum current steepness increases by the increasing of $n$.

Although using the previous simplification of power function would overcome most of the problems, but a much simpler function have been proposed that can do the job easier. This function has been proposed by F. Heidler named by Heidler function. Heidler Function is very frequent used in several standards:

$$
\begin{equation*}
I(t)=\frac{I_{\max }}{\eta} * \frac{\left(\frac{t}{\tau_{1}}\right)^{n}}{1+\left(\frac{t}{\tau_{1}}\right)^{n}} * e^{-\frac{t}{\tau_{2}}} \tag{3-21}
\end{equation*}
$$

Now let's do the same analysis as we did for the Double Exponential function by deriving the correction factor using of the first derivative:

$$
\begin{gather*}
\frac{d I(t)}{d t}=\frac{I_{\max }}{\eta} *\left[\begin{array}{c}
\left(\frac{n}{\tau_{1}} *\left(\frac{t}{\tau_{1}}\right)^{n-1} * \frac{1}{1+\left(\frac{t}{\tau_{1}}\right)^{n}} * e^{-\frac{t}{\tau_{2}}}\right)-\left(\frac{\left(\frac{t}{\tau_{1}}\right)^{n}}{1+\left(\frac{t}{\tau_{1}}\right)^{n}} * e^{-\frac{t}{\tau_{2}}} * \frac{1}{\tau_{2}}\right) \\
-\left(\frac{1}{\left(1+\left(\frac{t}{\tau_{1}}\right)^{n}\right)^{2}} * \frac{n}{\tau_{1}} *\left(\frac{t}{\tau_{1}}\right)^{n-1} *\left(\frac{t}{\tau_{1}}\right)^{n} * e^{-\frac{t}{\tau_{2}}}\right)
\end{array}\right] \\
\frac{d I(t)}{d t}=\frac{I_{\max }}{\eta} * \frac{\left(\frac{t}{\tau_{1}}\right)^{n}}{1+\left(\frac{t}{\tau_{1}}\right)^{n}} * e^{-\frac{t}{\tau_{2}} *\left[\frac{n}{\tau_{1}} * \frac{\tau_{1}}{t}-\frac{1}{\tau_{2}}-\frac{n * t^{n-1}}{\left(1+\left(\frac{t}{\tau_{1}}\right)^{n}\right) * \tau_{1}^{n}}\right]} \\
\frac{d I(t)}{d t}=\frac{I_{\max }}{\eta} * \frac{\left(\frac{t}{\tau_{1}}\right)^{n}}{1+\left(\frac{t}{\tau_{1}}\right)^{n}} * e^{-\frac{t}{\tau_{2}} *\left[\frac{n}{t}-\frac{1}{\tau_{2}}-\frac{n * t^{n-1}}{\left(\tau_{1}^{k}+t^{n}\right)}\right]} \\
\frac{d I(t)}{d t}=I(t) *\left[\frac{n}{t}-\frac{1}{\tau_{2}}-\frac{n * t^{n-1}}{\left(\tau_{1}{ }^{n}+t^{n}\right)}\right] \tag{3-22}
\end{gather*}
$$

It is very obvious that the first derivative equals to zero at the onset time $t=0$. The maximum current occurs at $\mathrm{t}_{\text {max }}$ having the first derivative of the current equal to zero:

$$
\frac{d I\left(t_{\max }\right)}{d t}=0
$$

$$
I\left(t_{\max }\right) *\left[\frac{n}{t_{\max }}-\frac{1}{\tau_{2}}-\frac{n * t_{\max }^{n-1}}{\left(\tau_{1}^{n}+t_{\max }^{n}\right)}\right]=0
$$

And since $\mathrm{I}\left(\mathrm{t}_{\text {max }}\right)$ does not equal zero, then:

$$
\begin{align*}
& \frac{n}{t_{\max }}-\frac{1}{\tau_{2}}-\frac{n * t_{\max }^{n-1}}{\left(\tau_{1}^{n}+t_{\max }^{n}\right)}=0 \\
& \frac{n}{t_{\max }}=\frac{1}{\tau_{2}}+\frac{n * t_{\max }{ }^{n-1}}{\left(\tau_{1}^{n}+t_{\max }^{n}\right)} \\
& n=\frac{t_{\max }}{\tau_{2}}+\frac{n * t_{\max }{ }^{n}}{\left(\tau_{1}{ }^{n}+t_{\max }{ }^{n}\right)} \\
& n=\frac{t_{\max }}{\tau_{2}}+\frac{n}{\left(\frac{\tau_{1}^{n}}{t_{\max }{ }^{n}}+1\right)} \\
& \frac{t_{\max }}{\tau_{2}}=n\left(1-\frac{1}{\left(\frac{\tau_{1}^{n}}{t_{\max ^{n}}}+1\right)}\right) \\
& \frac{t_{\max }}{\tau_{2}}=n\left(\frac{\frac{\tau_{1}{ }^{n}}{t_{\max }{ }^{n}}}{\left(\frac{\tau_{1}{ }^{n}}{t_{\max }{ }^{n}}+1\right)}\right) \\
& \frac{\frac{\tau_{1}{ }^{n}}{t_{\max }{ }^{n}}+1}{\frac{\tau_{1}{ }^{n}}{t_{\max }{ }^{n}}}=\frac{n * \tau_{2}}{t_{\max }} \\
& \frac{1}{\frac{\tau_{1}{ }^{n}}{t_{\max }{ }^{n}}}+1=\frac{n * \tau_{2}}{t_{\max }} \\
& \frac{t_{\max }{ }^{n}}{\tau_{1}{ }^{n}}+1=\frac{n * \tau_{2}}{t_{\max }} \\
& \frac{t_{\max }{ }^{n+1}}{\tau_{1}{ }^{n+1}}+\frac{t_{\max }}{\tau_{1}}=\frac{n * \tau_{2}}{\tau_{1}} \tag{3-23}
\end{align*}
$$

In order to get a general expression for $\mathrm{t}_{\text {max }}$ we need to solve the previous equation, but unfortunately there is no general solution for such equation. So instead we will use the iterative method to get and approximate expression for $\mathrm{t}_{\text {max }}$ :

$$
\begin{equation*}
t_{\max _{j+1}}=\tau_{1} * \sqrt[n+1]{\frac{n * \tau_{2}}{\tau_{1}}-\frac{t_{\max _{j}}}{\tau_{1}}} \tag{3-24}
\end{equation*}
$$

Assuming the starting value of $\mathrm{t}_{\text {max }}=0$ :

$$
\begin{gathered}
t_{\text {max }_{1}}=\tau_{1} * \sqrt[n+1]{\frac{n * \tau_{2}}{\tau_{1}}} \\
t_{\max _{2}}=\tau_{1} * \sqrt[n+1]{\frac{n * \tau_{2}}{\tau_{1}}-\frac{t_{m a x_{1}}}{\tau_{1}}}=\tau_{1} * \sqrt[n+1]{\frac{n * \tau_{2}}{\tau_{1}}-\sqrt[n+1]{\frac{n * \tau_{2}}{\tau_{1}}}}
\end{gathered}
$$

So the general form of the $t_{\text {max }}$ :

$$
\begin{equation*}
t_{\max }=\tau_{1} * \sqrt[n+1]{\frac{n * \tau_{2}}{\tau_{1}}-\sqrt[n+1]{\frac{n * \tau_{2}}{\tau_{1}}-\sqrt[n+1]{\frac{n * \tau_{2}}{\tau_{1}}-\ldots \ldots}}} \tag{3-25}
\end{equation*}
$$

Substituting in the current function to get the correction factor:

$$
\begin{align*}
I\left(t_{\max }\right) & =\frac{I_{\max }}{\eta} * \frac{\left(\frac{t_{\max }}{\tau_{1}}\right)^{n}}{1+\left(\frac{t_{\max }}{\tau_{1}}\right)^{n}} * e^{-\frac{t_{\max }}{\tau_{2}}} \\
\eta & =\frac{\left(\frac{t_{\max }}{\tau_{1}}\right)^{n}}{1+\left(\frac{t_{\max }}{\tau_{1}}\right)^{n}} * e^{-\frac{t_{\max }}{\tau_{2}}} \tag{3-26}
\end{align*}
$$

To get the exact value of $\mathrm{t}_{\max }$ is so complicated so instead a simplification is considered using the first term only which gives reasonable correction factor value with very small error, putting into consideration that $\mathrm{n}>3$ and $\frac{\tau_{2}}{\tau_{1}}>10$. By using that approximation the correction factor will be:

$$
\begin{gather*}
\frac{t_{\max }^{n+1}}{\tau_{1}^{n+1}}+\frac{t_{\max }}{\tau_{1}}=\frac{n * \tau_{2}}{\tau_{1}} \\
\frac{t_{\max }^{n+1}}{\tau_{1}^{n+1}} \gg 1 \\
t_{\max }=\tau_{1} * \sqrt{\frac{n+1}{\tau_{1}}} \tag{3-27}
\end{gather*}
$$

Substituting in the correction factor equation (3-26):

$$
\begin{gather*}
\eta=\frac{\left(\frac{t_{\max }}{\tau_{1}}\right)^{n}}{1+\left(\frac{t_{\max }}{\tau_{1}}\right)^{n}} * e^{-\frac{t_{\max }}{\tau_{2}}} \\
\eta=e^{-\frac{\tau_{1}}{\tau_{2}} * \sqrt[n+1]{\frac{n * \tau_{2}}{\tau_{1}}}} \tag{3-28}
\end{gather*}
$$



Figure 3.7: Current simulation using Heidler function with $\mathrm{I}=10 \mathrm{kA}, \tau_{1}=0.3 \mu \mathrm{~s}, \tau_{2}=3 \mu \mathrm{~s}$ and $\mathrm{n}=7$


Figure 3.8: Current derivative simulation using Heidler derivative function with $\mathrm{I}=10 \mathrm{kA}, \tau_{1}=0.3 \mu \mathrm{~s}, \tau_{2}=3 \mu \mathrm{~s}$ and

$$
\mathrm{n}=7
$$

Starting with the first derivative result, differentiae it one more time with respect to to derive the $\mathrm{t}_{\mathrm{ms}}$ :

$$
\begin{aligned}
& \frac{d^{2} I(t)}{d t^{2}}=I(t) *\left[\frac{n}{t}-\right.\left.\frac{1}{\tau_{2}}-\frac{n * t^{n-1}}{\left(\tau_{1}{ }^{n}+t^{n}\right)}\right]^{2}+I(t) *\left[-\frac{n}{t^{2}}-\frac{n *(n-1) t^{n-2}}{\left(\tau_{1}{ }^{n}+t^{n}\right)}-\frac{\left(n * t^{n-1}\right)^{2}}{\left(\tau_{1}^{n}+t^{n}\right)}\right] \\
& \frac{d^{2} I(t)}{d t^{2}}=I(t) * {\left[\left[\frac{n^{2}}{t^{2}}-\frac{n}{t * \tau_{2}}-\frac{n^{2} * t^{n-2}}{\left(\tau_{1}{ }^{n}+t^{n}\right)}-\frac{n}{t * \tau_{2}}+\frac{1}{\tau_{2}{ }^{2}}+\frac{n * t^{n-1}}{\tau_{2} *\left(\tau_{1}^{n}+t^{n}\right)}\right.\right.} \\
&\left.-\frac{n^{2} * t^{n-2}}{\left(\tau_{1}^{n}+t^{n}\right)}+\frac{n * t^{n-1}}{\tau_{2} *\left(\tau_{1}{ }^{n}+t^{n}\right)}+\left(\frac{n * t^{n-1}}{\left(\tau_{1}{ }^{n}+t^{n}\right)}\right)^{2}\right] \\
&\left.+\left[-\frac{n}{t^{2}}-\frac{n *(n-1) t^{n-2}}{\left(\tau_{1}{ }^{n}+t^{n}\right)}-\frac{\left(n * t^{n-1}\right)^{2}}{\left(\tau_{1}{ }^{n}+t^{n}\right)}\right]\right] \\
& \frac{d^{2} I(t)}{d t^{2}}=I(t) *\left[\frac{n^{2}}{t^{2}}-\frac{2 n}{t * \tau_{2}}-\frac{2 n^{2} * t^{n-2}}{\left(\tau_{1}^{n}+t^{n}\right)}+\frac{1}{\tau_{2}^{2}}+\frac{2 n * t^{n-1}}{\tau_{2} *\left(\tau_{1}^{n}+t^{n}\right)}-\frac{n}{t^{2}}\right. \\
&\left.-\frac{n *(n-1) t^{n-2}}{\left(\tau_{1}^{n}+t^{n}\right)}\right]
\end{aligned}
$$

$$
\begin{equation*}
\frac{d^{2} I(t)}{d t^{2}}=I(t) *\left[\frac{n}{t} *\left(\frac{n-1}{t}-\frac{2}{\tau_{2}}\right)-\frac{n * t^{n-1}}{\left(\tau_{1}{ }^{n}+t^{n}\right)} *\left(\frac{3 n-1}{t}+\frac{2}{\tau_{2}}\right)+\frac{1}{\tau_{2}^{2}}\right] \tag{3-29}
\end{equation*}
$$

It is very obvious to notice that the second derivative equals to zero at the onset time $t=$ 0 . This makes the Heidler function suitable to analytical represents the lightning return-stroke current.

$$
\begin{gather*}
\frac{d^{2} I\left(t_{m s}\right)}{d t^{2}}=0 \\
I\left(t_{m s}\right) *\left[\frac{n}{t_{m s}} *\left(\frac{n-1}{t_{m s}}-\frac{2}{\tau_{2}}\right)-\frac{n * t_{m s}^{n-1}}{\left(\tau_{1}{ }^{n}+t_{m s}{ }^{n}\right)} *\left(\frac{3 n-1}{t_{m s}}+\frac{2}{\tau_{2}}\right)+\frac{1}{\tau_{2}{ }^{2}}\right]=0 \\
\frac{n}{t_{m s}} *\left(\frac{n-1}{t_{m s}}-\frac{2}{\tau_{2}}\right)-\frac{n * t_{m s}^{n-1}}{\left(\tau_{1}{ }^{n}+t_{m s}{ }^{n}\right)} *\left(\frac{3 n-1}{t_{m s}}+\frac{2}{\tau_{2}}\right)+\frac{1}{\tau_{2}{ }^{2}}=0 \tag{3-30}
\end{gather*}
$$

In order to get the general from of the time of occurrence of the maximum steepness, equation (3-30) should be solved. And since there is no general solution for that equation, numerical method will be used to get the value for $\mathrm{t}_{\mathrm{ms}}$. Having the maximum steepness of the current occurs at the rising edge so the current can be simplified by assuming that decay part of the current function $y(t)=1$ :

$$
\begin{gather*}
I(t)=\frac{I_{\max }}{\eta} * \frac{\left(\frac{t}{\tau_{1}}\right)^{n}}{1+\left(\frac{t}{\tau_{1}}\right)^{n}}  \tag{3-31}\\
\frac{d I(t)}{d t}=\frac{I_{\max }}{\eta} *\left[\begin{array}{c}
\left(\frac{n}{\tau_{1}} *\left(\frac{t}{\tau_{1}}\right)^{n-1} * \frac{1}{1+\left(\frac{t}{\tau_{1}}\right)^{n}} * e^{-\frac{t}{\tau_{2}}}\right) \\
\left.-\left(\frac{1}{\left(1+\left(\frac{t}{\tau_{1}}\right)^{n}\right)^{2}} * \frac{n}{\tau_{1}} *\left(\frac{t}{\tau_{1}}\right)^{n-1} *\left(\frac{t}{\tau_{1}}\right)^{n} * e^{-\frac{t}{\tau_{2}}}\right)\right] \\
\frac{d I(t)}{d t}=\frac{I_{\max }}{\eta} * \frac{\left(\frac{t}{\tau_{1}}\right)^{n}}{1+\left(\frac{t}{\tau_{1}}\right)^{n}} * e^{-\frac{t}{\tau_{2}}} *\left[\frac{n}{t}-\frac{n * t^{n-1}}{\tau_{1}^{n}+t^{n}}\right]
\end{array},\right.
\end{gather*}
$$

$$
\begin{aligned}
& \frac{d I(t)}{d t}=I(t) *\left[\frac{n}{t}-\frac{n * t^{n-1}}{\tau_{1}{ }^{n}+t^{n}}\right] \\
& \frac{d^{2} I(t)}{d t^{2}}=I(t) *\left[\frac{n}{t}-\frac{n * t^{n-1}}{\tau_{1}{ }^{n}+t^{n}}\right]^{2}+I(t) *\left[-\frac{n}{t^{2}}-\left(\frac{n *(n-1) * t^{n-2}}{\tau_{1}{ }^{n}+t^{n}}-\frac{\left(n * t^{n-1}\right)^{2}}{\left(\tau_{1}{ }^{n}+t^{n}\right)^{2}}\right)\right] \\
& \frac{d^{2} I(t)}{d t^{2}}=I(t) *\left[\left(\frac{n}{t}-\frac{n * t^{n-1}}{\tau_{1}{ }^{n}+t^{n}}\right)^{2}-\frac{n}{t^{2}}-\frac{n *(n-1) * t^{n-2}}{\tau_{1}{ }^{n}+t^{n}}+\frac{\left(n * t^{n-1}\right)^{2}}{\left(\tau_{1}{ }^{n}+t^{n}\right)^{2}}\right] \\
& \frac{d^{2} I\left(t_{\mathrm{max}} \text { steepness } \stackrel{\text { def }}{=} t_{m s}\right)}{d t^{2}}=0 \\
& \left(\frac{n}{t_{m s}}-\frac{n * t_{m s}{ }^{n-1}}{\tau_{1}{ }^{n}+t_{m s}{ }^{n}}\right)^{2}-\frac{n}{t_{m s}{ }^{2}}-\frac{n *(n-1) * t_{m s}{ }^{n-2}}{\tau_{1}^{n}+t_{m s}{ }^{n}}+\frac{\left(n * t_{m s}{ }^{n-1}\right)^{2}}{\left(\tau_{1}^{n}+t_{m s}{ }^{n}\right)^{2}}=0 \\
& \left(\frac{n}{t_{m s}}\right)^{2}-2 \frac{n^{2} * t_{m s}{ }^{n-2}}{\tau_{1}{ }^{n}+t_{m s}{ }^{n}}+\left(\frac{n * t_{m s}{ }^{n-1}}{\tau_{1}{ }^{n}+t_{m s}{ }^{n}}\right)^{2}-\frac{n}{t_{m s}{ }^{2}}-\frac{n *(n-1) * t_{m s}{ }^{n-2}}{\tau_{1}{ }^{n}+t_{m s}{ }^{n}} \\
& +\left(\frac{n * t_{m s}^{n-1}}{t_{m s}^{n}+t_{m s}^{n}}\right)^{2}=0 \\
& \frac{n}{t_{m s}{ }^{2}}(n-1)-\frac{n *(3 n-1) * t_{m s}^{n-2}}{\tau_{1}^{n}+t_{m s}{ }^{n}}+2\left(\frac{n * t_{m s}^{n-1}}{\tau_{1}^{n}+t_{m s}{ }^{n}}\right)^{2}=0 \\
& \frac{n}{t_{1}{ }^{2}}(n-1) *\left(\tau_{1}{ }^{n}+t_{m s}{ }^{n}\right)^{2}-n * t_{m s}{ }^{n-2} *(3 n-1) *\left(\tau_{1}{ }^{n}+t_{m s}{ }^{n}\right) \\
& +2\left(n * t_{m s}{ }^{n-1}\right)^{2}=0 \\
& \left(\frac{n^{2}}{t_{m s}{ }^{2}}-\frac{n}{t_{m s}{ }^{2}}\right) *\left(\tau_{1}{ }^{2 n}+2 * \tau_{1}{ }^{n} * t_{m s}{ }^{n}+t_{m s}{ }^{2 n}\right)-\left(3 n^{2} * t_{m s}{ }^{n-2}-n * t_{m s}{ }^{n-2}\right) \\
& \text { * }\left(\tau_{1}{ }^{n}+t_{m s}{ }^{n}\right)+2\left(n * t_{m s}{ }^{n-1}\right)^{2}=0
\end{aligned}
$$

$$
\begin{align*}
& \left(\frac{n * \tau_{1}{ }^{n}}{t_{m s}}\right)^{2}+2 * n^{2} * \tau_{1}{ }^{n} * t_{m s}{ }^{n-2}+\left(n * t_{m s}{ }^{n-1}\right)^{2}-n *\left(\frac{\tau_{1}{ }^{n}}{t_{m s}}\right)^{2}-2 * n * \tau_{1}{ }^{n} \\
& * t_{m s}{ }^{n-2}-n\left(t_{m s}{ }^{n-1}\right)^{2}-3 n^{2} * t_{m s}{ }^{n-2} * \tau_{1}{ }^{n}+n * t_{m s}{ }^{n-2} * \tau_{1}{ }^{n} \\
& -3\left(n * t_{m s}{ }^{n-1}\right)^{2}+n\left(t_{m s}{ }^{n-1}\right)^{2}+2\left(n * t_{m s}{ }^{n-1}\right)^{2}=0 \\
& n *\left(\frac{\tau_{1}{ }^{n}}{t_{m s}}\right)^{2}(n-1)-n^{2} * \tau_{1}{ }^{n} * t_{m s}{ }^{n-2}-n * \tau_{1}{ }^{n} * t_{m s}{ }^{n-2}=0 \\
& n * \tau_{1}{ }^{2 n}(n-1)-n^{2} * \tau_{1}{ }^{n} * t_{m s}{ }^{n}-n * \tau_{1}{ }^{n} * t_{m s}{ }^{n}=0 \\
& \tau_{1}{ }^{n}(n-1)-n * t_{m s}{ }^{n}-t_{m s}{ }^{n}=0 \\
& t_{m s}{ }^{n} *(n+1)=\tau_{1}{ }^{n}(n-1) \\
& t_{m s}^{n}=\frac{\tau_{1}{ }^{n}(n-1)}{(n+1)} \\
& t_{\text {max steepness }} \stackrel{\text { def }}{=} t_{m s}=\tau_{1} * \sqrt[n]{\frac{n-1}{n+1}} \tag{3-34}
\end{align*}
$$

Let $f_{l}=\sqrt[n]{\frac{n-1}{n+1}}$ :

$$
t_{m s}=\tau_{1} * f_{1}
$$

By substituting the $\mathrm{t}_{\mathrm{ms}}$ at the approximated current derivative equation (3-32), the value of the maximum steepness will be:

$$
\frac{d I\left(t_{m s}\right)}{d t}=\frac{I_{\max }}{\eta} * \frac{\left(\frac{\left.\tau_{1} * \sqrt[n]{\frac{n-1}{n+1}}\right)^{n}}{\tau_{1}}\right.}{1+\left(\frac{\tau_{1} * \sqrt[n]{\frac{n-1}{n+1}}}{\tau_{1}}\right)^{n}} *\left[\frac{n}{\tau_{1} * \sqrt[n]{\frac{n-1}{n+1}}}-\frac{n *\left(\tau_{1} * \sqrt[n]{\frac{n-1}{n+1}}\right)^{n-1}}{\tau_{1}{ }^{n}+\left(\tau_{1} * \sqrt[n]{\frac{n-1}{n+1}}\right)^{n}}\right]
$$

$$
\begin{gather*}
\frac{d I\left(t_{m s}\right)}{d t}=\frac{I_{\max }}{\eta} * \frac{\frac{n-1}{n+1}}{1+\frac{n-1}{n+1}} *\left[\frac{n}{\tau_{1} * \sqrt[n]{\frac{n-1}{n+1}}}-\frac{n *\left(\sqrt[n]{\left.\frac{n-1}{n+1}\right)^{n-1}}\right.}{\tau_{1} *\left(1+\frac{n-1}{n+1}\right)}\right] \\
\frac{d I\left(t_{m s}\right)}{d t}=\frac{I_{\max }}{\eta} * \frac{n-1}{n+1+n-1} *\left[\frac{n}{\tau_{1} * \sqrt[n]{\frac{n-1}{n+1}}}-\frac{n}{\tau_{1} * \sqrt[n]{\frac{n-1}{n+1}} *\left(1+\frac{n+1}{n-1}\right)}\right] \\
\frac{d I\left(t_{m s}\right)}{d t}=\frac{I_{\max }}{\eta} * \frac{n-1}{2 * \tau_{1} * \sqrt[n]{\frac{n-1}{n+1}}} *\left[1-\frac{1}{\left(1+\frac{n+1}{n-1}\right)}\right] \\
\frac{d I\left(t_{m s}\right)}{d t}=\frac{I_{\max }}{\eta} * \frac{n-1}{2 * \tau_{1} * \sqrt[n]{\frac{n-1}{n+1}} *\left[1-\frac{n-1}{(n-1+n+1)}\right]} \\
\frac{d I\left(t_{m s}\right)}{d t}=\frac{I_{\max }}{\eta} * \frac{n-1}{2 * \tau_{1} * \sqrt[n]{\frac{n-1}{n+1}}} *\left[1-\frac{n-1}{2 n}\right] \\
\frac{d I\left(t_{m s}\right)}{d t}=\frac{I_{\max }}{\eta} * \frac{n-1}{2 * \tau_{1} * \sqrt[n]{\frac{n-1}{n+1}}} * \frac{n+1}{2 n} \\
\frac{d I\left(t_{m s}\right)}{d t}=\frac{I_{\max }}{\eta} * \frac{n^{2}-1}{4 n * \tau_{1}} * \sqrt[n]{\frac{n+1}{n-1}} \tag{3-35}
\end{gather*}
$$

Let $f_{2}=\frac{n^{2}-1}{n^{2}} * \sqrt[n]{\frac{n+1}{n-1}}$ :

$$
\frac{d I\left(t_{m s}\right)}{d t}=\frac{I_{\max }}{\eta} * \frac{n}{4 \tau_{1}} f_{2}
$$

Now let's find the value of the current at the time of occurrence of the maximum steepness:

$$
\begin{align*}
& I\left(t_{m s}\right)=\frac{I_{\max }}{\eta} * \frac{\left(\frac{t_{m s}}{\tau_{1}}\right)^{n}}{1+\left(\frac{t_{m s}}{\tau_{1}}\right)^{n}} \\
& I\left(t_{m s}\right)=\frac{I_{\max }}{\eta} * \frac{\frac{n-1}{n+1}}{1+\frac{n-1}{n+1}} \\
& I\left(t_{m s}\right)=\frac{I_{\max }}{\eta} * \frac{n-1}{2 n} \tag{3-36}
\end{align*}
$$

Let $f_{3}=\frac{n-1}{2 n}$ :

$$
I\left(t_{m s}\right)=\frac{I_{\max }}{\eta} * f_{3}
$$

The time of occurrence of the maximum steepness, the value of maximum steepness and the current value all together depends on the value of $k$ as shown the $f_{1,}, f_{2}, f_{3}$. If $n$ is very large:

$$
\begin{gather*}
f_{1}=1--------\rightarrow t_{m s}=\tau_{1}  \tag{3-37}\\
f_{2}=1-----------\rightarrow \frac{d I\left(t_{m s}\right)}{d t}=\frac{I_{\max }}{\eta} * \frac{n}{4 \tau_{1}} \tag{3-38}
\end{gather*}
$$

Or it can be simply proved by substituting $t_{m s}=\tau_{1}$ at $\frac{d I\left(t_{m s}\right)}{d t}$ :

$$
\begin{gathered}
\frac{d I\left(t_{\text {max steepness }} \stackrel{\text { def }}{=} t_{m s}\right)}{d t}=I\left(t_{m s}\right) *\left[\frac{n}{t_{1}}-\frac{n * t_{m s}^{n-1}}{\tau_{1}^{n}+t_{m s}^{n}}\right] \\
\frac{d I\left(t_{m s}\right)}{d t}=\frac{I_{m a x}}{\eta} * \frac{\left(\frac{t_{m s}}{\tau_{1}}\right)^{n}}{1+\left(\frac{t_{m s}}{\tau_{1}}\right)^{n}} *\left[\frac{n}{t_{m s}}-\frac{n * t_{m s}^{n-1}}{\tau_{1}^{n}+t_{m s}{ }^{n}}\right] \\
\frac{d I\left(t_{m s}\right)}{d t}=\frac{I_{\max }}{\eta} * \frac{1}{2} *\left[\frac{n}{\tau_{1}}-\frac{n * \tau_{1}^{n-1}}{\tau_{1}{ }^{n}+\tau_{1}^{n}}\right] \\
\frac{d I\left(t_{m s}\right)}{d t}=\frac{I_{m a x}}{\eta} * \frac{1}{2} *\left[\frac{n}{\tau_{1}}-\frac{n * \tau_{1}^{n-1}}{2 \tau_{1}^{n}}\right]
\end{gathered}
$$

$$
\begin{gather*}
\frac{d I\left(t_{m s}\right)}{d t}=\frac{I_{\max }}{\eta} * \frac{1}{2} *\left[\frac{n}{\tau_{1}}-\frac{n}{2 \tau_{1}}\right] \\
\frac{d I\left(t_{m s}\right)}{d t}=\frac{I_{\max }}{\eta} * \frac{1}{2} * \frac{n}{2 \tau_{1}} \\
\frac{d I\left(t_{m s}\right)}{d t}=\frac{I_{\max }}{\eta} * \frac{n}{4 \tau_{1}} \\
f_{3}=\frac{1}{2}-----------\rightarrow I\left(t_{m s}\right)=\frac{I_{\max }}{2 \eta} \tag{3-39}
\end{gather*}
$$

Although estimating the current maximum steepness was done using an approximation, it gives an over view about how the $t_{m s}$ and maximum steepness value behaves, also it gives an indication regarding the dependency between $n$ and time of occurrence of the maximum steepness and its value.

### 3.4 Pulse Function

The pulse function is a new function that is used to represent the lightning return-stroke current [24], and it is mathematically defined by:

$$
\begin{equation*}
I(t)=\frac{I_{\max }}{\eta} *\left(1-e^{-\frac{t}{\tau_{1}}}\right)^{n} * e^{-\frac{t}{\tau_{2}}} \tag{3-40}
\end{equation*}
$$

Where:
$I_{\max }$ : Maximum Current.
$\eta$ : Correction Factor
$\tau_{1}, \tau_{2}$ : Time Constants for determination of the current rise and decay time


Figure 3.9: Current simulation using Pulse function with $\mathrm{I}=10 \mathrm{kA}, \tau_{1}=0.5 \mu \mathrm{~s}, \tau_{2}=10 \mu \mathrm{~s}$ and $\mathrm{n}=5$

Similar to Heidler function, the correction factor of the pulse can be estimated by using the first derivative to get the location of the peak of the current $\mathrm{t}_{\text {max }}$ and by substituting with the $\mathrm{t}_{\text {max }}$ at the current function (3-40) we get the correction factor.

$$
\begin{gather*}
\frac{d I(t)}{d t}=\frac{I_{\max }}{\eta} *\left[\left(\frac{n}{\tau_{1}} * e^{-\frac{t}{\tau_{1}}}\right) *\left(1-e^{-\frac{t}{\tau_{1}}}\right)^{n-1} * e^{-\frac{t}{\tau_{2}}}-\frac{1}{\tau_{2}} *\left(1-e^{-\frac{t}{\tau_{1}}}\right)^{n} * e^{-\frac{t}{\tau_{2}}}\right] \\
\frac{d I(t)}{d t}=\frac{I_{\max }}{\eta} *\left(1-e^{-\frac{t}{\tau_{1}}}\right)^{n} * e^{-\frac{t}{\tau_{2}}} *\left[\frac{n}{\tau_{1}} * \frac{e^{-\frac{t}{\tau_{1}}}}{\left(1-e^{-\frac{t}{\tau_{1}}}\right)}-\frac{1}{\tau_{2}}\right] \\
\frac{d I(t)}{d t}=I(t) *\left[\frac{n}{\tau_{1}} * \frac{e^{-\frac{t}{\tau_{1}}}}{\left(1-e^{-\frac{t}{\tau_{1}}}\right)}-\frac{1}{\tau_{2}}\right]  \tag{3-41}\\
\frac{d I\left(t_{\max }\right)}{d t}=I\left(t_{\max }\right) *\left[\frac{n}{\tau_{1}} * \frac{e^{-\frac{t_{\max }}{\tau_{1}}}}{\left(1-e^{-\frac{t_{\max }}{\tau_{1}}}\right)}-\frac{1}{\tau_{2}}\right]
\end{gather*}
$$

The pulse function does not have discontinuity of the current derivative at the onset time same as Heilder function.

$$
\begin{align*}
& I\left(t_{\max }\right) \neq 0 \\
& \frac{n}{\tau_{1}} * \frac{e^{-\frac{t_{\max }}{\tau_{1}}}}{\left(1-e^{-\frac{t_{\max }}{\tau_{1}}}\right)}-\frac{1}{\tau_{2}}=0 \\
& \frac{n}{\tau_{1}} * \frac{e^{-\frac{t_{\max }}{\tau_{1}}}}{\left(1-e^{-\frac{t_{\max }}{\tau_{1}}}\right)}=\frac{1}{\tau_{2}} \\
& \frac{e^{-\frac{t_{\max }}{\tau_{1}}}}{\left(1-e^{-\frac{t_{\max }}{\tau_{1}}}\right)}=\frac{\tau_{1}}{n * \tau_{2}} \\
& e^{-\frac{t_{\max }}{\tau_{1}}}=\frac{\tau_{1}}{n * \tau_{2}} *\left(1-e^{-\frac{t_{\max }}{\tau_{1}}}\right) \\
& e^{-\frac{t_{\max }}{\tau_{1}}} *\left(1+\frac{\tau_{1}}{n * \tau_{2}}\right)=\frac{\tau_{1}}{n * \tau_{2}} \\
& e^{-\frac{t_{\max }}{\tau_{1}}}=\frac{\tau_{1}}{\tau_{1}+n * \tau_{2}} \\
& \frac{t_{\max }}{\tau_{1}}=\ln \left(\frac{\tau_{1}+n * \tau_{2}}{\tau_{1}}\right) \\
& t_{\text {max }}=\tau_{1} * \ln \left(1+\frac{n * \tau_{2}}{\tau_{1}}\right) \tag{3-42}
\end{align*}
$$

Substitute with $\mathrm{t}_{\text {max }}$ at the current expression (3-40):

$$
\begin{gathered}
I\left(t_{\max }\right)=\frac{I_{\max }}{\eta} *\left(1-e^{-\frac{t_{\max }}{\tau_{1}}}\right)^{n} * e^{-\frac{t_{\max }}{\tau_{2}}} \\
I_{\max }=\frac{I_{\max }}{\eta} *\left(1-e^{-\frac{t_{\max }}{\tau_{1}}}\right)^{n} * e^{-\frac{t_{\max }}{\tau_{2}}} \\
\eta=\left(1-\frac{\tau_{1}}{\tau_{1}+n * \tau_{2}}\right)^{n} * e^{-\frac{t_{\max }}{\tau_{1}} * \frac{\tau_{1}}{\tau_{2}}}
\end{gathered}
$$

$$
\begin{equation*}
\eta=\left(\frac{n * \tau_{2}}{\tau_{1}+n * \tau_{2}}\right)^{n} *\left(\frac{\tau_{1}}{\tau_{1}+n * \tau_{2}}\right)^{\frac{\tau_{1}}{\tau_{2}}} \tag{3-43}
\end{equation*}
$$

Pulse Function Derivative Simulation


Figure 3.10: Current derivative simulation using Pulse function derivative with $\mathrm{I}=10 \mathrm{kA}, \tau_{1}=0.5 \mu \mathrm{~s}, \tau_{2}=10 \mu \mathrm{~s}$ and

$$
\mathrm{n}=5
$$

$$
\begin{aligned}
\frac{d^{2} I(t)}{d t^{2}}=I(t) & *\left[\frac{n}{\tau_{1}} * \frac{e^{-\frac{t}{\tau_{1}}}}{\left(1-e^{-\frac{t}{\tau_{1}}}\right)}-\frac{1}{\tau_{2}}\right]^{2}+I(t) \\
& *\left[\frac{n}{\tau_{1}} *\left(\frac{\frac{-e^{-\frac{t}{\tau_{1}}}}{\tau_{1}} *\left(1-e^{-\frac{t}{\tau_{1}}}\right)-e^{-\frac{t}{\tau_{1}}} * \frac{e^{-\frac{t}{\tau_{1}}}}{\tau_{1}}}{\left(1-e^{-\frac{t}{\tau_{1}}}\right)^{2}}\right)\right]
\end{aligned}
$$

$$
\begin{align*}
\frac{d^{2} I(t)}{d t^{2}}=I(t) & *\left[\left(\frac{n}{\tau_{1}} * \frac{e^{-\frac{t}{\tau_{1}}}}{\left(1-e^{-\frac{t}{\tau_{1}}}\right.}\right)\right)^{2}-\frac{2 n}{\tau_{1} * \tau_{2}} * \frac{e^{-\frac{t}{\tau_{1}}}}{\left(1-e^{-\frac{t}{\tau_{1}}}\right)}+\frac{1}{\tau_{2}{ }^{2}}-\frac{n}{\tau_{1}{ }^{2}} \\
* & \left.\frac{e^{-\frac{t}{\tau_{1}}} *\left(1-e^{-\frac{t}{\tau_{1}}}\right)}{\left(1-e^{-\frac{t}{\tau_{1}}}\right)^{2}}-\frac{n}{\tau_{1}{ }^{2}} * \frac{e^{-\frac{2 t}{\tau_{1}}}}{\left(1-e^{-\frac{t}{\tau_{1}}}\right)^{2}}\right] \\
\frac{d^{2} I(t)}{d t^{2}}= & I(t) *\left[\left(\frac{n}{\tau_{1}} * \frac{e^{-\frac{t}{\tau_{1}}}}{\left(1-e^{-\frac{t}{\tau_{1}}}\right)}\right)^{2}-\frac{2 n}{\tau_{1} * \tau_{2}} * \frac{e^{-\frac{t}{\tau_{1}}}}{\left(1-e^{-\frac{t}{\tau_{1}}}\right)}\right.  \tag{3-44}\\
& \left.+\frac{1}{\tau_{2}^{2}}-\frac{n}{\tau_{1}^{2}} * \frac{e^{-\frac{t}{\tau_{1}}}}{\left(1-e^{-\frac{t}{\tau_{1}}}\right)^{2}}\right]
\end{align*}
$$

Also we can conclude the second derivative posses no discontinuities at the onset time. Therefore both functions Heidler and the Pulse satisfies the conditions for analytically representation of the lightning return-stroke current.

$$
\left.\begin{array}{c}
\frac{d^{2} I\left(t_{m s}\right)}{d t^{2}}=I\left(t_{m s}\right) *\left[\begin{array}{c}
\left(\frac{n}{\tau_{1}} * \frac{e^{-\frac{t_{m s}}{\tau_{1}}}}{\left(1-e^{-\frac{t_{m s}}{\tau_{1}}}\right.}\right)
\end{array}\right)^{2}-\frac{2 n}{\tau_{1} * \tau_{2}} * \frac{e^{-\frac{t_{m s}}{\tau_{1}}}}{1-e^{-\frac{t_{m s}}{\tau_{1}}}}+\frac{1}{\tau_{2}{ }^{2}} \\
-\frac{n}{\tau_{1}^{2}} * \frac{e^{-\frac{t_{m s}}{\tau_{1}}}}{\left(1-e^{-\frac{t_{m s}}{\tau_{1}}}\right)^{2}}
\end{array}\right]
$$

$$
\begin{align*}
& \left(\frac{n}{\tau_{1}}\right)^{2} * e^{-\frac{2 t_{m s}}{\tau_{1}}}-\frac{2 n}{\tau_{1} * \tau_{2}} * e^{-\frac{t_{m s}}{\tau_{1}}} *\left(1-e^{-\frac{t_{m s}}{\tau_{1}}}\right)+\frac{\left(1-e^{-\frac{t_{m s}}{\tau_{1}}}\right)^{2}}{\tau_{2}{ }^{2}}-\frac{n}{\tau_{1}^{2}} * e^{-\frac{t_{m s}}{\tau_{1}}}=0 \\
& e^{-\frac{2 t_{m s}}{\tau_{1}}}\left(\left(\frac{n}{\tau_{1}}\right)^{2}+\frac{2 n}{\tau_{1} * \tau_{2}}+\frac{1}{\tau_{2}^{2}}\right)-e^{-\frac{t_{m s}}{\tau_{1}}}\left(\frac{2 n}{\tau_{1} \tau_{2}}+\frac{2}{\tau_{2}^{2}}+\frac{n}{\tau_{1}{ }^{2}}\right)+\frac{1}{\tau_{2}{ }^{2}}=0 \tag{3-45}
\end{align*}
$$

Since previous equation (3-45) does not have a general form, the expression of the time of occurrence of the maximum steepness cannot be estimated. The same approximation established in Heidler function can be used for the Pulse function, in which the $t_{m s}$ occurs at the rising edge and the decay part of the function can be approximated to $\mathrm{y}(\mathrm{t})=1$.

$$
\begin{gather*}
I(t)=\frac{I_{\max }}{\eta} *\left(1-e^{-\frac{t}{\tau_{1}}}\right)^{n}  \tag{3-46}\\
\frac{d I(t)}{d t}=I(t) * \frac{n}{\tau_{1}} * \frac{e^{-\frac{t}{\tau_{1}}}}{\left(1-e^{-\frac{t}{\tau_{1}}}\right)} \\
\frac{d I(t)}{d t}=I(t) * \frac{n}{\tau_{1}} * \frac{1}{e^{\frac{t}{\tau_{1}}}-1}  \tag{3-47}\\
\frac{d^{2} I(t)}{d t^{2}}=I(t) *\left(\frac{n}{\tau_{1}}\right)^{2} * \frac{1}{\left(e^{\frac{t}{\tau_{1}}}-1\right)^{2}}-I(t) * \frac{n}{\tau_{1}} * \frac{e^{\frac{t}{\tau_{1}}}}{\tau_{1}} * \frac{1}{d t^{2}}=I(t) * \frac{1}{\left(e^{\frac{t}{\tau_{1}}}-1\right)^{2}} \\
\left.\frac{d^{2} I\left(t_{m s}\right)}{d t^{2}}=I\left(t_{m s}\right) * \frac{n}{\tau_{1}}-1\right)^{2}  \tag{3-48}\\
\left(e^{\frac{t_{m s}}{\tau_{1}}}-1\right)^{2}
\end{gather*} \frac{1}{\tau_{1}} *\left(\frac{n}{\tau_{1}}-\frac{e^{\frac{t}{\tau_{1}}}}{\tau_{1}}\right),(3)
$$

By substituting the $t_{\mathrm{ms}}$ at the approximated current derivative equation (3-47), the value of the maximum steepness will be:

$$
\begin{gather*}
\frac{d I\left(t_{m s}\right)}{d t}=I\left(t_{m s}\right) * \frac{n}{\tau_{1}} * \frac{e^{-\frac{t_{m s}}{\tau_{1}}}}{\left(1-e^{-\frac{t_{m s}}{\tau_{1}}}\right.} \\
\frac{d I\left(t_{m s}\right)}{d t}=\frac{I_{m a x}}{\eta} *\left(1-e^{-\frac{t_{m s}}{\tau_{1}}}\right)^{n} * \frac{n}{\tau_{1}} * \frac{e^{-\frac{t_{m s}}{\tau_{1}}}}{\left(1-e^{-\frac{t_{m s}}{\tau_{1}}}\right)} \\
\frac{d I\left(t_{m s}\right)}{d t}=\frac{I_{\max }}{\eta} *\left(1-e^{-\frac{t_{m s}}{\tau_{1}}}\right)^{n-1} * \frac{n}{\tau_{1}} * e^{-\frac{t_{m s}}{\tau_{1}}} \\
\frac{d I\left(t_{m s}\right)}{d t}=\frac{I_{m a x}}{\eta} *\left(1-e^{-\frac{\tau_{1} * \ln (n)}{\tau_{1}}}\right)^{n-1} * \frac{n}{\tau_{1}} * e^{-\frac{\tau_{1} * \ln (n)}{\tau_{1}}} \\
\frac{d I\left(t_{m s}\right)}{d t}=\frac{I_{m a x}}{\eta} *\left(1-\frac{1}{n}\right)^{n-1} * \frac{1}{\tau_{1}} \tag{3-50}
\end{gather*}
$$

The value of the current at the time of occurrence of the maximum steepness:

$$
\begin{gather*}
I\left(t_{m s}\right)=\frac{I_{\max }}{\eta} *\left(1-e^{-\frac{t_{m s}}{\tau_{1}}}\right)^{n} \\
I\left(t_{m s}\right)=\frac{I_{\max }}{\eta} *\left(1-\frac{1}{n}\right)^{n} \tag{3-51}
\end{gather*}
$$

## Chapter 4

## Curve Fitting Technique

### 4.1 Curve Fitting

Data is often given for a discrete value along a continuum. However, estimates may be required at points between the discrete values or mathematical models may be needed to predict values that would have been obtained if the experiment had been conducted using larger values of the independent variables than used in the experiment.

The process of finding a mathematical description that fits a given set of data is known as "Curve Fitting." There are two general approaches for curve fitting that are distinguished from each other on the basis of the amount of error associated with the data.

First, when the data exhibit significant degree of error or noise, in this case the strategy is to derive a curve that represents a general trend of data. Because any individual data point may be incorrect there is no need for the line to intersect with these points. An example of this nature is "Least Square Regression."

Second, where the data is known to be very precise, the basic approach is to fit a curve or a series of curves that pass directly through each of the points. The estimation of values between well-known discrete points is called Interpolation.

In many cases, having a set of data points derived from an experiment where a substantial error is associated with the data, the second approach (interpolation) is not useful and may yield to unsatisfactory results when used to estimate the intermediate points or predict further future points. The lightning return-stroke current measured at the CN Tower is such type of data and the first approach (Least Square Regression) will be used.

### 4.1.1 Least Square Regression

One strategy for fitting a "best" line through the data would be to minimize the sum of the residual for all available data, as in:

$$
\begin{equation*}
\sum_{i=1}^{n}\left(y_{i}-f\left(x_{i}\right)\right) \tag{4-1}
\end{equation*}
$$

Where:
n is the total number of points
$y_{i}$ is the $y$ value of data points corresponding to $x_{i}$
$f\left(x_{i}\right)$ is the value of the model formula evaluated at $\mathrm{x}_{\mathrm{i}}$

However, this is an inadequate criterion, as shown in Figure 4.1 which depicts the fit of a straight line to two points. Obviously the best fit is the line connecting the points. However, any straight line passing thorough the midpoint of the connecting line results in a minimum value of equation 4-1.


Figure 4.1: Example of the criteria for "best fit" based on minimizing the sum of the residual

Therefore, another logical criterion might be to minimize the sum of the absolute values of the discrepancies, as in:

$$
\begin{equation*}
\sum_{i=1}^{n}\left|y_{i}-f\left(x_{i}\right)\right| \tag{4-2}
\end{equation*}
$$

Figure 4.2 demonstrates why this criterion is also inadequate. For the four points shown, any straight line falling within the dashed line will minimize the sum of the absolute values. Thus, this criterion also does not yield a unique best fit.


Figure 4.2: Example of the criteria for "best fit" based on minimizing the sum of the absolute value of the residual

A third strategy for fitting a best line is the minimax criterion. In this technique, the line is chosen to minimize the maximum distance that an individual point falls from the line. As depicted in Figure 4.3, this strategy is illustrated for regression because it gives undue influence to an outlier, that is, a single point with a large error. It should be noted that the minimax principle is sometimes well suited for fitting a sample function as well as a complicated function.

A strategy that overcomes the shortcoming of the aforementioned approaches is to minimize the sum of the squares of the residuals between the measured y and the y calculated.

This criterion has a number of advantages, including the fact that it yields a unique solution for a given set of data.

$$
\begin{equation*}
\sum_{i=1}^{n}\left(y_{i}-f\left(x_{i}\right)\right)^{2} \tag{4-3}
\end{equation*}
$$



Figure 4.3: Example of the criteria for "best fit" based on minimizing the maximum error of any individual point

### 4.1.2 Non Linear Regression

There are many cases in engineering field where nonlinear models must be fitted to the data. In the present context, these models are defined as those that have a nonlinear dependence on their parameters. As with linear least square, nonlinear regression is based on determining the values of the parameters that minimize the sum of the square of the residuals. However, for the nonlinear case the solution must proceed in an iterative way.

The Gauss-Newton method is one algorithm for minimizing the sum for the squares of the residual between data and a nonlinear equation. The key concept underlying the technique is that a Taylor expansion is used to express the original nonlinear equation in an approximate, linear form. Then, least square theory can be used to obtain new estimates of the parameters that move in the direction of minimizing the residual.

To illustrate how this is done, first the relationship between the nonlinear equation and the data can be expressed generally as:

$$
\begin{equation*}
y_{i}=f\left(x_{i}: a_{0}, a_{1}, \ldots \ldots, a_{n}\right)+e_{i} \tag{4-4}
\end{equation*}
$$

Where:
$f\left(x_{i}: a_{0}, a_{1}, \ldots \ldots, a_{n}\right)$ is a function of the independent variable $\mathrm{x}_{\mathrm{i}}$ and nonlinear function of the parameters $a_{0}, a_{1}, \ldots \ldots, a_{n}$.
$e_{i}$ is a random error

The nonlinear model can be expanded in a Taylor series around the parameter values and curtailed after the first derivative:

$$
\begin{equation*}
f\left(x_{i}\right)_{j+1}=f\left(x_{i}\right)_{j}+\frac{\partial f\left(x_{i}\right)_{j}}{\partial a_{0}} \Delta a_{0}+\frac{\partial f\left(x_{i}\right)_{j}}{\partial a_{1}} \Delta a_{1}+\ldots \ldots \ldots \tag{4-5}
\end{equation*}
$$

Where:

$$
\begin{array}{ll}
\mathrm{j} \text { is the initial guess } & \mathrm{j}+1 \text { is the prediction } \\
\Delta a_{0}=a_{0_{j+1}}-a_{0_{j}} & \Delta a_{1}=a_{1_{j+1}}-a_{1_{j}}
\end{array}
$$

Thus, the original model is linearized with respect to the parameters and substituting it on equation 4-4:

$$
\begin{equation*}
y_{i}-f\left(x_{i}\right)_{j}=\frac{\partial f\left(x_{i}\right)_{j}}{\partial a_{0}} \Delta a_{0}+\frac{\partial f\left(x_{i}\right)_{j}}{\partial a_{1}} \Delta a_{1}+\ldots \ldots+e_{i} \tag{4-6}
\end{equation*}
$$

It can be reformatted in the matrix form:

$$
\begin{equation*}
\{D\}=\left[Z_{j}\right]\{\Delta A\}+\{E\} \tag{4-7}
\end{equation*}
$$

Where:
$\left[Z_{j}\right]=\left[\begin{array}{ccc}\frac{\partial f_{1}}{\partial a_{0}} & \frac{\partial f_{1}}{\partial a_{1}} & \frac{\partial f_{1}}{\partial a_{n}} \\ \frac{\partial f_{2}}{\partial a_{0}} & \frac{\partial f_{2}}{\partial a_{1}} & \frac{\partial f_{2}}{\partial a_{n}} \\ \cdot & \cdot & \cdot \\ \frac{\partial f_{n}}{\partial a_{0}} & \frac{\partial f_{n}}{\partial a_{2}} & \frac{\partial f_{n}}{\partial a_{n}}\end{array}\right] \quad\{D\}=\left\{\begin{array}{c}y_{1}-f\left(x_{1}\right) \\ y_{2}-f\left(x_{2}\right) \\ \cdot \\ \cdot \\ y_{n}-f\left(x_{n}\right)\end{array}\right\} \quad\{\Delta A\}=\left\{\begin{array}{c}\Delta a_{0} \\ \Delta a_{1} \\ \cdot \\ \cdot \\ \Delta a_{n}\end{array}\right\}$
$\left[Z_{j}\right]$ is the matrix of partial derivatives of the function evaluated at the guess j .
$\frac{\partial f_{i}}{\partial a_{k}}$ is the partial derivative of the function with respect to $\mathrm{k}^{\text {th }}$ parameter evaluated at the $\mathrm{i}^{\text {th }}$ data point.
$\{D\}$ is a vector contains the differences between the measurements and the function values
$\{\Delta A\}$ contains the changes of the parameter values.

Applying the least square theory on equation 4-7 results in the following normal equation:

$$
\begin{equation*}
\left[\left[Z_{j}\right]^{T}\left[Z_{j}\right]\right]\{\Delta A\}=\left\{\left[Z_{j}\right]^{T}\{D\}\right\} \tag{4-8}
\end{equation*}
$$

Thus, the approach consists of solving equation 4-8 for $\{\Delta A\}$, which can be employed to compute improved values for the parameters, as in:

$$
\begin{aligned}
& a_{0, j+1}=a_{0, j}+\Delta a_{0} \\
& a_{1, j+1}=a_{1, j}+\Delta a_{1}
\end{aligned}
$$

This procedure is repeated until the solution converges and until it falls below an acceptable stopping criterion.

$$
\begin{equation*}
\left|\varepsilon_{a}\right|_{k}=\left|\frac{a_{k, j+1}-a_{k, j}}{a_{k, j+1}}\right| 100 \% \tag{4-9}
\end{equation*}
$$

### 4.1.3 Quality of Fit

In general, Curve fitting using Least Square Regression is a process of finding a mathematical model that fits a given set of data points. The term "Model" in this context means a mathematical formula for a line or curve optimally passes through or near the data points. In Least Square curve fitting, the condition for the optimal solution is the sum of the squares of the difference between the model and the data points is minimized.

After fitting the data with one or more models, the comparison between models should be based on the goodness of the model to the fitted data. There are several ways to investigate the goodness of the fit. But two methods are commonly used which are visual inspection and coefficient of determination or $\mathrm{R}^{2}$.
$R^{2}$ measures how successful the fit is in explaining the variation of the data. It is deifned as the square of the correlation between the response values and the predicted response values. It is also called the square of the multiple correlation coefficient and the coefficient of multiple determination. $\mathrm{R}^{2}$ is defined as the ratio of the sum of squares of the regression (SSR) and the total sum of squares (SST).

$$
\begin{equation*}
R^{2}=\frac{S S R}{S S T}=\frac{\sum_{i=1}^{n}\left(\widehat{y}_{l}-\bar{y}\right)^{2}}{\sum_{i=1}^{n}\left(y_{i}-\bar{y}\right)^{2}} \tag{4-10}
\end{equation*}
$$

Where:
$\widehat{y}_{l}$ is the y value predicted by the model for $\mathrm{x}_{\mathrm{i}}$
$\bar{y}$ is the mean of the observed data $y$ values

R-square can take on any value between 0 and 1 , with a value closer to 1 indicating that a greater proportion of variance is accounted for by the model. For example, an R-square value of 0.8234 means that the fit explains $82.34 \%$ of the total variation in the data about the average.

### 4.2 Modeling Environment

As mentioned before in the previous chapter, there are several functions that can be used to model lightning return-stroke currents for tall structures, some have major problems because the discontinuity in the current derivative at the onset time, such as the double exponential function and its Jones modification. However, other functions like the Pulse and Heidler functions do not suffer from such problems. This makes the Pulse and Heidler functions qualified to simulate the lightning return-stroke current.

A comparison between the Pulse and Heilder Functions is to be performed to determine which function most fit the measured data and can be used to describe the return-stroke current for tall structure. In order to do such comparison, the first step is to build a modeling environment for each function individually which can be described as lightning return-stroke current parameters extraction system for the analytical function. This system which represents an automated approach for extracting the parameters of the analytical functions is capable to best fit the function with the measured data. The estimation of the analytical parameters transforms the discrete data into a continuous signal, from which the current waveform parameters can be calculated. This parameters extraction system is recognized as a curve fitting system.

Although the measured current signal has been used for simulation, the possibility of using the measured current derivative signal for simulation is considered. As the current derivative represents the behavior of how the current changes with time. Any error in the current usually leads to lager error in the current derivative.

### 4.2.1 Artificial Digital Signals

The measured data recorded form the CN Tower exhibit some problems:

- Noise problem: No electric measurement is free of noise that is why measurement systems are equipped with different stages of filters. Even the final digitized data for computerized systems need to have its data filtered.
- Reflection problem: This is due to the discontinuities of the structure where the lightning current flows. The return stroke current will be diluted by reflections,
causing the deformation of the measured current. Example: the reflection from the space deck, the restaurant and the ground base of the CN tower.
- The resolution (accuracy) problem: This is a major problem appearing in any measurement system in any area of research. It can be divided in to two categories: horizontal time resolution and vertical measured data resolution.

Due to the previous problems, uncertainty exists in the modeling environment of each function for producing the best results. So this modeling environment needs to be evaluated. Instead of getting through the measured signal, using artificial signal would be productive to evaluate the modeling environment through which a comparison between different simulation functions could be preformed, allowing further development to produce a tool that can be used to estimate the required current waveform parameters from the measured signal.

The artificial signal can be defined as a signal free of any reflections and noise. It can be produced the same way as in case of the measured signal in which a signal source producing a continuous signal then passing through digitizer with digitization rate same as in the measuring system to produce the artificial signal. The signal source in case of the artificial signal is that produced by the analytical functions with their analytical parameters being set.


Figure 4.4: Artificial signal production simulation for current and current derivative with rate 10 ns

### 4.2.2 Modeling Using Artificial Signals

On one hand, using the system to model the artificial signal with the function by which it was produced, an $R^{2}$ value of 1.0 is achieved for both $R_{c}{ }^{2}\left(R^{2}\right.$ resulting from current signal) and $R_{d}{ }^{2}$ ( $R^{2}$ resulting from current derivative signal). For example, we chose a Pulse function with certain analytical parameters and then we digitized it to create an artificial signal, then we used a Pulse function with unknown parameters to fit the created artificial signal. The curve fitting produced a $100 \% \mathrm{R}^{2}$ accuracy (Fig. 4.5). Similarly, using a Heidler function with known parameters, to create an artificial digital signal and then fit it with a general Heidler function again produced a $100 \% \mathrm{R}^{2}$ accuracy (Fig. 4.6). The fitting was also performed on the derivative of the Pulse function or the derivative of Heidler function using artificial digital current
derivative signals (Figs. 4.5 and 4.6). This means that the system was able to retrieve the exact value of the analytical function parameters from the digitized signal.

Since the artificial signal is formed from the same function being used for fitting and with condition that the minimum number of points in the signal equal to or greater than the number of the function unknowns (4 parameters values in each function), then the exact solution can be obtained. This means the accuracy of the parameters value and the fitting result do not depend on the number of points in the signal being used or the rate of digitization except the previous condition is satisfied. Both the Pulse function and Heidler function give the same accuracy in the result.


Figure 4.5: Curve fitting results using Pulse function artificial signal with $\mathrm{I}=10 \mathrm{kA}, \tau_{1}=0.07 \mu \mathrm{~s}, \tau_{2}=1 \mu \mathrm{~s}$ and $\mathrm{n}=6$


Figure 4.6: Curve fitting results using Heidler function artificial signal with $\mathrm{I}=10 \mathrm{kA}, \tau_{1}=0.1 \mu \mathrm{~s}, \tau_{2}=2 \mu \mathrm{~s}$ and $\mathrm{n}=5$
On the other hand, having the artificial data produced from one function and simulated using the other function represents a real fitting with artificial signal. Figure 4.7 shows the fitting of the pulse function to the artificial current signal produced from Heidler function and the simulation of the corresponding current derivative. It can be concluded that fitting the current produce better accuracy $\left(\mathrm{R}_{\mathrm{c}}{ }^{2}=0.9997\right)$ than fitting the current derivative $\left(\mathrm{R}_{\mathrm{d}}{ }^{2}=0.9954\right)$.


Figure 4.7: Current fitting results using Pulse current function and the corresponding current derivative simulation with Heidler artificial signal of parameters $\mathrm{I}=10 \mathrm{kA}, \tau_{1}=0.1 \mu \mathrm{~s}, \tau_{2}=2 \mu \mathrm{~s}$ and $\mathrm{n}=5$

Performing the same fitting technique on the current derivative is shown in Figure 4.8. The fitting results showed a decrease in comparison with fitting the current $\left(R_{c}{ }^{2}=0.9970\right.$ and $R_{d}{ }^{2}$ $=0.9513$ ). It can be recognized from Fig. 4.8 that the fit to the maximum current derivative was not achieved, which leads to lower current peak.


Figure 4.8: Current derivative fitting results using Pulse current derivative function and the corresponding current simulation with Heidler artificial signal of parameters $\mathrm{I}=10 \mathrm{kA}, \tau_{1}=0.1 \mu \mathrm{~s}, \tau_{2}=2 \mu \mathrm{~s}$ and $\mathrm{n}=5$

The current derivative curve fitting results can be improved by setting a better initial value for each parameter using the values of the parameters obtained from the current fitting. Figure 4.9, in addition to $\mathrm{R}_{\mathrm{c}}{ }^{2}=0.9993$ and $\mathrm{R}_{\mathrm{d}}{ }^{2}=0.99649$, shows the best fit for the current derivative. Table 1 presents the results of the different fitting approaches of the Pulse function including fitting of the current, current derivative, current derivative with its parameters being adjusted along the value of the parameters that is used to set the Heidler function to produce the artificial signal.

In case of the current derivative fitting with the initial values being adjusted, the error in the estimated parameters value and the value from current fitting is small. On one hand, this can be understood as a fine tuning for the results obtained from current fitting. On the other hand, the current fitting can be used to evaluate the nearest initial values of the parameters for current derivative fitting. It can be concluded that when fitting the current derivative, the corresponding current simulation becomes very reasonable and the R-square for the current is high while the
reverse process is not correct. Also, the pulse function derivative fitting is proved to be very sensitive to the initial value of its parameters.


Figure 4.9: Current derivative fitting results using Pulse current derivative function and the corresponding current simulation with Heidler artificial signal of parameters $\mathrm{I}=10 \mathrm{kA}, \tau_{1}=0.1 \mu \mathrm{~s}, \tau_{2}=2 \mu \mathrm{~s}$ and $\mathrm{n}=5$

Table 4.1: Simulation results of the current, current derivative and current derivative with adjusted initial value of the parameters using the Pulse function

|  | $\mathrm{I}(\mathrm{kA})$ | $\tau_{1}(\mu \mathrm{~s})$ | $\tau_{2}(\mu \mathrm{~s})$ | n | $\mathrm{R}_{\mathrm{c}}$ | $\mathrm{R}_{\mathrm{d}}$ |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| Heidller (Artificial signal with <br> 10 ns digitization rate) | 10 | 0.1 | 2 | 5 | ------ | ----- |
| Pulse function current fitting | 10.024 | 0.02892 | 2.126 | 20.98 | 0.99997 | 0.99537 |
| Pulse function current <br> derivative fitting | 9.6 | 0.03834 | 1.729 | 9.417 | 0.99702 | 0.95139 |
| Pulse function current <br> derivative fitting with current <br> fitting result initial condition | 9.8 | 0.02835 | 2.103 | 23.24 | 0.99929 | 0.99649 |

In case of using the pulse function to produce the artificial signal and Heidler function represent the fitting function, a similar behavior is obtained. For the current fitting shown in Figure 4.10, the resultant $R_{c}$ is 0.9999 and $R_{d}$ is 0.9946 . For the current derivative fitting shown in Figure 4.11, the resultant $R_{c}$ is 0.9969 and $R_{d}$ is 0.9979 . The values of the parameters for the current and current derivative fitting are shown in Table 4.2. On using the analytical parameters values obtained from the current fitting to set their initial values for the current derivative fitting, it gives the same result as obtained from the current normal derivative fitting. Heidler function current derivative fitting is not sensitive to the initial values of the parameters, which makes the Heidler function derivative fit the signal much easier than using the Pulse function derivative.


Figure 4.10: Current fitting results using Heidler current derivative function and the corresponding current simulation with Pulse artificial signal of parameters $\mathrm{I}=10 \mathrm{kA}, \tau_{1}=0.07 \mu \mathrm{~s}, \tau_{2}=1 \mu \mathrm{~s}$ and $\mathrm{n}=6$

We can conclude form all the previous simulating results, including the use of the artificial signals, that the quality of the fit for the simulated current and the current derivative in case of fitting the current derivative is the best, resulting in highest RD and reasonably high RC. Also, the Pulse function is found to be very sensitive to the initial point of the parameters while Heidler function is not for current derivative fitting


Figure 4.11: Current derivative fitting results using Heidler current function and the corresponding current simulation with Pulse artificial signal of parameters $I=10 \mathrm{kA}, \tau_{1}=0.07 \mu \mathrm{~s}, \tau_{2}=1 \mu \mathrm{~s}$ and $\mathrm{n}=6$

Table 4.2: Simulation results of the current and current derivative using Heidler function

|  | $\mathrm{I}(\mathrm{kA})$ | $\tau_{1}$ (us) | $\tau_{2}$ (us) | n | $\mathrm{R}_{\mathrm{c}}$ | $\mathrm{R}_{\mathrm{d}}$ |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| Pulse (Artificial signal <br> with 10 ns digitization <br> rate) | 10 | 0.07 | 1 | 6 | ------ | ----- |
| Heidler function <br> current fitting | 9.98 | 0.1606 | 0.9125 | 3.206 | 0.9999 | 0.9946 |
| Heidler function <br> current derivative <br> fitting | 10.053 | 0.1593 | 0.9304 | 3.168 | 0.9969 | 0.9979 |

### 4.3 Initial Conditions

The parameters extraction system is recognized as a curve fitting system. It uses four analytical function parameters to simulate the measured signal. For any curve fitting technique the initial value of each parameter and the feasible region where its optimal value exists should be specified. The more accurate the initial value accompanied by tighter feasible region, the easier and faster the optimal value can be estimated. In order to do so, it is very important when looking for the initial value of each parameter to make it independent on the rest of the parameters as any small error in independent parameters lead to significant error in the dependant parameters.

As defined in the previous chapter, Heidler function analytical parameters are the same as those for the Pulse function. These parameters are: $I_{\max }, \tau_{1}, \tau_{2}$ and $n$. The initial value of $I_{\max }$ for both simulating functions can be estimated using the measured current directly, in which the peak is its value. The initial value of the rest of the parameters can be deduced using the current functions for both Heidler and the Pulse functions.

Since the current fitting leads to parameter values very near to the optimal ones, in addition to its simplicity, the use of the current function is highly recommended for the initial value search. Also, the approximation by equating the decay part $y(t)$ to unity during the rise portion produces a separation, which helps in finding the initial values of the parameters $\tau_{1}$ and $n$ from $\tau_{2}$.

### 4.3.1 Initial Conditions for the Pulse Function

In case of the Pulse function, we use the approximated current equation 3-46:

$$
I(t)=\frac{I_{\max }}{\eta} *\left(1-e^{-\frac{t}{\tau_{1}}}\right)^{n}
$$

Although the value of $I_{\text {max }}$ is known from the measured current and knowing two other data points is in principal enough for calculating $\tau_{1}$ and $n$, but for better accuracy three points will be used to eliminate $I_{\max }$ and $\eta$ from the equations. The three points should be chosen such that they occur before the maximum current for the approximation to be applicable.

$$
\begin{align*}
& I\left(t_{1}\right)=\frac{I_{\max }}{\eta} *\left(1-e^{-\frac{t_{1}}{\tau_{1}}}\right)^{n}  \tag{4-11}\\
& I\left(t_{2}\right)=\frac{I_{\max }}{\eta} *\left(1-e^{-\frac{t_{2}}{\tau_{1}}}\right)^{n}  \tag{4-12}\\
& I\left(t_{3}\right)=\frac{I_{\max }}{\eta} *\left(1-e^{-\frac{t_{3}}{\tau_{1}}}\right)^{n} \tag{4-13}
\end{align*}
$$

Dividing equation (4-11) over (4-12) to eliminate $I_{\max }$ and $\eta$ :

$$
\begin{equation*}
\frac{I\left(t_{1}\right)}{I\left(t_{2}\right)}=\left(\frac{1-e^{-\frac{t_{1}}{\tau_{1}}}}{1-e^{-\frac{t_{2}}{\tau_{1}}}}\right)^{n} \tag{4-14}
\end{equation*}
$$

The same procedure is done for equation (4-12) and (4-13) to eliminate $I_{\max }$ and $\eta$ :

$$
\begin{equation*}
\frac{I\left(t_{2}\right)}{I\left(t_{3}\right)}=\left(\frac{1-e^{-\frac{t_{2}}{\tau_{1}}}}{1-e^{-\frac{t_{3}}{\tau_{1}}}}\right)^{n} \tag{4-15}
\end{equation*}
$$

Having two equations in two unknowns, simplification is needed to eliminate one of the parameters. This can be done by taking normal logarithmic for both sides of both equations (414) and (4-15):

$$
\begin{align*}
& \ln \left(\frac{I\left(t_{1}\right)}{I\left(t_{2}\right)}\right)=n * \ln \left(\frac{1-e^{-\frac{t_{1}}{\tau_{1}}}}{1-e^{-\frac{t_{2}}{\tau_{1}}}}\right)  \tag{4-16}\\
& \ln \left(\frac{I\left(t_{2}\right)}{I\left(t_{3}\right)}\right)=n * \ln \left(\frac{1-e^{-\frac{t_{2}}{\tau_{1}}}}{1-e^{-\frac{t_{3}}{\tau_{1}}}}\right) \tag{4-17}
\end{align*}
$$

Divide (4-16) by (4-17) to eliminate $n$ :

$$
\begin{equation*}
\frac{\ln \left(\frac{I\left(t_{1}\right)}{I\left(t_{2}\right)}\right)}{\ln \left(\frac{I\left(t_{2}\right)}{I\left(t_{3}\right)}\right)}=\frac{\ln \left(\frac{1-e^{-\frac{t_{1}}{\tau_{1}}}}{1-e^{-\frac{t_{2}}{\tau_{1}}}}\right)}{\ln \left(\frac{1-e^{-\frac{t_{2}}{\tau_{1}}}}{1-e^{-\frac{t_{3}}{\tau_{1}}}}\right)} \tag{4-18}
\end{equation*}
$$

Equation (4-18) represents one equation in one unknown $\tau_{1}$. Since it does not have a general solution for $\tau_{1}$, $\tau_{1}$ can be calculated numerically. Substituting $\tau_{1}$ in equation (4-16), n will be in the form of:

$$
\begin{equation*}
n=\frac{\ln \left(\frac{I\left(t_{1}\right)}{I\left(t_{2}\right)}\right)}{\ln \left(\frac{1-e^{-\frac{t_{1}}{\tau_{1}}}}{1-e^{-\frac{t_{2}}{\tau_{1}}}}\right)} \tag{4-19}
\end{equation*}
$$

Using the same procedure for estimating $\tau_{2}$ by applying the original Pulse function equation (3-40) on two data points where the data points time of occurrence should be after the peak of the current time of occurrence during the decay portion.

$$
\begin{align*}
& I\left(t_{4}\right)=\frac{I_{\max }}{\eta} *\left(1-e^{-\frac{t_{4}}{\tau_{1}}}\right)^{n} * e^{-\frac{t_{4}}{\tau_{2}}}  \tag{4-20}\\
& I\left(t_{5}\right)=\frac{I_{\max }}{\eta} *\left(1-e^{-\frac{t_{5}}{\tau_{1}}}\right)^{n} * e^{-\frac{t_{5}}{\tau_{2}}} \tag{4-21}
\end{align*}
$$

Dividing the previous two equations to eliminate the $I_{\max }$ and $\eta$ :

$$
\begin{equation*}
\frac{I\left(t_{4}\right)}{I\left(t_{5}\right)}=\left(\frac{1-e^{-\frac{t_{4}}{\tau_{1}}}}{1-e^{-\frac{t_{5}}{\tau_{1}}}}\right)^{n} * \frac{e^{-\frac{t_{4}}{\tau_{2}}}}{e^{-\frac{t_{5}}{\tau_{2}}}} \tag{4-22}
\end{equation*}
$$

Simplify the equation (4-22) to get $\tau_{2}$ in terms of the measured data points and the calculated other parameters $\tau_{1}$ and $n$ :

$$
\begin{align*}
& \frac{I\left(t_{4}\right)}{I\left(t_{5}\right)}=\left(\frac{1-e^{-\frac{t_{4}}{\tau_{1}}}}{1-e^{-\frac{t_{5}}{\tau_{1}}}}\right)^{n} * e^{\frac{t_{5}-t_{4}}{\tau_{2}}}  \tag{4-23}\\
& e^{\frac{t_{5}-t_{4}}{\tau_{2}}}=\frac{\frac{I\left(t_{4}\right)}{I\left(t_{5}\right)}}{\left(\frac{1-e^{-\frac{t_{4}}{\tau_{1}}}}{1-e^{-\frac{t_{5}}{\tau_{1}}}}\right)^{n}}  \tag{4-24}\\
& \frac{t_{5}-t_{4}}{\tau_{2}}=\ln \left(\frac{\frac{I\left(t_{4}\right)}{I\left(t_{5}\right)}}{\left(\frac{1-e^{-\frac{t_{4}}{\tau_{1}}}}{1-e^{-\frac{t_{5}}{\tau_{1}}}}\right)^{n}}\right)  \tag{4-25}\\
&\left.\tau_{2}=\frac{\left.\ln \left(\frac{t_{5}-t_{4}}{\left(\frac{1\left(t_{4}\right)}{I\left(t_{5}\right)}\right.}\right)_{\left.1-e^{-\frac{t_{4}}{\tau_{1}}}\right)^{-\frac{t_{5}}{\tau_{1}}}}^{1-}\right)}{( }\right) \tag{4-26}
\end{align*}
$$

The technique used to estimate the initial value of the function analytical parameters needs to be evaluated to make sure that the approximation used does not affect significantly the initial values of the estimated parameters. It will also evaluate the technique used in terms of its deviation from the optimum value through which the feasible region can be determined.

This evaluation is done using the artificial signal (produced by the Pulse function). Two types of evaluation will be carried out for each of the three parameters $\tau_{1}, \tau_{2}$ and $n$. The first type is by studying the relative error in the value of estimated parameter while changing its value as
shown in Figure 4.12. The second type is by studying the relative error in the value of estimated parameter while changing the values of the other two parameters as shown in Figures 4.13-4.15.


Figure 4.12: Relative error value for the estimated values of $\tau_{1}, \tau_{2}$ and $n$ with the change of its value individually using the Pulse function initial condition estimation

It can be concluded from the previous figure that the values of the relative error in the estimation of $\tau_{2}$ are relatively small compared to the relative error in each of $\tau_{1}$ and $n$. Also, the values of the relative error due to the estimation of parameters vary proportionally with the variation of the parameters.


Figure 4.13: Relative error value for the estimated values of $n$ with the change of $\tau_{1}$ and $\tau_{2}$ using the Pulse function initial condition estimation


Figure 4.14: Relative error value for the estimated values of $\tau_{2}$ with the change of $\tau_{1}$ and $n$ using the Pulse function initial condition estimation


Figure 4.15: Relative error value for the estimated values of $\tau_{1}$ with the change of $\tau_{2}$ and $n$ using the Pulse function initial condition estimation

For studying the effect of varying $\tau_{1}$ and $\tau_{2}$ on the relative error of $n$, as shown in Figure 4.13, the value of the relative error of n is highly dependent on the changing of $\tau_{2}$ while changing $\tau_{1}$ does not have significant effect. For studying the effect of varying $\tau_{1}$ and $n$ on the relative error of $\tau_{2}$, as shown in Figure 4.14, the value of the relative error of $\tau_{2}$ is dependent on the change in n while changing $\tau_{1}$ does not have any effect. Finally, for studying the effect of varying n and $\tau_{2}$ on the relative error of $\tau_{1}$, as shown in Figure 4.15, the value of the relative error of $\tau_{1}$ is dependent on n and highly dependent on $\tau_{2}$.

### 4.3.2 Initial Conditions for Heidler Function

In case of the Heidler function, we use the approximated current equation 3-31:

$$
I(t)=\frac{I_{\max }}{\eta} * \frac{\left(\frac{t}{\tau_{1}}\right)^{n}}{1+\left(\frac{t}{\tau_{1}}\right)^{n}}
$$

Although the value of $\mathrm{I}_{\text {max }}$ known from the measured current and knowing two other data points is in principle enough for calculating $\tau_{1}$ and $n$, but for better accuracy three points will be used to eliminate $I_{\max }$ and $\eta$ from the equations:

$$
\begin{align*}
& I\left(t_{1}\right)=\frac{I_{\max }}{\eta} * \frac{\left(\frac{t_{1}}{\tau_{1}}\right)^{n}}{1+\left(\frac{t_{1}}{\tau_{1}}\right)^{n}}  \tag{4-27}\\
& I\left(t_{2}\right)=\frac{I_{\max }}{\eta} * \frac{\left(\frac{t_{2}}{\tau_{1}}\right)^{n}}{1+\left(\frac{t_{2}}{\tau_{1}}\right)^{n}}  \tag{4-28}\\
& I\left(t_{3}\right)=\frac{I_{\max }}{\eta} * \frac{\left(\frac{t_{3}}{\tau_{1}}\right)^{n}}{1+\left(\frac{t_{3}}{\tau_{1}}\right)^{n}} \tag{4-29}
\end{align*}
$$

Divide equation (4-27) by (4-28) to eliminate $I_{\max }$ and $\eta$ :

$$
\begin{equation*}
\frac{I\left(t_{1}\right)}{I\left(t_{2}\right)}=\left(\frac{t_{1}}{t_{2}}\right)^{n} * \frac{\tau_{1}{ }^{n}+t_{2}{ }^{n}}{\tau_{1}{ }^{n}+t_{1}{ }^{n}} \tag{4-30}
\end{equation*}
$$

Simplify this equation to get single parameter in one side and the other parameters along with the data on the other side:

$$
\begin{equation*}
\frac{I\left(t_{1}\right)}{I\left(t_{2}\right)} *\left(\frac{t_{2}}{t_{1}}\right)^{n}=\frac{\tau_{1}^{n}+t_{2}{ }^{n}}{\tau_{1}{ }^{n}+t_{1}{ }^{n}} \tag{4-31}
\end{equation*}
$$

$$
\begin{gather*}
\frac{I\left(t_{1}\right)}{I\left(t_{2}\right)} *\left(\frac{t_{2}}{t_{1}}\right)^{n} *\left(\tau_{1}^{n}+t_{1}^{n}\right)=\tau_{1}^{n}+t_{2}^{n}  \tag{4-32}\\
\tau_{1}^{n} *\left(\frac{I\left(t_{1}\right)}{I\left(t_{2}\right)} *\left(\frac{t_{2}}{t_{1}}\right)^{n}-1\right)=t_{2}^{n} *\left(1-\frac{I\left(t_{1}\right)}{I\left(t_{2}\right)}\right)  \tag{4-33}\\
\tau_{1}^{n} *\left(\frac{I\left(t_{1}\right)}{t_{1}^{n}}-\frac{I\left(t_{2}\right)}{t_{2}^{n}}\right)=I\left(t_{2}\right)-I\left(t_{1}\right)  \tag{4-34}\\
\tau_{1}^{n}=\frac{I\left(t_{2}\right)-I\left(t_{1}\right)}{\frac{I\left(t_{1}\right)}{t_{1}^{n}}-\frac{I\left(t_{2}\right)}{t_{2}^{n}}} \tag{4-35}
\end{gather*}
$$

The separation between the $\tau_{1}$ and n is achieved in equation (4-35). The same technique carried out, divide equation (4-28) by (4-29) and simplify to get the second equation of $\tau_{1}$ and $n$ :

$$
\begin{equation*}
\tau_{1}{ }^{n}=\frac{I\left(t_{3}\right)-I\left(t_{2}\right)}{\frac{I\left(t_{2}\right)}{t_{2}^{n}}-\frac{I\left(t_{3}\right)}{t_{3}^{n}}} \tag{4-36}
\end{equation*}
$$

Having two equations in two unknowns, simplification is needed to eliminate one of the parameters. This can be done by taking normal logarithmic for both sides of both equations (4$35)$ and (4-36) then dived both equations to eliminate $n$ :

$$
\begin{equation*}
\frac{I\left(t_{2}\right)-I\left(t_{1}\right)}{\frac{I\left(t_{1}\right)}{t_{1}{ }^{n}}-\frac{I\left(t_{2}\right)}{t_{2}{ }^{n}}}=\frac{I\left(t_{3}\right)-I\left(t_{2}\right)}{\frac{I\left(t_{2}\right)}{t_{2}^{n}}-\frac{I\left(t_{3}\right)}{t_{3}{ }^{n}}} \tag{4-37}
\end{equation*}
$$

Simplify the previous equation:

$$
\begin{equation*}
\frac{I\left(t_{3}\right)-I\left(t_{2}\right)}{I\left(t_{2}\right)-I\left(t_{1}\right)}=\frac{\frac{I\left(t_{2}\right)}{t_{2}^{n}}-\frac{I\left(t_{3}\right)}{t_{3}{ }^{n}}}{\frac{I\left(t_{1}\right)}{t_{1}^{n}}-\frac{I\left(t_{2}\right)}{t_{2}^{n}}} \tag{4-38}
\end{equation*}
$$

$$
\begin{gather*}
\frac{I\left(t_{2}\right)-I\left(t_{3}\right)}{I\left(t_{2}\right)-I\left(t_{1}\right)}=\frac{\frac{I\left(t_{2}\right)}{t_{2}{ }^{n}}-\frac{I\left(t_{3}\right)}{t_{3}{ }^{n}}}{\frac{I\left(t_{2}\right)}{t_{2}{ }^{n}}-\frac{I\left(t_{1}\right)}{t_{1}{ }^{n}}}  \tag{4-39}\\
\frac{1-\frac{I\left(t_{3}\right)}{I\left(t_{2}\right)}}{1-\frac{I\left(t_{1}\right)}{I\left(t_{2}\right)}}=\frac{1-\frac{I\left(t_{3}\right)}{I\left(t_{2}\right)} *\left(\frac{t_{2}}{t_{3}}\right)^{n}}{1-\frac{I\left(t_{1}\right)}{I\left(t_{2}\right)} *\left(\frac{t_{2}}{t_{1}}\right)^{n}} \tag{4-40}
\end{gather*}
$$

The previous equation represents one equation with one unknown n. Since it does not have a general solution for n , n can be calculated numerically. Substituting in the n on of the previous equation, $\tau_{1}$ will be in the form of:

$$
\begin{equation*}
\tau_{1}=\sqrt[n]{\frac{I\left(t_{2}\right)-I\left(t_{1}\right)}{\frac{I\left(t_{1}\right)}{t_{1}^{n}}-\frac{I\left(t_{2}\right)}{t_{2}^{n}}}} \tag{4-41}
\end{equation*}
$$

Using the same procedure for finding $\tau_{2}$ by applying the original current Heidler function equation (3-21) on two data points where the data points time of occurrence should be after the peak current time of occurrence during the decay portion.

$$
\begin{align*}
& I\left(t_{4}\right)=\frac{I_{\max }}{\eta} * \frac{\left(\frac{t_{4}}{\tau_{1}}\right)^{n}}{1+\left(\frac{t_{4}}{\tau_{1}}\right)^{n}} * e^{-\frac{t_{4}}{\tau_{2}}}  \tag{4-42}\\
& I\left(t_{5}\right)=\frac{I_{\max }}{\eta} * \frac{\left(\frac{t_{5}}{\tau_{1}}\right)^{n}}{1+\left(\frac{t_{5}}{\tau_{1}}\right)^{n}} * e^{-\frac{t_{5}}{\tau_{2}}} \tag{4-43}
\end{align*}
$$

Dividing the two equations to eliminate the $I_{\max }$ and $\eta$ :

$$
\begin{equation*}
\frac{I\left(t_{4}\right)}{I\left(t_{5}\right)}=\frac{\left(\frac{t_{4}}{\tau_{1}}\right)^{n} *\left(1+\left(\frac{t_{5}}{\tau_{1}}\right)^{n}\right)}{\left(1+\left(\frac{t_{4}}{\tau_{1}}\right)^{n}\right) *\left(\frac{t_{5}}{\tau_{1}}\right)^{n}} * \frac{e^{-\frac{t_{4}}{\tau_{2}}}}{e^{-\frac{t_{5}}{\tau_{2}}}} \tag{4-44}
\end{equation*}
$$

Simplify the previous equation to get $\tau_{2}$ in terms of the measured data points and the calculated other parameters $\tau_{1}$ and n :

$$
\begin{gather*}
\frac{I\left(t_{4}\right)}{I\left(t_{5}\right)}=\left(\frac{t_{4}}{t_{5}}\right)^{n} *\left(\frac{\tau_{1}^{n}+t_{5}^{n}}{\tau_{1}^{n}+t_{4}{ }^{n}}\right) * e^{\frac{t_{5}-t_{4}}{\tau_{2}}}  \tag{4-45}\\
e^{\frac{t_{5}-t_{4}}{\tau_{2}}}=\frac{I\left(t_{4}\right)}{I\left(t_{5}\right)} *\left(\frac{\tau_{1}^{n}+t_{4}{ }^{n}}{\tau_{1}^{n}+t_{5}{ }^{n}}\right) *\left(\frac{t_{5}}{t_{4}}\right)^{n}  \tag{4-46}\\
\frac{t_{5}-t_{4}}{\tau_{2}}=\ln \left(\frac{I\left(t_{4}\right)}{I\left(t_{5}\right)} *\left(\frac{\tau_{1}^{n}+t_{4}{ }^{n}}{\tau_{1}^{n}+t_{5}{ }^{n}}\right) *\left(\frac{t_{5}}{t_{4}}\right)^{n}\right)  \tag{4-47}\\
\tau_{2}=\frac{t_{5}-t_{4}}{\ln \left(\frac{I\left(t_{4}\right)}{I\left(t_{5}\right)} *\left(\frac{\tau_{1}{ }^{n}+t_{4}{ }^{n}}{\tau_{1}{ }^{n}+t_{5}{ }^{n}}\right) *\left(\frac{t_{5}}{t_{4}}\right)^{n}\right)} \tag{4-48}
\end{gather*}
$$

The evaluation is done using the artificial signal (Produced by Heidler function). The same two types of evaluation done for the initial value of the analytical parameters of the Pulse function will be repeated for the analytical parameters of Heidler function. The first type results are shown in Figure 4.16. The second type results are shown in Figures 4.17, 4.18 and 4.19.


Figure 4.16: Relative error value for the estimated values of $\tau_{1}, \tau_{2}$ and $n$ with the change of its value individually using Heidler function initial condition estimation

It can be concluded from the previous figure that the values of the relative error in estimation of $\tau_{2}$ are negligible. It is clear that the relative error due to the estimation of n is inversely proportional with the change in n , while the relative error due to the estimation of $\tau_{1}$ is directly proportional with the change in $\tau_{1}$. However, the relative error in $\tau_{2}$ follows a non uniform behavior with the change in $\tau_{2}$.


Figure 4.17: Relative error value for the estimated values of $n$ with the change of $\tau_{1}$ and $\tau_{2}$ using Heidler function initial condition estimation


Figure 4.18: Relative error value for the estimated values of $\tau_{2}$ with the change of $\tau_{1}$ and $n$ using Heidler function initial condition estimation


Figure 4.19: Relative error value for the estimated values of $\tau_{1}$ with the change of $\tau_{2}$ and $n$ using Heidler function initial condition estimation

For studying the effect of varying $\tau_{1}$ and $\tau_{2}$ on the relative error of n , as shown in Figure 4.17, the value of the relative error of n is inversely proportional to the change of $\tau_{2}$ while directly proportional with the change of $\tau_{1}$. For studying the effect of varying $\tau_{1}$ and n on the relative error of $\tau_{2}$, as shown in Figure 4.18, the value of the relative error of $\tau_{2}$ is directly proportional with high values of $\tau_{1}$ and inversely proportional to low values of n and is constant otherwise. Finally, In the case of studying the effect of varying $n$ and $\tau_{2}$ on the relative error of $\tau_{1}$ as shown in Figure 4.19, the value of the relative error of $\tau_{1}$ is inversely proportional to the change in both n and $\tau_{2}$.

It is very clear that the relative error value for the estimated initial value of the parameters is very small and acceptable for both functions. Also the effect of changing other parameters on the relative error of the initial value of the estimated parameter vanishes as the parameters value increases. In general the approximation used to estimate the initial values of the parameters is acceptable.

## Chapter 5

## Results \& Discussions

In Chapter 4, the initial value estimation of the analytical parameters of the functions and the idea of using the artificial digital signals for evaluating the analytical parameters estimation system were discussed. The evaluation of the initial value estimation process has proved the success of the used method and showed how near the initial values of the analytical parameters is with respect to the optimal ones. Since the initial value estimation process resulted in analytical parameters values very near to the optimal one and the least square technique is used to concise this value to the optimal, then the system can be applied on the measured signal.

As discussed before, the measured signal has some problems, but the major ones are the reflections due to the discontinuities in the tower structure and the limited duration of the measured signal, is less than the duration of the full signal. Previously, researches used to study only the first impulse of the measured current derivative signal just before the current peak, as shown in Figure 5.1, before the arrival of reflections in the measured signal. The first current derivative impulse represents the rising edge of the current waveform. This led to some limitations in calculating the current waveform parameters, specially the decay time, the current pulse width at half peak level and the charge. In order to calculate all the current waveform parameters, the fitting function should fit the first impulse of the derivative and should have the same decay behavior as in the measured current signal.


Figure 5.1: First impulse of the measured signal
Due to the existence of the mentioned problems, it is necessary when judging the quality of the fit to use some guidelines along with $\mathrm{R}^{2}$. These guidelines are the maximum steepness time of occurrence, zero crossing time of the current derivative and the peak value of the current waveform. Applying the initial value estimation for the analytical parameters along with the least square curve fitting method and using both functions as simulating functions, keeping in mind the guidelines and $\mathrm{R}^{2}$ for judging the quality of the fit, the comparison between the Pulse function and Heidler function can be performed on fitting the current derivative signal measured at the CN Tower through which the current waveform parameters can be calculated.

A complete analysis is carried out for each function individually to understand its behavior while fitting the measured signal. The effect of the reflections on the fitting process has been investigated while applying a set of conditions on the fitting process until the best fitting is achieved. The last step after achieving the best fitting for each function is the comparison to determine which one fits better the CN Tower measured signal, and then use the better function to calculate the current waveform parameters.

Figures 5.2 and 5.3 show the result of applying the derivative of the Pulse and Heidler functions to fit the current derivative measured signal at the CN Tower and the simulation of the corresponding current with the integration of the measured signal. The initial values of the analytical parameters, the values of the analytical parameters after the fitting process and the corresponding $\mathrm{R}^{2}$ are listed in Table 5.1. It is clear that the initial values of $\mathrm{I}_{\max }, \tau_{1}$ and n are far from the final values.

Using the curve fitting process, it is very clear that the maximum steepness value and time of occurrence have not been achieved, the zero crossing is shifted away which leads to wrong risetime value. Consequently the guidelines for judging the quality of the fit were not fulfilled. Also the decay of the fitted signal is very slow with respect to the measured signal which leads to very large decay time and charge on calculating from the fitting functions. As a result, the fitting process needs to be improved for both functions.


Figure 5.2: Fitting results of the Pulse and Heidler functions for the measured current derivative signal and the corresponding current


Figure 5.3: Zoomed fitting result of the Pulse and Heidler functions for the measured current derivative signal and the corresponding current

Table 5.1: Parameters initial and final values for the Pulse and Heidler functions fitting process and the associated $\mathrm{R}^{2}$

|  |  | $\mathrm{I}_{\max }(\mathrm{kA})$ | $\tau_{1}(\mu \mathrm{~s})$ | $\tau_{2}(\mathrm{~ms})$ | n | $\mathrm{R}^{2}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Pulse | Initial Value | 9.388 | 0.1038 | 0.1274 | 235 | 0.6706 |
|  | Fitting Results | 7.911 | 0.09607 | 0.1759 | 368.2 |  |
|  | Initial Value | 9.94455 | 0.6762 | 0.1134 | 1.931 | 0.6877 |
|  | Fitting Results | 7.845 | 0.5685 | 0.207 | 10.81 |  |

### 5.1 Estimating $\boldsymbol{\tau}_{2}$

As explained in Chapter 3, the structure of Heidler function and the Pulse function is divided into current rise portion $x(t)$ and current decay portion $y(t)$ as in equation (3-14). By studying the effect of the rise portion $x(t)$ for both functions when $t \gg \tau_{1}$, during the decay of the signal, the functions seem to be insensitive to the rise portion $(x(t) \approx 1)$. For further investigation, Figure 5.4 shows the simulation of the rising portion of the function using parameters values from Table 4.2. It is very clear that the rising portion of the function saturates just after reaching the peak of the current. It reaches $99 \%$ of the maximum value at $0.6 \mu$ s for the Pulse function and at $0.65 \mu \mathrm{~s}$ for Heidler function. In this case the decay portion of the function will be the dominating part.


Figure 5.4: Fitting the rising portion of the Pulse and Heidler functions

During the decay of the signal, The fitting functions will be simplified to:

$$
\begin{equation*}
I(t)=A * e^{-\frac{t}{\tau_{2}}} \tag{5-1}
\end{equation*}
$$

where:
A : Magnitude factor
$\tau_{2}$ : Time constant for determination of the decay time
The current derivative is:

$$
\begin{equation*}
\frac{d I(t)}{d t}=-\frac{A}{\tau_{2}} * e^{-\frac{t}{\tau_{2}}} \tag{5-2}
\end{equation*}
$$

The decay time constant $\tau_{2}$ can be estimated separately from other parameters. In order to do so the measured signal will be divided into three parts. The first part is the initial impulse of the measured signal before the arrival of reflections; the second part is the decaying part of the measurment including the reflections and finally the third part which is less affected by the reflections of the tower discontinuities in comparison to the second part as shown in Figure 5.5.


Figure 5.5: Classifications of the measured current signal parts

The current derivative measured signal contains high frequency noise, which results in the failure of estimating $\tau_{2}$ from the current derivative directly. Therefore, we use the current, which is the integration of the measured current derivative signal, for the estimation of $\tau_{2}$. The process of integrating the current derivative signal filters the high frequency noise without affecting the decay behavior of the current. Using this method to estimate $\tau_{2}$ reduces the unknown analytical parameters of the function to three, namely $\tau_{1}, \mathrm{n}$ and $\mathrm{I}_{\text {max }}$.

Applying the curve fitting process on the decay part of the signal using the fitting function of equation (5-1), A and $\tau_{2}$ can be estimated. Figure 5.6 shows the result of the fitting process with $\tau_{2}$ value of 0.1239 ms and A value of 9.72 kA . The magnitude factor A is not important because its value is affected by the reflection while $\tau_{2}$ represents the decay behavior of the current, which is the integration of the measured signal.


Figure 5.6: Simulation of the fitting result of the Pulse and Heidler functions on the integration of the measured signal decay part (current decay part)

The rest of the parameters can be estimated by applying the simulating functions on the first impulse of the current derivative as shown in Figures 5.7 and 5.8. The decay part for both
functions has been substantially improved. For the Pulse function, the current derivative fitting is acceptable but the corresponding current has lower values, the peak current is not achieved and its time of occurrence is shifted. For Heidler function, although the current derivative fitting is not as good as the Pulse function but the corresponding current is much better than that of the Pulse function. The problem in Heidler function fitting is that the value of the peak current and its time of occurrence have not been simultaneously achieved.

Table 5.2 contains the initial value of the analytical parameters and the value of the analytical parameters after the fitting process and the corresponding $R^{2}$. It is very important to note that the value of $\tau_{2}$ is fixed before and after the fitting process of the first impulse as it was obtained from the corresponding decay part of the signal. Using the value of $\tau_{2}$ in the estimation of the initial value of other analytical parameters initial value yields very accurate initial values for $\tau_{1}$ and $\mathrm{I}_{\mathrm{max}}$, close to the optimal one.


Figure 5.7: Simulation of fitting result of the Pulse and Heidler functions for the measured current derivative signal with fixed $\tau_{2}$ and the corresponding current signal


Figure 5.8: Zoomed simulation of fitting result of the Pulse and Heidler functions for the measured current derivative signal with fixed $\tau_{2}$ and the corresponding current signal

Table 5.2: Parameters' initial and final values for the Pulse and Heidler functions fitting process with fixed $\tau_{2}$ and the associated $\mathrm{R}^{2}$

|  |  | $\mathrm{I}_{\max }(\mathrm{kA})$ | $\tau_{1}(\mu \mathrm{~s})$ | $\tau_{2}(\mathrm{~ms})$ | n | $\mathrm{R}^{2}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Pulse | Initial Value | 7.6 | 0.07679 | 0.1239 | 876.3 | 0.9006 |
|  | Fitting Results | 7.5 | 0.07775 | 0.1239 | 1108 |  |
|  | Initial Value | 7.586 | 0.559 | 0.1239 | 11.54 | 0.0 .8963 |
|  | Fitting Results | 7.667 | 0.5684 | 0.1239 | 10.83 |  |

### 5.2 Estimating $\mathbf{I}_{\text {max }}$

Since the value of the peak of the current is a common problem, it can be adjusted by forcing the value of the peak of the current $I_{\text {max }}$ to have a certain value. This value can be estimated by visual investigation of the integration of the measured current derivative signal. By assigning the value of $\mathrm{I}_{\text {max }}$, the number of analytical parameters will be reduced to two: n and $\tau_{1}$.

Using the value of $\tau_{2}$, previously obtained, and assigning to $\mathrm{I}_{\max }$ its measured value (7.9 kA ), and applying the curve fitting process on the first impulse, the other analytical parameters can easily estimated. As shown in Figure 5.9, although the Pulse function current improves, the maximum steepness value and its time of occurrence, as well as the zero crossing of the current derivative need to be improved to best fit the measured signal. Heidler function shows the same problems as the Pulse function.

The values of the analytical parameters, before and after the fitting process and the corresponding $\mathrm{R}^{2}$ are listed in Table 5.3. $\mathrm{I}_{\max }$ and $\tau_{2}$ values are the same as the fitting process involved on n and $\tau_{1}$. It is very clear that $\mathrm{R}^{2}$ of both functions is reduced slightly but the current simulation is improved.


Figure 5.9: Zoomed simulation of fitting result of the Pulse and Heidler functions for the measured current derivative signal with fixed $\tau_{2}$ and $\mathrm{I}_{\max }$ and the corresponding current signal

Table 5.3: Parameters' initial and final values for the Pulse and Heidler functions fitting process with fixed $\tau_{2}$ and $I_{\text {max }}$ and the associated $\mathrm{R}^{2}$

|  |  | $\mathrm{I}_{\max }(\mathrm{kA})$ | $\tau_{1}(\mu \mathrm{~s})$ | $\tau_{2}(\mathrm{~ms})$ | n | $\mathrm{R}^{2}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Pulse | Initial Value | 7.9 | 0.06962 | 0.1239 | 1410 | 0.8986 |
|  | Fitting Results | 7.9 | 0.0773 | 0.1239 | 1166 |  |
|  | Initial Value | 7.9 | 0.5611 | 0.1239 | 14.44 | 0.8959 |
|  | Fitting Results | 7.9 | 0.5689 | 0.1239 | 10.6 |  |

### 5.3 Applying Constraints

Another way for improving the fitting process is by introducing a constraint to force the analytical parameters to reach their optimal values. This constraint is applied by adjusting either the time of occurrence of the zero crossing of the current derivative or the time of occurrence of the maximum steepness of the current to its measured value. The analytical analysis preformed in chapter 3 can be used to define the suitable constraints for each function. We refer to these constraints as time-forcing constraints.

Using the known values of $\tau_{2}$ and $\mathrm{I}_{\text {max }}$, which were previously estimated, $\tau_{1}$ can be expressed in terms of $n$ or vice versa. Accordingly, the number of analytical parameters of the functions will be reduced to one.

### 5.3.1 Condition of Current Derivative Zero Crossing

Firstly, the time-forcing constraint will be applied to adjust the zero crossing of the current derivative to its measured value. For the Pulse function, applying the constraint using equation (3-42), $n$ can be expressed in terms of $\tau_{1}$ :

$$
\begin{equation*}
n=\frac{\tau_{1}}{\tau_{2}} *\left(e^{\frac{t_{\max }}{\tau_{1}}}-1\right) \tag{5-3}
\end{equation*}
$$

By substituting $n$ in the Pulse function equation (3-40) and its derivative, equation (3-41), the only unknown analytical parameter will be $\tau_{1}$. The function and its derivative can be reformatted as follows:

$$
\begin{equation*}
I(t)=\frac{I_{\max }}{\eta} *\left(1-e^{-\frac{t}{\tau_{1}}}\right)^{\frac{\tau_{1}}{\tau_{2}} *\left(e^{\frac{t_{\max }}{\tau_{1}}}-1\right)} * e^{-\frac{t}{\tau_{2}}} \tag{5-4}
\end{equation*}
$$

$$
\begin{align*}
\frac{d I(t)}{d t}=\frac{I_{\max }}{\eta} & \left.*\left(1-e^{-\frac{t}{\tau_{1}}}\right)^{\frac{\tau_{1}}{\tau_{2}} *\left(e^{\frac{t_{\max }}{\tau_{1}}}-1\right.}\right) * e^{-\frac{t}{\tau_{2}}} \\
& *\left[\frac{\left(e^{\frac{t_{\max }}{\tau_{1}}}-1\right)}{\tau_{2}} * \frac{e^{-\frac{t}{\tau_{1}}}}{\left(1-e^{-\frac{t}{\tau_{1}}}\right)}-\frac{1}{\tau_{2}}\right] \tag{5-5}
\end{align*}
$$

For Heidler function, applying the constraint using equation (3-27), $\tau_{1}$ can be expressed in terms of n :

$$
\begin{equation*}
\tau_{1}=\sqrt[n]{\frac{t_{\max }{ }^{n+1}}{n * \tau_{2}}} \tag{5-6}
\end{equation*}
$$

By substituting $n$ in Heidler function equation (3-21) and its derivative equation (3-22), the only unknown analytical parameter will be $n$. The function and its derivative can be reformatted as follows:

$$
\begin{align*}
& I(t)=I_{\max } *\left(\frac{t}{t_{\max }}\right)^{n} * \frac{t_{\max }{ }^{n+1}+n * \tau_{2} * t_{\max }^{n}}{t_{\max }^{n+1}+n * \tau_{2} * t^{n}} * e^{-\frac{t-t_{\max }}{\tau_{2}}}  \tag{5-7}\\
& \quad \frac{d I(t)}{d t}=I_{\max } *\left(\frac{t}{t_{\max }}\right)^{n} * \frac{t_{\max }{ }^{n+1}+n * \tau_{2} * t_{\max }{ }^{n}}{t_{\max }{ }^{n+1}+n * \tau_{2} * t^{n}} \\
&  \tag{5-8}\\
& *\left[\frac{n}{t}-\frac{1}{\tau_{2}}-\frac{n * t^{n-1}}{\left(\frac{t_{\max }^{n+1}}{n * \tau_{2}}+t^{n}\right)}\right] * e^{-\frac{t-t_{\max }}{\tau_{2}}}
\end{align*}
$$

The equations of the simulating functions (5-5) and (5-8), representing the Pulse and Heidler functions after using the time forcing constraint for the zero crossing, respectively, are applied on the measured current derivative first impulse to find the optimal value of $\tau_{1}$ or $n$. An overshoot for the maximum steepness value of the current signal occurred leading to a very fast rise in the current signal as shown in Figure 5.10. Consequently the zero crossing constraint was not possible to apply to achieve the optimal value of the analytical parameters.

The resultant $\mathrm{R}^{2}$ value of the fitting process is very poor and proves that such condition causes the analytical parameters to deviate from their optimal values. The initial values of the analytical parameters, values of the parameters after the fitting process and the corresponding $\mathrm{R}^{2}$ are listed in Table 5.4.


Figure 5.10: Zoomed simulation of fitting result of the Pulse and Heidler functions for the measured current derivative signal with fixed $\tau_{2}, \mathrm{I}_{\max }$ and with forcing time constraints of the zero crossing of the current derivative signal and the corresponding current signal

Table 5.4: Parameters' initial and final values for the Pulse and Heidler functions fitting process with fixed $\tau_{2}, I_{\text {max }}$ and with forcing time constraints of the zero crossing of the current derivative signal and the associated $\mathrm{R}^{2}$

|  |  | $\mathrm{I}_{\text {max }}(\mathrm{kA})$ | $\tau_{1}(\mu \mathrm{~s})$ | $\tau_{2}(\mathrm{~ms})$ | n | $\mathrm{R}^{2}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Pulse | Initial Value | 7.9 | 0.02476 | 0.1239 | ------------- | 0.216 |
|  | Fitting Results | 7.9 | 0.03476 | 0.1239 | ------------- |  |
| Heidler | Initial Value | 7.9 | ------------ | 0.1239 | 22.81 | 0.4424 |
|  | Fitting Results | 7.9 | ------------- | 0.1239 | 20.38 |  |

### 5.3.2 Time of Maximum Steepness Condition

Secondly, the time-forcing constraint will be applied to adjust the time of occurrence of the maximum steepness of the current signal to its measured value. For the Pulse function, applying the constraint using equation (3-49), $n$ can be expressed in terms of $\tau_{1}$ :

$$
\begin{equation*}
n=e^{\frac{t_{m s}}{\tau_{1}}} \tag{5-9}
\end{equation*}
$$

By substituting $n$ in the Pulse function equation (3-40) and its derivative equation (3-41), the only unknown parameter will be $\tau_{1}$. The function and its derivative can be reformatted as follows:

$$
\begin{align*}
& I(t)= \frac{I_{\max }}{\eta} *\left(1-e^{-\frac{t}{\tau_{1}}}\right)^{e^{\frac{t_{m s}}{\tau_{1}}}} * e^{-\frac{t}{\tau_{2}}}  \tag{5-10}\\
& \frac{d I(t)}{d t}=\frac{I_{\max }}{\eta} *\left(1-e^{-\frac{t}{\tau_{1}}}\right)^{e^{\frac{t_{m s}}{\tau_{1}}}} * e^{-\frac{t}{\tau_{2}}} \\
& *\left[\frac{e^{\frac{t_{m s}}{\tau_{1}}}}{\tau_{1}} * \frac{e^{-\frac{t}{\tau_{1}}}}{\left(1-e^{-\frac{t}{\tau_{1}}}\right)}-\frac{1}{\tau_{2}}\right] \tag{5-11}
\end{align*}
$$

Similarly, for Heidler function, applying this constraint using equation (3-34), $\tau_{1}$ can be expressed in terms of $n$ :

$$
\begin{equation*}
\tau_{1}=t_{m s} * \sqrt[n]{\frac{n+1}{n-1}} \tag{5-12}
\end{equation*}
$$

By substituting n in Heidler function equation (3-21) and its derivative equation (3-22), the only unknown parameter will be $n$. The function and its derivative can be reformatted as follows:

$$
\begin{gather*}
I(t)=\frac{I_{\max }}{\eta} * \frac{\left(\frac{t}{t_{m s}}\right)^{n} * \frac{n-1}{n+1}}{1+\left(\frac{t}{t_{m s}}\right)^{n} * \frac{n-1}{n+1}} * e^{-\frac{t}{\tau_{2}}}  \tag{5-13}\\
\frac{d I(t)}{d t}=\frac{I_{\max }}{\eta} * \frac{\left(\frac{t}{t_{m s}}\right)^{n} * \frac{n-1}{n+1}}{1+\left(\frac{t}{t_{m s}}\right)^{n} * \frac{n-1}{n+1}} * e^{-\frac{t}{\tau_{2}}}  \tag{5-14}\\
*\left[\frac{n}{t}-\frac{1}{\tau_{2}}-\frac{n * t^{n-1}}{\left(t_{m s}^{n} * \frac{n+1}{n-1}+t^{n}\right)}\right]
\end{gather*}
$$

Equations (5-11) and (5-14) representing the Pulse and Heidler functions after using the time-forcing constraint for the time of occurrence of the maximum steepness, respectively, are applied on the measured current derivative first impulse to find the optimal value of $\tau_{1}$ or n . The best fit is achieved for both functions as shown in Figure 5.11.

Table 5.5 contains the initial values of the analytical parameters, the values of the analytical parameters after the fitting process and the associated $\mathrm{R}^{2}$ of the fitting process. The Pulse function followed the guidelines and the highest $R^{2}$ value supported this guidelines. It is very clear that the Pulse function better fits the measured current derivative and it also better simulates the current signal than those of Heidler function.


Figure 5.11: Zoomed simulation of fitting result of the Pulse and Heidler functions for the measured current derivative signal with fixed $\tau_{2}, \mathrm{I}_{\max }$ and with forcing time constraints of the maximum steepness of current signal and the corresponding current signal

Table 5.5: Parameters' initial and final values for the Pulse and Heidler functions fitting process with fixed $\tau_{2}, \mathrm{I}_{\max }$ and with forcing time constraints of the the maximum steepness of current signal and the associated $\mathrm{R}^{2}$

|  |  | $\mathrm{I}_{\text {max }}(\mathrm{kA})$ | $\tau_{1}(\mu \mathrm{~s})$ | $\tau_{2}(\mathrm{~ms})$ | n | $\mathrm{R}^{2}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Pulse | Initial Value | 7.9 | 0.05181 | 0.1239 | ------------- | 0.8771 |
|  | Fitting Results | 7.9 | 0.07797 | 0.1239 | ------------- |  |
| Heidler | Initial Value | 7.9 | ------------- | 0.1239 | 12.36 | 0.8371 |
|  | Fitting Results | 7.9 | ------------ | 0.1239 | 9.629 |  |

### 5.4 Current Waveform Parameters

The described fitting process represents the best method to reach the optimal fit for simulating the measured signal. The objective is not only simulating the full measured signal including the decay part but also calculating all current waveform parameters. Using this fitting process for the Pulse function, all current waveform parameters can be calculated, which leads to the establishment of extensive cumulative statistics regarding each waveform parameter. For the current derivative signal, shown in Figure 5.11, the waveform parameters calculated from the simulation using the Pulse function and those obtained manually, if possible, are compared in Table 5.6

Table 5.6: Current waveform parameters value calculated from the Pulse function

| Current waveform parameter | Manual | Pulse Function |
| :---: | :---: | :---: |
| Current peak ( kA ) | 7.9 | 7.9 |
| Maximum Steepness ( $\mathrm{kA} / \mu \mathrm{s}$ ) | 36.98 | 38.5735 |
| Risetime ( $\mu \mathrm{s}$ ) | 0.25 | 0.2343 |
| Decay time ( $\mu \mathrm{s}$ ) | ------------ | 285.97 |
| Current pulse width half maximum ( $\mu \mathrm{s}$ ) | ------------ | 89.204 |
| Charge ( C ) | ------------ | 0.9393 |

It is very important to emphasize the better accuracy and the easiness achieved when using the simulated function to calculate the current waveform parameters in comparison with estimating the parameters using traditional manual methods. Also, having the simulating function at hand it was possible to determine other parameters that were never determined before using manual methods, such as the decay time, the charge and the waveform pulse width at half the current peak.

### 5.5 Cumulative Statistical Distributions

The described fitting process is applied on 15 flashes, containing 31 return strokes. The calculated current waveform parameters were used to form statistics to determine the probability distribution of the value of each parameter, including the range and the $50 \%$ probability level, which is fundamental in building lightning protection systems.

### 5.5.1 Current Peak

The cumulative distribution of the current peak of the lightning return-stroke current for the analyzed signals is shown in figure 5.12. The current peak, determined from the simulated function, varies from a minimum value of 2.59 kA to a maximum value of 11.08 kA . The average value of the current peak was found to be 5.57 kA . It was observed that in $50 \%$ of the recorded lightning strokes, the current peak exceeded 5.73 kA , in $5 \%$ of recorded signals, the current peak exceeded 10.21 kA and in $95 \%$ of captured strokes the current peak exceeded 2.98 kA.


Figure 5.12: Cumulative distribution of the current peak

### 5.5.2 Maximum Steepness

For the maximum steepness, its cumulative distribution is shown in Figure 5.13. The lightning signals analyzed had a minimum value of $11.65 \mathrm{kA} / \mu \mathrm{s}$ and maximum value of 54.68 $\mathrm{kA} / \mu \mathrm{s}$ for the maximum steepness. The average value for the maximum steepness was found to be $31.33 \mathrm{kA} / \mu \mathrm{s}$. In $95 \%$ of the signals, the maximum steepness came to be higher than 14.31 $\mathrm{kA} / \mu \mathrm{s}$ and in $5 \%$ of the cases the maximum steepness was higher than $51.21 \mathrm{kA} / \mu \mathrm{s}$. The $50 \%$ probability value for the maximum steepness was found to be $34.31 \mathrm{kA} / \mu \mathrm{s}$.


Figure 5.13: Cumulative distribution of the maximum steepness

### 5.5.3 Risetime to the Current Peak

The $10 \%$ to $90 \%$ risetime to the current peak for each of the analyzed signals was determined from the simulated function. Figure 5.14 shows the cumulative distribution of the current risetime. The minimum recorded risetime to the current peak was 124 ns while the maximum recorded value was 524 ns with an average of 242.3 ns . It is important realize this large range of variation in the current risetime when designing lightning protection systems. For the analyzed signals, the current risetime exceeds $143 \mathrm{~ns}, 211 \mathrm{~ns}$ and 481 ns in $95 \%, 50 \%$ and $5 \%$ of the cases, respectively.


Figure 5.14: Cumulative distribution of the risetime to the current peak

### 5.5.4 Decay Time from the Current Peak

The decay time of the current peak is an important waveform parameter. The $90 \%$ to $10 \%$ current decay time from the current peak for each of the analyzed signals was determined. The cumulative distribution of the current decay time is shown in Figure 5.15. The minimum value of the current decay time was found to be $65.88 \mu$ s with maximum value of $537.13 \mu \mathrm{~s}$. The average value of the current decay time is $255.64 \mu$ s. In $95 \%$ of the cases, the current decay time exceeds $75.26 \mu \mathrm{~s}$ and in $5 \%$ of the cases, it exceeds $523.6 \mu \mathrm{~s}$. The $50 \%$ probability value of the current decay time is $261.3 \mu \mathrm{~s}$.


Figure 5.15: Cumulative distribution of the decay time from the current peak

### 5.5.5 Current Pulse Width

The current pulse width at the half current peak level for each of the analyzed signals was calculated by determining the time difference between the time at the $50 \%$ level of current peak at the decay side of the current pulse and the $50 \%$ level of the current peak at the rising side of the current pulse. Figure 5.16 shows the cumulative distribution of the current pulse width. The minimum recorded current pulse width is $21.47 \mu$ s and the maximum recorded value is 157.51 $\mu \mathrm{s}$, with an average of $83.99 \mu \mathrm{~s}$. The large range of variation in current pulse width is important to take into consideration by engineers working on lightning protection methods. For the analyzed signals, the current pulse width $75.23 \mu \mathrm{~s}, 261.3 \mu \mathrm{~s}$ and $512.3 \mu \mathrm{~s}$ in $95 \%, 50 \%$ and $5 \%$ of the cases, respectively.


Figure 5.16: Cumulative distribution of the current pulse width

### 5.5.6 Charge

The charge is one of the most important waveform parameters in the area of protection. The charge for each of the analyzed signals was obtained by integrating the current. The cumulative distribution of the charge is shown in Figure 5.15. The minimum value of the charge is 0.1187 C , while the maximum value is 1.897 C . The average value of the charge is 0.6642 C . In $95 \%$ of the cases, the charge exceeds 0.154 C and in $5 \%$ of the cases, the charge exceeds 1.671 C . In $50 \%$ of the analyzed signals, the charge exceeds 0.599 C .


Figure 5.17: Cumulative distribution of the charge

Table 5.7 summarize all the current waveform parameters cumulative statics disruption results including the values of the waveform parameters at $95 \%, 50 \%$ and $5 \%$ of the cases. Also it covers the maximum value, the minimum value, and the average value of the waveform parameters.

Table 5.7: Summary of the current waveform parameters cumulative statistics distribution

| Waveform Parameter | Minimum | Maximum | Average | $5 \%$ | $50 \%$ | $95 \%$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Current Peak (kA ) | 2.895 | 11.08 | 5.572 | 10.21 | 5.73 | 2.98 |
| Maximum Steepness <br> $(\mathrm{kA} / \mu \mathrm{s})$ | 11.65 | 54.68 | 31.3261 | 51.21 | 34.31 | 14.31 |
| Risetime to Current Peak <br> $(\mu \mathrm{s})$ | 0.124 | 0.524 | 0.2423 | 0.481 | 0.211 | 0.143 |
| Decay Time from current <br> peak ( $\mu \mathrm{s})$ | 65.88 | 537.13 | 255.64 | 523.6 | 261.3 | 75.26 |
| Current Pulse Width <br> $(\mu \mathrm{L})$ | 21.47 | 157.51 | 83.99 | 512.3 | 261.3 | 75.23 |
| Charge ( C ) | 0.1187 | 1.897 | 0.6642 | 1.671 | 0.5986 | 0.1542 |

## Chapter 6

## Conclusions and Recommendations

The main purpose for the study was to model the lightning return-stroke current, measured at the CN Tower through which the current waveform parameters are calculated. Although many simulating functions have been proposed, Heidler function and the Pulse function were considered for the lightning return-stroke current modeling.

From the previous results it was concluded that the current derivative fitting yield better results for simulating the measured signal than the current fitting. The initial values of the analytical parameters of a given function were estimated. Using artificial digital signals, free of noise and reflection, proved to be very efficient in the process of initial value estimations and the evaluation simulation functions.

Previously, the fitting process involved the first impulse only of the measured current derivative signal ignoring the decaying part to avoid reflections due to the tower's structure discontinuities. The decoupling behavior of the Pulse and Heidler functions helped in dividing the signal into three parts and in avoiding the use of the second part of the signal, which is highly distorted by the reflection, in the fitting process. The third part was used for estimating the decay time constant $\left(\tau_{2}\right)$ and the other parameters were estimated from the first part of the signal (mainly the rise portion of the signal). Results showed that $\tau_{2}$ and $\mathrm{I}_{\text {max }}$ have the same value for both simulating functions (the Pulse and Heidler functions).

Finally, using a time forcing constraint for the maximum steepness of the current for fitting the rising part of the signal with the estimated values of $\tau_{2}$ and $I_{\text {max }}$, the fitting process was reduced to a single parameter estimation and the comparison between the Pulse and Heidler functions was carried out. Although Heidler function is widely used in fitting the return-stroke current signal, the Pulse function proved to be the better fit for the lightning return-stroke current measured at the CN Tower.

The described fitting process represents the best method to reach the optimal value for simulating the measured signal, not only by fitting the full measured signal including the decay part but also by calculating all current waveform parameters. Using this fitting process for the Pulse function all current waveform parameters were calculated. The calculated current
waveform parameters for 15 flashes, containing 31 return strokes, were used to form statistics to determine the probability distribution of the value of each parameter, including the range and the $50 \%$ probability level, which is fundamental in building lightning protection systems.

The cumulative distribution can be improved by analyzing more lightning return-stroke current signals. Using the lighting return-stroke currents collected from the tower since 1991, a more extensive statistics can be achieved.

Also, the Pulse function can be improved for better fitting of the lighting return-stroke current, which results in more accurate current waveform parameter values. This can be done by introducing another term of forming a modified Pulse function, which will have eight analytical parameters.

Although this work was preformed for modelling the CN Tower lightning return-stroke current signals, it can be equally used for other lightning-related signals. Furthermore, this research can be applied to a variety of signals, including those encountered in biomedical applications, such as the heart-related electrical signals.

## References

[1] F. Rachidi, W.Janischewskyj, A. M. Hussein, C. A. Nucci, S. Guerrieri, B. Kordi, and J. S. Chang, "Current and Electromagnetic field associated with lightning return-stroke to tall tower", IEEE Transaction on Electromagnetic Compatibility, vol. 43, no. 3, pp. 356-367, August 2001.
[2] V. A. Rakov and M. A. Uman, "Review and evaluation of lightning return stroke models including some aspects of their application," IEEE Transactions on Electromagnetic Compatibility, vol. 40, no. 4, pp. 403-426, November 1998.
[3] Z. Feizhou and L. Shanghe, "A New Function to Represent the Lightning Return-Stroke Current", IEEE Transactions on Electromagnetic Compatibility, vol. 44, no. 4, pp. 595597, November 2002.
[4] M. Milewski and A. M. Hussein, "Evaluation of tall-structure lightning return-stroke model using CN Tower data," $19^{\text {th }}$ International Conference on Electromagnetic Disturbances, pp. 180-185, Bialystok, Poland, Sept. 23-25, 2009.
[5] M. Milewski and A.M. Hussein, "Lightning Return-Stroke Transmission Line Model Based on CN Tower Lightning Data and Derivative of Heidler Function", IEEE Canadian Conference on Electrical and Computer Engineering, pp. 1861-1866, Niagara Falls, Ontario, May 4-7, 2008.
[6] K. Bitner and A.M. Hussein, "Modelling of the CN Tower lightning return-stroke current derivative," $28^{\text {th }}$ International Conference on Lightning Protection, pp. 261-266, Kanazawa, Japan, September 2006.
[7] R.D. Jones, "On the use of tailored return-stroke current representation to simplify the analysis of lightning effect on systems", Transactions of the American Institute of Electrical Engineers, Electromagnetic Compatibility, pp. 95-96, 1977.
[8] R.L. Gardner, Baker L., Baum C.E., Andersh D.J., "Comparison of lightning with public domain HEMP waveforms on the surface of an aircraft", $6^{\text {th }}$ EMC Symposium, pp.175-180, Zurich, 1985.
[9] A.M. Hussein, W. Janischewskyj, M. Milewski, V. Shostak, W. Chisholm, J.S. Chang "Current waveform parameters of CN Tower lightning return strokes," Journal of Electrostatics, vol. 60, pp.149-162, March 2004.
[10] Vladimir A. Rakov, Martin A. Uman, Lightning Physics and Effects, Cambridge University Press, Florida, 2003.
[11] Martin A. Uman, The Lightning Discharge, Academic Press Inc., Orlando, Florida, 1987.
[12] Earle R. Williams, "The Electrification of Thunderstorm," Scientific America, no. 11, pp. 88-99, November 1988.
[13] Martin A. Uman, Understanding Lightning, Bek Technical Publications Inc., Carnegie, Pennsylvania, 1971.
[14] R.B. Anderson, A.J. Eriksson, "Lightning Parameters for Engineering Applications," Electra, vol. 96, pp.65-105, 1980.
[15] W. Janischewskyj, A.M. Hussein, V. Shostak and P. Dzurevych, "Characterization of the Current Wavefront Parameters of Lightning Strikes to the CN Tower in Toronto," $8^{\text {th }}$ International Symposium on High Voltage Engineering, Yokohama, Japan, 1993.
[16] K.B. McEachron, "Lightning to the Empire State Building," Journal of the Franklin Institute, vol. 227, pp. 149-217, 1939.
[17] K.B. McEachron, "Lightning to the Empire State Building," Transactions AIEE, vol. 60, pp. 885-889, September 1941.
[18] A.M. Hussein, W. Janischewskyj, J.S. Chang, V. Shostak, W.A. Chisholm, P. Dzurevych and Z.I. Kawasaki, "Simulation Measurment of Lightning Paramters for Strokes to the Toronto Canadian National Tower," Journal of Geophysical Reaserch, vol. 100, no. D5, pp. 8853-8861, May 1995.
[19] W. Janischewskyj, A.M. Hussein, V. Shostak, I. Rusan, J.-X. Li and J.S. Chang, "Statistics of Lightning Strikes to the Toronto National Canadian Tower (1978-1995)," IEEE Transactions on Power Delivery, vol. 12, no. 3, pp. 1210-1221, July 1997.
[20] A.M. Hussein, W. Janischewskyj, M. Milewski, V. Shostak, J.S. Chang and W. Chishlom, "Return-Stroke Current Waveform Parameters of Lightning to the CN Tower (19922001)," International Conference of Lightning Protection, pp. 161-166, Cracow, Poland, September 2003.
[21] V. Shostak, M. Wiacek and D. Tam, Report on the New Rogowski Coil Calibration, The University of Toronto, Toronto, Canada, May 1998.
[22] F. Heidler, J.M. Cavetic, B.V. Santic, "Calculation of Lightning current parameters", IEEE Transactions on Power Delivery. Vol. 14, No. 2, pp. 399-404, April 1999.
[23] F. Heidler, J.M. Cavetic, "A Class of Analytical Functions to Study the Lightning Effects Associated with the Current Front", European Transaction on Electrical Power. Vol. 12, No. 2, pp. 141 - 150, April 2002.
[24] Chen Yazhou, Liu Shanghe, Wu Xiaorong, Zhang Feizhou, "A New Kind of Channel-base Current Function," $3^{\text {rd }}$ International symposium on Electromagnetic Compatibility, pp. 304-307, May 2002.

