TRAJECTORY BASED MARKET MODELS FOR TWO STOCKS

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A thesis presented to Ryerson University
in partial fulfillment of the
requirements for the degree of
Master of Science
in the program of
Applied Mathematics

Toronto, Ontario, Canada, 2019
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Abstract:

This paper studies the explicit calculation of the set of superhedging (and underhedging) portfolios where one asset is used to superhedge another in a discrete time setting. A general operational framework is proposed and trajectory models are defined based on a class of investors characterized by how they operate on financial data leading to potential portfolio rebalances. Trajectory market models will be specified by a trajectory set and a set of portfolios. Beginning with observing charts in an operationally prescribed manner, our trajectory sets will be constructed by moving forward recursively, while our superhedging portfolios are computed through a backwards recursion process involving a convex hull algorithm. The models proposed in this thesis allow for an arbitrary number of stocks and arbitrary choice of numeraire. Although price bounds, $\underline{V}_0(\mathbf{X}_0, X^2, \mathcal{M}) \leq \overline{V}_0(\mathbf{X}_0, X^2, \mathcal{M})$, will never yield a market misprice, our models will allow an investor to determine the amount of risk associated with an initial investment v.

ACKNOWLEDGEMENTS

Throughout the writing of this thesis I have recieved a great deal of support and assistance. I would like to first thank my supervisor, Professor Sebastian Ferrando, whom always steered me in the right direction and encouraged me whenever I required it. Without him I would not have been able to further develop my skills to complete this thesis to this extent.

I would also like to thank each person involved in the Department of Mathematics at Ryerson University. It was a pleasure to spend the duration of my master's building the relationships I have.

Finally, I would like to thank my family, friends, and Camille for always being there with an open ear and always giving their patience and understanding of the amount of work required to complete this thesis. I love you all and am truly grateful for having you in my life.

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Notation

Here we describe the notation used throughout the thesis. We indicate parameters used to define trajectory models, historical estimation, and model construction.

Chapter 2 - Notation used to define trajectory market models:

- $\mathcal{S}$: Trajectory set consisting of *undiscounted* assets $S_i = (S_i^0, S_i^1, ..., S_i^d)$, portfolio rebalancing times T_i and a financial observable W_i. (see Definition 1)
- S_i^j: The price of asset j at the i'th portfolio rebalancing, where $j = 0, 1, ..., d$. (see Definition 1)
- $\mathcal{X}$: Trajectory set of *discounted* assets $X_i = (X_i^1, X_i^2, ..., X_i^d)$. (see Section 2.1.1)
- X_i^j: The value of asset j at the i'th portfolio rebalancing when using S^0 as a numeraire, where $j = 1, ..., d$. (see Section 2.1.1)
- H: sequences of functions representing portfolios. $H = \{(H_i^0, H_i^1, ..., H_i^d)\}_{i \geq 0}$, where H_i^j represents the amount of holdings in asset j at the i'th portfolio rebalancing.
- $\mathcal{H}$: The portfolio set, where portfolios $H \in \mathcal{H}$. (see Definition 2)
- $\mathcal{M}$: Trajectory based market, where $\mathcal{M} = \mathcal{X} \times \mathcal{H}$. (see Definition 3)
- $V_k^\Phi(\mathbf{X})$: Portfolio value for the trajectory $\mathbf{X}$ at rebalancing k. (see Equation (2.2))
- $G_k^\Phi(\mathbf{X})$: Profits generated for the trajectory $\mathbf{X}$ at rebalancing k.(see Equation (2.3))
- $\mathcal{X}_{(\mathbf{X},k)}$: Trajectory set consisting of trajectories conditioned at node $(\mathbf{X}, k)$. (see Section 2.1.3)

- $\Delta X(\mathcal{X}_{(\mathbf{X},k)})$: Set of changes in value from rebalancing k to $k+1$ conditioned on node $(\mathbf{X}, k)$. (see Equation (2.8))

- $ri(co(\Delta X(\mathcal{X}_{(\mathbf{X},k)})))$: The relative interior of the convex hull of the set of changes in value $\Delta X(\mathcal{X}_{(\mathbf{X},k)})$. (see Proposition 4)

- $cl(co(\Delta X(\mathcal{X}_{(\mathbf{X},k)})))$: The closure of the convex hull of the set of changes in value $\Delta X(\mathcal{X}_{(\mathbf{X},k)})$. (see Proposition 4)

- $\overline{V}_k(\mathbf{X}, X^2, \mathcal{M})$: The upper price bound of discounted asset X^2 along trajectory path $\mathbf{X}$ for the market $\mathcal{M}$. (see Definition 8)

- $\underline{V}_k(\mathbf{X}, X^2, \mathcal{M})$: The lower price bound of discounted asset X^2 along trajectory path $\mathbf{X}$ for the market $\mathcal{M}$. (see Definition 8)

Chapter 3 - Notation used to define historical estimation methods:

- $\mathcal{T}$: Historical time interval which an investor has access to chart values $s(t)$, where $t \in \mathcal{T}$. (see Section 3.1)

- $s(t)$: Matrix of time series of undiscounted asset prices, where $s(t) = \big(s^0(t), s^1(t), ..., s^d(t)\big)$. (see Section 3.1)

- $s^j(t)$: time series of the undiscounted price of asset j, $j = 0, 1, ..., d$. (see Section 3.1)

- $x(t)$: Matrix of time series of discounted asset prices, where $x(t) = \big(x^1(t), ..., x^d(t)\big)$. We refer to $x(t)$ as a chart. (see Section 3.1)

- $x^j(t)$: Time series of the discounted price of asset j, $j = 1, 2, ..., d$. (see Section 3.1)

- δ_0: Investor calibrated parameter which provides an investor with historical chart sampling times. (see Definition 10)

- $\{r_l\}$: The set of historical rebalancing times (for a specific interval $[t_0, t_0+T] \in \mathcal{T}$). (see Definition 10)

- δ: Investor calibrated parameter which provides an investor with historical portfolio rebalancing times, t_i, in our δ-uncorrelated models. (see Definition 11)

- $\delta_u p$ and δ_{down}: Investor calibrated parameters which provides an investor with historical portfolio rebalancing times, t_i, in our δ-correlated models. (see Definition 13)

- $\{t_i\}$: The set of historical rebalancing times (for a specific interval $[t_0, t_0+T] \in \mathcal{T}$). (see Definition 11 and Definition 13)

- $N(x, [t_0, t_0 + T])$: The number of portfolio rebalances that occurs for a given chart $x(t)$ in time interval $[t_0, t_0 + T] \subseteq \mathcal{T}$ with calibrated parameter δ (or δ_{up} and δ_{down}). (see Definition 12)

- m_i^1 and m_i^2 are the number of $\hat{\delta}^1$ and $\hat{\delta}^2$ value changes of assets $x^1(t)$ and $x^2(t)$, respectively, between portfolio rebalances i and $i+1$. (see Equation (3.6))

- q_i is the number of time intervals of size Δ between two consecutive portfolio rebalances i and $i+1$. (see Equation (3.8))

- P_i is the number of $\hat{\nu}_0$ changes of the accumulated vector variation between two consecutive portfolio rebalances i and $i+1$. (see Equation (3.7))

- $\mathcal{N}_E(x, [t_0, t_0 + T])$ is a collection of all (m_i^1, m_i^2, q_i, P_i) values that occur in the interval $[t_0, t_0 + T]$. (see Definition 14)

- i^* is the maximum number of possible portfolio rebalances that occur historically in the interval $[t_0, t_0 + T]$. This is only used in Type 0 models (first introduced in 4.2) to terminate the recursive creation of trajectory paths. (see Definition 15)

- $X^*(x, \mathcal{T}, i)$ and $X_*(x, \mathcal{T}, i)$ for $i \geq 0$ represent the maximum and minimum ratio of normed vector changes that occurs at the i'th δ-movement within the charts $x(t)$, respectively. This constraint will limit the amount our trajectory asset values may fluctuate since an initial portfolio rebalancing ($i = 0$). (see Definition 16)

- $N^*(x, \mathcal{T}, \rho)$ and $N_*(x, \mathcal{T}, \rho)$ represent the maximum and minimum portfolio rebalances that occur within $x(t)$. This is used to limit the number of portfolio rebalances that occur after time $\rho \in \Delta\mathbb{N}$ has elapsed. The investor will then not rebalance a portfolio more (and less) often than they would have historically. (see Definition 17)

- $N^*(x, \mathcal{T}, w)$ and $N_*(x, \mathcal{T}, w)$ for $w = w(x, [t_0, t_i]) \geq 0$ represent the maximum and minimum portfolio rebalances that occur within $x(t)$ after a chart has accumulated $w(x, [t_0, t_i])$ amount of variation at the i'th rebalancing. (see Definition 21)

- $T^*(x, \mathcal{T}, i)$ and $T_*(x, \mathcal{T}, i)$ represent the maximum and minimum amount of time elapsed after the i'th portfolio rebalancing. This restricts the investor to perform the i'th portfolio rebalancing at times which they would have done so historically. (see Definition 18)

- $T^*(x, \mathcal{T}, w)$ and $T_*(x, \mathcal{T}, w)$ represent the maximum and minimum amount of time elapsed after $w = w(x, [t_0, t_i])$ amount of variation is accumulated after the i'th portfolio rebalancing. This restricts the investor to perform the i'th portfolio rebalancing at times which they would have done so historically. (see Definition 22)

- $W^*(x, \mathcal{T}, i)$ and $W_*(x, \mathcal{T}, i)$ for $i \geq 0$ represent the maximum and minimum amount of accumulated variation after the i'th portfolio rebalancing time. This is used to limit the amount that model asset values X^1, X^2 can vary up to the i'th portfolio rebalance. (see Definition 19)

- $W^*(x, \mathcal{T}, \rho)$ and $W_*(x, \mathcal{T}, \rho)$ for $\rho \in [0, T]$ represent the maximum and minimum amount of accumulated variation between historical portfolio rebalancing times. This is used to limit the amount that model asset values X^1, X^2 can vary after time ρ has elapsed. (see Definition 20)

Chapter 4 - Notation used to construct trajectory market models (note that we utilize a similar, yet different, notation to parameters introduced in Chapter 3. This enables Chapter 4 to not rely on previously introduced notation pertaining to charts):

- $\mathcal{N}_E$: Set of changes used to construct trajectory sets. (see Section 4.1)

- $\mathcal{N}_A(\mathbf{X}_i)$: Admissible set of $\mathbf{X}_{i+1}$ values for a given $\mathbf{X}_i$. (see Section 4.1)

- $N(\mathbf{X})$: Maximum number of portfolio rebalances in a trajectory. (see Section 4.2.1)

- $\overline{X}(i)$ and $\underline{X}(i)$: Model pruning constraints; maximum and minimum vector percent change for X at rebalancing i. (see Sections 4.2 and 4.2.2)

- $\overline{N}(\rho)$ and $\underline{N}(\rho)$: Model pruning constraints; maximum and minimum number of rebalances after $\rho \in [0, T]$ time has elapsed. (see Sections 4.2 and 4.2.3)

- $\overline{N}(w)$ and $\underline{N}(w)$: Model pruning constraints; maximum and minimum number of rebalances after w amount of vector variation has been accumulated. (see Sections 4.2 and 4.2.4)

- $\overline{T}(i)$ and $\underline{T}(i)$: Model pruning constraints; maximum and minimum amount of time elapsed after the i'th portfolio rebalancing. (see Sections 4.2 and 4.2.3)

- $\overline{T}(w)$ and $\underline{T}(w)$: Model pruning constraints; maximum and minimum amount of time elapsed after w amount of vector variation has been accumulated. (see Sections 4.2 and 4.2.4)

- $\overline{W}(i)$ and $\underline{W}(i)$: Model pruning constraints; maximum and minimum amount of accumulated vector variation after the i'th portfolio rebalancing. (see Sections 4.2 and 4.2.4)

- $\overline{W}(\rho)$ and $\underline{W}(\rho)$: Model pruning constraints; maximum and minimum amount of accumulated vector variation after $\rho \in [0, T]$ time has elapsed. (see Sections 4.2 and 4.2.4)

Chapter 5 - Notation used for worst-case estimates:

- $\mathcal{N}_E(x, \mathcal{T})$: worst-case estimate of a set of emprically measured chart changes. (see Section 5.4)

- NOTE: *worst-case* estimates of pruning constraints in model building ($\overline{X}(i)$, $\underline{X}(i)$, $\overline{N}(\rho)$, $\underline{N}(\rho)$, $\overline{N}(W_i)$, $\underline{N}(W_i)$, $\overline{T}(i)$, $\underline{T}(i)$, $\overline{T}(W_i)$, $\underline{T}(W_i)$, $\overline{W}(i)$, $\underline{W}(i)$, $\overline{W}(T_i)$, and $\underline{W}(T_i)$) are defined in Section 3.4.

Chapter 1

Introduction

The theory of asset pricing has been studied extensively throughout the academic and financial literature, with a large emphasis being on the study of options pricing. A classic setting of asset pricing utilizes a Black-Scholes model, where stock prices evolve by a Geometric Brownian Motion. It is shown in Eberlein and Jacod [1997] that this model in particular has serious deficiencies from the point of view of the distribution of returns as well as from the point of view of its path properties. For example, the paper states that returns of stochastic processes may not be observable quantities. Then to overcome such deficiencies, market models will often incorporate assumptions which cause an investor to enter the realm of incomplete models. If models are constructed with the absence of arbitrage opportunities such market models will yield non-unique prices.

That is, Eberlein and Jacod [1997] discuss that assumptions used in order to construct these models and obtain asset prices, models yield non-informative super- and sub-replication bounds. The obtained asset prices degenerate down to *absolute bounds*, where the term *absolute bounds* refers to super- and under-replication bounds that hold for any possible no-arbitrage model; hence rendering the bounds useless or non-informative. The reference Pfleiderer [2014] mentions that although theoretical models are necessary to understand our financial systems, it is often found that models in finance 'cherry pick' assumptions in order to force models to produce given results. If these assumptions are difficult, or impossible, to relate to the real world, models based on such premises may be unreliable. The aforementioned papers should show that although stochastic models are powerful tools in understanding how financial systems work, they can also mislead.

Contrary to literature mainly focussing on replicating portfolios and options pricing, the literature on superhedging risky assets with a portfolio of risky assets is not extensive. More so, there is not an extensive amount of literature on the use of risky numeraires in dynamic asset pricing. When the numeraire is a random process, the pricing of a claim whose value has been transformed under change of numeraire, e.g. under a change of currency, has to take into account the risks existing on the foreign market (see Privault [2019] for this claim along with examples of changes of numeraire where the numeraire is a random process). In order to perform a fair pricing, one must determine a valid probability measure, under which the transformed process will be martingale.

Filipovic [2007] goes to show that when pricing assets there is no optimal numeraire, or rather, there exists no optimal numeraire that yields lower solvency capital requirements than any other numeraire. This means that there does not exist a numeraire which enables the investor to obtain an expected return *greater* than a risk-free rate. While being aware of the fact from Filipovic [2007], the setting we provide allows for an arbitrary numeraire to be chosen. We then observe the geometric effect of changing numeraire and our ability to construct market models that yield informative price bounds.

This paper addresses the same issue explored in Ferrando et al. [2019a]: can we justify simple models for asset evolution where we can evaluate sub- and super-replication prices and, in such a way that these quantities have useful risk-rewards characteristics? In contrast to Ferrando et al. [2019a], which has as a main focus derivative pricing while using a simple bank account as numeraire, we focus our methodologies on pricing one risky asset in terms of another one while incorporating an arbitrary numeraire

An algorithm construction, and its mathematical justifcation, that evaluates the sub- and super-replication bounds in a *probability-free* setting has been given in Degano et al. [2018]. It has been applied in Ferrando et al. [2019a] to obtain call option prices. This same paper also argues that the reason for price bounds degenerating to absolute bounds may be due to assumptions used to construct stochastic processes. These processes utilize an assumed probability distribution which implicity incorporates analytical constraints, leading to price paths not comparable to realistic outcomes. Thus, one may hypothesize that the construction of a market model through an operational, non-probabilistic point of view may provide an investor with informative worst-case price bounds which resemble realistic price paths more closely.

1.1 Logical constraints from adopting an operational setting and a superhedging methodology.

We adopt the following general point of view: we consider an investor that looks for an investment opportunity by trading one asset x^1 against another asset x^2. The investor's model may signal a relative misprice between the assets. Under such conditions, the investor will sell (or short) an asset, say x^2, while the other asset, say x^1, is used to superhedge x^2. We expand on these financial operations below once further needed concepts are introduced.

In this Introduction, we rely on an informal, descriptive, meaning for several technical words (estimation, worst case, calibration etc). These words will be given a more precise meaning for the specific models that we introduce later on. In any case, in this Introduction, we try our best to provide a context and a preliminary meaning for the use we will make of several technical terms and notions. We rely on some intended repetition to better describe the overall approach.

The main two components for constructing our models are: an *operational setting* and a *superhedging methodology*. The former notion assumes an investor with a well defined portfolio rebalancing strategy who operates on data through measurements and portfolio rebalances. For us, such setup mostly implies how historical data will be used to construct future scenarios. There are several alternatives to proceed i.e., the methodology does not prescribe a unique way of constructing a model. On the other hand, the superhedging approach is a natural method to adopt once we decline to assign probabilities to future scenarios (as we generally do in the thesis).

By construction, and as we argue below, the proposed models will never signal an investment opportunity which is an arbitrage opportunity. Therefore, in order to invest, there is the need to take in risk. To explain how we approach the notion of risk we indicate that we split the model construction in two stages. **1**) In the first stage, a worst case methodology proposes a set of possible future scenarios (trajectories). In a close analogy with stochastic processes, one can think that one is proposing the support of a process. Such set of trajectories can be shrunk by means of historical frequencies; these are the measured ratios of occurrence of the event in question over the total number of historical possibilities. Such a form of taking in risk could be labeled *worst case risk*. **2**) In a second stage, once the set of trajectories is available, there is the reasonable possibility to place a probability distribution on such set. This we consider a prerogative of our investor and leads to the possibility of providing a risk analysis for the profit and loss (P&L) function. Such function provides the gains and losses as a function of individual trajectories and associated to the superhedging strategy but with an initial investment being different, smaller, that the superhedging upper bound price.

The word *calibration* refers to setting the values of parameters that are under the control of the investor. Criteria for calibration are model specific and so, will be described later in the thesis. On the other hand, *estimation* here means setting the values of parameters that are not under the control of the investor; those parameters are set through the observation of historical trajectories. The main criteria for estimation is worst case historical estimation which, we expect, should be quite clear for each of our models. The other criteria is *risk-*

taking estimation which is a graduation from worst case by means of historical frequencies (this is what we called above “worst case risk”).

It should be clear that many of our model choices could also be considered as other sources of risk. In particular, there is the (perennial) question: how reliable are these historical frequencies to account for the likelihood of future events? This seems an impossible question to address in general; in particular, the answer depends on specific characteristics of the model construction, estimation of some parameters and calibration of other parameters. One possibility to address such question could be to develop a measure of robustness across the different modeling constructions that we face. These kinds of questions fall outside the scope of the thesis which concentrates in implementing a class of models and illustrating their risk-reward trade offs.

The proposed models have a direct financial meaning and are based on a general methodology. We emphasize that all model's components have an empirical meaning and it is this characteristic that makes them suitable for empirical testing or to be used in conjunction with machine learning. This is in contrast to the usual models driven by an apriori prescribed noise which are not easily falsifiable. Also, using operational-type models for investment decisions, allows to attach a definite meaning to each such decision. For example, we will know that taking into a certain level of risk amounts to neglecting the future occurrence of a specific set of trajectories and we also will know their historical measured frequency. Most importantly, as future trajectories unfold, we know in real time if they belong to the model or not.

To go back to our opening paragraph and to better describe the overall goal of our thesis: we want to evaluate the quality of our models used to signal investment opportunities. The thesis studies the quality of our proposed models in a rather qualitative and indirect way (as opposed to performing out of sample testing). The fact that the models have a direct empirical meaning allow us to analyze features of the models that can be interpreted in investment terms.

In this thesis, when we use the words "operational setting", we mean a model construction methodology where an investor has adopted a certain investment pattern (e.g., the investor may trade after five minutes have elapsed). Given such an "operational" setting we then look to construct future scenarios that reflect such constraints. Say, we look at possible, historical, stock changes that took place in five minute intervals. We remark that fixing an operational setting does not uniquely imply a model construction method (this straightforward fact will be apparent once we introduce the models' constructions). In fact there may be several possibilities to proceed for model construction. Interestingly, the methodology can be deployed also in several financial situations. For example: Do we look for investment opportunities several trading steps into the future or after a single trade? Do we consider overnight effects? What are the effects of a time horizon and sampling frequency? How does changing the numeraire affects the model's implications? Due to the number of parameters required to create our models, analyzing each and every one of these aspects is out of the scope of this thesis. We do however wish to observe the effect of choosing arbitrary numeraires on overall asset values.

We emphasize that our adopted investment strategy is to superhedge and underhedge one stock by means of trading on another stock. This leads to values of super and under pricing. That is, we obtain price bounds for the price of one stock relative to another stock, of course the roles of the stocks can be reversed (we also have the flexibility of using a third stock as numeraire). That is, our model provides two prices today, one leading to superhedging and another one to underheging. These properties are guaranteed to hold for all trajectories in the model. For simplicity, in many cases, we may only refer to superhedging and omit mentioning underhedging. Notice that superheging is a very stringent investment strategy as it guarantees the existence of a portfolio in terms of x^1 that will upperbound, for all trajectories in the model, x^2 at a future time. Given that the models are estimated under a worst case point of view, it follows that the initial investment, denoted by $\overline{V}_0(\mathbf{X}_0, X^2, \mathcal{M})$, required to set up the superhedging portfolio will be high ($\mathcal{M}$ indicates the particular model). This fact implies that a sizable risk needs to be taken to reduce this value to just x_0^2. As the models can be deployed in different ways and at different market conditions, the amount of risk and the superhedging rewards, will vary. In essence, the investor can then analyze the market as a casino where the odds change and they can be assessed relative to an investor based model that is objective and interpretable in financial terms.

The reference Ferrando et al. [2019a] superhedged European options with the underlying stock while here we superhedge a given stock with another one. In particular, in the present setting, there is no need to target the superhedging at a pre-specified future time. Another key difference between the setting of our thesis and Ferrando et al. [2019a] is that the market price of the option at present time (i.e. at the time when the model is being set up) was not needed as part of the trajectory market construction in Ferrando et al. [2019a]. This implies that there is the possibility that the superheding and underhedging model prices may uncover an arbitrage opportunity involving the selling/buying of the option and trading on the stock. In other words, from the model's point of view, the market could misprice an option leading to an arbitrage opportunity (from the point of view of the model and when trading with the stock and option). The latter means that in Ferrando et al. [2019a] there could be an investment opportunity without the need to take in risk.

In contrast to the situation described above, and in the setting of the present thesis, we construct vector valued trajectory models for two stocks and both of their initial market given prices are used for the model construction. The off-shoot is that the model will never predict a (riskless) mispricing leading to an arbitrage opportunity. This can be seen by the fact that if $\underline{V}_0(\mathbf{X}_0, X^2, \mathcal{M}) \leq \overline{V}_0(\mathbf{X}_0, X^2, \mathcal{M})$ is the model price interval we can then prove that $\underline{V}_0(\mathbf{X}_0, X^2, \mathcal{M}) \leq x_0^2 \leq \overline{V}_0(\mathbf{X}_0, X^2, \mathcal{M})$. This result follows from the no arbitrage property (actually it also follows from the more general 0-neutral property) that holds for the constructed vector valued model. Therefore, in order to search for desirable investment opportunities, we look for risk-taking investments where we sell (or short) stock x^2 and invest $v \leq x_0^2$ in our superhedging strategy. Given that the actual investment required in order to superhedge along all model trajectories is the amount $\overline{V}_0(\mathbf{X}_0, X^2, \mathcal{M})$, and $\overline{V}_0(\mathbf{X}_0, X^2, \mathcal{M}) > v$, we will then face the risk of not being able to superhedge along a certain subset of the model trajectories. One concludes that such subset contributes to the existence of the gap $[v, \overline{V}_0(\mathbf{X}_0, X^2, \mathcal{M})]$.

1.2 Possibilities to Deploy the General Methodology

Here we indicate, in a practical and direct way and without specific details, several possibilities and guidelines to construct and deploy our general methodology.

We propose future scenarios (trajectories) generated from historical events which reflect the operational setting of the investor. We also confine ourselves to a time horizon of one day used as a reference to complete our portfolio transactions. The reason for choosing a time horizon of one day, is that this allows the investor to neglect any overnight effects which might otherwise occur in stock prices due to overnight trading or investor's sentiments created from newly released news. One could deploy the methodology over time horizons of several days as well, however, in this case one should adjust the estimation and calibration methodology accordingly (we try to present our methods using general criteria which can hopefully be transported to different regimes).

These future scenarios could represent a single step or a multiple set of steps. Here a *step* means a potential portfolio rebalance. Multi-step trajectories are constructed in a combinatorial way from single step historical gathered events. Trajectories are vector valued $(X_i^1, X_i^2, \ldots)$ where the $\ldots$ indicate potential additional trajectory coordinates like some/all of: i, W_i, T_i. Here i is the number of steps along a given path, W_i the vector variation between rebalances and T_i the time of the i-th portfolio rebalance. Different models in this thesis are built by adding or removing some of these additional coordinates.

We remark that choosing which variables to include in the trajectory coordinates *is the crucial modeling decision* and we make no claim for special qualities for our selections besides simplicity. We only remark in passing that portfolio rebalances obey the inequality

$$||(X_{i+1}^1, X_{i+1}^2) - (X_i^1, X_i^2)|| \geq \delta \tag{1.1}$$

where δ is investor dependent and hence calibrated. In this thesis the use of the notation $||\cdot||$ indicates the Euclidean norm, and other choices of the norm are not explored. In the case that (X_t^1, X_t^2) is a martingale process, we remark that the number of times for which (1.1) holds, i.e. the number of steps, obeys Burkholder's δ-escape inequalities which are vector generalizations of Doob's upcrossing inequalities (Burkholder [1989]). The generality of Burkholder's inequalities suggests that the number of steps, over a certain time span, has some regularity i.e. stable behavior suitable for model building.

Forcing portfolio rebalances to obey Equation (1.1) allows for trajectory market models to be constructed as is done in Ferrando et al. [2019a]. While we do construct market models in this manner, in this paper we also introduce a new type of model construction. Instead of portfolio rebalances obeying Equation (1.1), we force them to obey the inequalities

$$\begin{aligned} 0 &\leq (X_{i+1}^2 - X_i^2) \leq \delta_{up}(X_{i+1}^1 - X_i^1), \\ 0 &\geq (X_{i+1}^2 - X_i^2) \geq \delta_{down}(X_{i+1}^1 - X_i^1), \end{aligned} \tag{1.2}$$

restricting the investor to movements where the asset X^1 moves in the same direction as the asset X^2. Models which obey (1.2) are expected to produce a smoother relationship since assets must always move together.

Let $N^*(x, \mathcal{T}, T_i)$ and $N_*(x, \mathcal{T}, T_i)$ represent the historical maximum and minimum number for which the inequality $||(x^1(t_i), x^2(t_i)) - (x^1(t_{i-1}), x^2(t_{i-1}))|| \geq \delta$ holds over historical time windows $[0, T_i] \subseteq \mathcal{T}$ where $\mathcal{T}$ is a historical time interval for which we have access to the data $(x^1(t), x^2(t)), t \in \mathcal{T}$. Here $x(t)$ denotes a chart, or rather, a multidimensional time series of values of a set of assets. The interval $[N_*(T_i), N^*(T_i)]$ will be used, during model construction, to curtail, or trim, the mentioned combination of one step events when building a multi-step trajectory. This interval can also be shrunk to remove trajectories (what we have called previously *worst case risk*). Other constraining intervals like $[W_*(i), W^*(i)]$, $[T_*(i), T^*(i)]$ etc are also available. A criteria that we use to assess the usefulness of a trajectory coordinate is to look at how, for example, $[N_*(T_i), N^*(T_i)]$ behaves as we aggregate more and more historical data: does $N^*(T_i) - N_*(T_i)$ becomes stable (i.e. does it stop growing as we aggregate historical data) and informative (i.e. of small enough value)? In other words, the worst case range $[N_*(T_i), N^*(T_i)]$ is informative regarding the coordinate i if its range provides information that allows to restrict the manifold of future possibilities for trajectory proliferation.

Chapter 2

Background Material

2.1 General, Discrete, Trajectory Based Models

The thesis relies on discrete time, one-dimensional, non-probabilistic market models as introduced in Ferrando et al. [2019b] and extended to the general multidimensional case in Degano et al. [2018]. Reference Ferrando et al. [2019a] gives a detailed outline for creating a trajectory based (1-dimensional) market model with operational assumptions which is then used to price a European call option on a stock. This paper aims to extend the research in Ferrando et al. [2019a] in order to develop a 2-dimensional trajectory based market model describing the joint movements of two stocks (each expressed in terms of a third numeraire stock). Each such 2-dimensional market model is then used to superhedge or underhedge one stock with respect to another. In this section we review the theoretical framework used to create that type of market.

We use the words asset and stock as synonymous and mostly refer to the "superhedging" operation while neglecting to mention, for simplicity, that we are also obtaining underhedging information (a underhedging portfolio and a underhedging price).

The general definitions start with a set of trajectories $\mathcal{S}$ with $\mathbf{S} \in \mathcal{S}$ being sequences $\mathbf{S} = \{\mathbf{S}_i\}_{i \geq 0}$ of stock prices (plus some additional coordinates as well) expressed in a currency numeraire. We then quickly allow to change numeraire and obtain sequences $\mathbf{X} = \{\mathbf{X}_i\}_{i \geq 0}$ the notation then changes from $\mathbf{X}(S) = \{\mathbf{X}_i(S)\}_{i \geq 0}$ to $\mathbf{X} = \{\mathbf{X}_i\}_{i \geq 0} \in \mathcal{X}$ ($\mathcal{X}$ the set of trajectories in a given numeraire units). So the set $\mathcal{S}$ is removed from further discussion as its explicit presence is not required in the remaining of the thesis as we explain next. The modeling set $\mathcal{S}$ is basic to Degano et al. [2018] and Ferrando et al. [2019a] as their analysis is in currency units, the reference Degano et al. [2018] does a theoretical analysis on the effect that a numeraire change may have on the no-arbitrage property, and other notions, and for this reason that reference requires to start with the original set $\mathcal{S}$ (satisfying certain properties). On the other hand, in the present thesis, we can start with a set $\mathcal{X}$ and construct it so that it satisfies the properties we require in order to have a well defined framework leading to price bounds and associated superhedging portfolios. Of course, our data, which we use to build $\mathcal{X}$, is given to us in currrency units but we do change the data to the required numeraire and so we can use this transformed data to construct $\mathcal{X}$.

2.1.1 Multidimensional Trajectorial Markets with Arbitrary Numeraire

This section provides a short summary of the formal definitions required to develop a trajectory based market model. We utilize the multidimensional definitions as given in Degano et al. [2019] however, following sections will only be concerned with $d = 2$.

Definition 1 (Trajectory Set)**.** *Given* $\boldsymbol{s}_0 = \{s_0^0, \dots, s_0^d\} \in \mathbb{R}^{d+1}$ *and* $w_0 \in \Omega_0$*, a trajectory based set* $\mathcal{S}$ *is a subset of the following:*

$$\mathcal{S}_\infty(s_0, w_0) \equiv \{\mathbf{S} \equiv \{\mathbf{S}_i \equiv (S_i, i, T_i, W_i)\}_{i \geq 0} : S_i \in \Sigma_i,\ W_i \in \Omega_i,$$

$$T_i \in \Delta\ \mathbb{N}_+,\ i \in \mathbb{N},\quad (S_0, W_0) = (s_0, w_0)\}$$

where $\Sigma = \{\Sigma_i\}$ *is a family of subsets of* $\mathbb{R}^{d+1}$*,* $\Omega = \{\Omega_i\}$ *is a family of sets and* $\Delta \in \mathbb{R}_+$*. Elements of* $\mathcal{S}$ *are called trajectories.*

The real numbers S_i^k in $S_i = (S_i^0, \dots, S_i^d)$ should be considered as having dimensional units relative to a currency numeraire, say $[S_i^k] = \$/[S^k]$ where \$ is one unit of the said currency and $[S^k]$ is a unit of asset S^k. The additional variables beyond the coordinate i, namely T_i and W_i do not play any role in trading considerations (in particular they do not play an explicit role in the computation of the superhedging and subhedging portfolios and associated price bounds). Moreover, as it will be detailed later in the thesis, these extra coordinates are included only in some of the models and for the sole purpose (but crucial) of constraining the combinatorial growth of the trajectory set. Their meaning is best described later in the thesis once further details have been introduced.

In financial theory it is important to observe the behaviour of asset prices with respect to the price of a separate asset, otherwise known as a *numeraire*. We then reserve S^0 to represent the numeraire used to obtain the discounted prices

$$X_i(S) \equiv (X_i^1(S), X_i^2(S), \dots, X_i^d(S)) \equiv \left(\frac{S_i^1}{S_i^0}, \frac{S_i^2}{S_i^0}, \dots, \frac{S_i^d}{S_i^0}\right),$$

$$D \equiv \{S_i^0 > 0\}.$$

$X_i^j(S)$ will then represent the value of asset j in units of the numeraire S^0. The numerical value of $X^j(S_i)$ (i.e. stripped from its units), is the number of units of the asset S^0, now the numéraire, which are required to acquire one unit of the S^j asset.

As explained in Section 2.1 our models construct the coordinates X_i^1, X_i^2 directly, that is, without first depending on the quantities S_i^1, S_i^2. For this reason our trajectories will drop any reference to elements of $\mathcal{S}$ and will be elements of a trajectory set denoted $\mathcal{X}$, its elements are sequences of the form $\mathbf{X}_i = (X_i, i, T_i, W_i)$ (construction of a variety of sets $\mathcal{X}$ is one of the major goals of the thesis). Notice that $X_i = (X_i^1, \ldots, X_i^d)$. It is also natural to change notations and denote our numeraire asset S^0 by X^0, something we do from now on; clearly $X_i^0 = 1$ for all i. We comment in passing that there is an abuse of notation at play here as W_i may be affected by the change of numeraire (i.e. going from currency units \$ to $[X^0]$ units). Nonetheless, this should not cause any problems in later developments in the thesis as W_i will naturally be also constructed in relation to the quantities X_i^1, X_i^2.

Definition 2 (Portfolio Set)**.** *A portfolio H is a sequence of functions $H \equiv \{\Phi_i = (H_i^0, H_i) = (H_i^0, H_i^1, \ldots, H_i^d)\}_{i \geq 0}$ and $H_i^0 : \mathcal{X} \to \mathbb{R}$, $H_i : \mathcal{X} \to \mathbb{R}^d$. Then:*

1. *A portfolio H is said to be admissible for the trajectory set $\mathcal{X}$ if for each $\mathbf{X} \in \mathcal{X}$ there exists an integer $N_H(\mathbf{X}) > 0$ such that $H_i(\mathbf{X}) = 0$ for all $i \geq N_H(\mathbf{X})$.*

2. *A portfolio H is said to be self-financing at $\mathbf{X} \in \mathcal{S}$ if for all $i \geq 0$ the following holds:*

$$H_i^0(\mathbf{X}) + H_i(\mathbf{X}) \cdot X_{i+1} = H_{i+1}^0(\mathbf{X}) + H_{i+1}(\mathbf{X}) \cdot X_{i+1} \tag{2.1}$$

3. *A portfolio H is called non-anticipative if for all $\mathbf{X}, \mathbf{X}' \in \mathcal{X}$, satisfying $X'_k = X_k$ for all $0 \leq k \leq i$, then it follows that $\Phi_i(\mathbf{X}) = \Phi_i(\mathbf{X}')$,*

where we have used the dot product notation $x \cdot y$, $x, y \in \mathbb{R}^d$.

To avoid confusion we provide a dimensional analysis of (2.1). Let $\mathbf{1}_Z$ denote one unit of asset Z and use the notation $[Z] = \mathbf{1}_Z$. Take $d = 1$ for simplicity, then the left and right hand side of (2.1) have the same units: $\mathbf{1}_{X^0} + \mathbf{1}_{X^1} \frac{\mathbf{1}_{X^0}}{\mathbf{1}_{X^1}} = \mathbf{1}_{X^0} + \mathbf{1}_{X^0}$. Notice that the number of shares $H_i^k(\mathbf{X})$ comes with units $[H_i^k(\mathbf{X})] = \mathbf{1}_{X^k}$

Definition 3 (Trajectory Based Market)**.** *Given $x_0 \in \mathbb{R}^d$, $w_0 \in \Omega_0$, a trajectory based set $\mathcal{X} \subseteq \mathcal{X}_\infty(x_0, w_0)$ and a portfolio set $\mathcal{H}$, we say that $\mathcal{M} = \mathcal{X} \times \mathcal{H}$ is a trajectory based market if it satisfies the following properties:*

1. *All $\Phi \in \mathcal{H}$ are self-financing and $\Phi = 0 \in \mathcal{H}$.*

2. *For each $(\mathbf{X}, \Phi) \in \mathcal{M}$ there exists $N_\Phi(\mathbf{X}) \in \mathbb{N}$ such that $\Phi_k(\mathbf{X}) = \Phi_{N_\Phi}(\mathbf{X}) = 0$ for all $k \geq N_\Phi(\mathbf{X})$.*

Given $\mathbf{X} \in \mathcal{X}$ and $k \geq 0$, we will use the notation $V_k^{\Phi}(\mathbf{X})$ for the value of the portfolio $\Phi \in \mathcal{H}$:

$$V_k^{\Phi}(\mathbf{X}) \equiv H_k^0(\mathbf{X}) + H_k(\mathbf{X}) \cdot \mathbf{X}_k. \tag{2.2}$$

$V_k^{\Phi}(\mathbf{X})$ can be interpreted as the value of the portfolio at the beginning of the stage k expressed in units of the numéraire. In addition, $G_k^{\Phi}(\mathbf{X})$ will denote the profits generated up to the stage k associated with $\Phi \in \mathcal{H}$ for a trajectory $\mathbf{X} \in \mathcal{X}$, that is

$$G_k^{\Phi}(\mathbf{X}) \equiv \sum_{i=0}^{k-1} H_i(\mathbf{X}) \cdot \Delta_i X \text{ for } \; k \geq 0 \; \text{ where } \; \Delta_i X = X_{i+1} - X_i. \tag{2.3}$$

$G_k^{\Phi}(\mathbf{X})$ reflects, in terms of the numéraire, the net gains accumulated by the portfolio Φ at the beginning of the k-th stage.

Remark 1. *Note that, for any portfolio Φ and any trajectory $\mathbf{X}$, it is true that $G_k^{\Phi}(\mathbf{X}) = -G_k^{-\Phi}(\mathbf{X})$.*

Proposition 1. *Let $\mathcal{X}$ be a space of trajectories, and let Φ be a portfolio on $\mathcal{X}$. Then the following statements are equivalent:*

1. *Φ is self-financing.*
2. *$H_{i-1}^0(\mathbf{X}) + H_{i-1}(\mathbf{X}) \cdot X_i = H_i^0(\mathbf{X}) + H_i(\mathbf{X}) \cdot X_)$ for all $\mathbf{X} \in \mathcal{X}$ and $i \geq 0$.*
3. *$V_k^{\Phi}(\mathbf{X}) = V_0^{\Phi} + G_k^{\Phi}(\mathbf{X}) = H_0^0 + H_0 \cdot X_0 + \sum_{i=0}^{k-1} H_i(\mathbf{X}) \cdot \Delta_i X$ for all $k \geq 0$.*

Remark 2. *From the previous Proposition, we know that the H^0 component of a self-financed portfolio Φ satisfies*

$$H_k^0(\mathbf{X}) - H_{k-1}^0(\mathbf{X}) = -(H_k(\mathbf{X}) - H_{k-1}(\mathbf{X})) \cdot X_k. \tag{2.4}$$

Given that

$$H_0^0 = V_0^{\Phi} - H_0 \cdot X_0, \tag{2.5}$$

the sequence H^0 is completely determined by the initial investment V_0^{Φ} and H by means of the previous equations.

We will say that the market $\mathcal{M} = \mathcal{X} \times \mathcal{H}$ is *semi-bounded* if for each $\Phi \in \mathcal{H}$ there is $n_{\Phi} \in \mathbb{N}$ such that $N_{\Phi}(\mathbf{X}) \leq n_{\Phi}$ for all $\mathbf{X} \in \mathcal{X}$ and it is *n-bounded*, for $n \in \mathbb{N}$, if $N_{\Phi}(\mathbf{X}) \leq n$ for each pair $(\mathbf{X}, \Phi) \in \mathcal{M}$.

The thesis considers $d = 2$, that is, we construct trajectory sets for two stocks (but a third stock is also present, but not modeled explicitly, and acting as numeraire). Despite this framework, the construction of superhedging and subhedging portfolios *involves a single asset*. That is, portfolios are built trading with a single stock (the numeraire, as is well known, is also a second asset being traded and plays the analogue role to a bank account with no interest rates). The choice of $d = 2$ and the ensuing implication of trading with only one asset was decided to keep the setting as simple as possible. Moreover, this restriction allows us to make use of results from Degano et al. [2018] where it is proven how to evaluate, by an algorithm, superhedging and subhedging portfolios and price bounds *in the case of trading with a single asset*. This dynamic programming based algorithm is referred to as the *convex hull* algorithm and is briefly summarized in Ferrando et al. [2019a] but fully developed in Degano et al. [2018]. We rely on this algorithm but do not provide much details besides some comments when needed. The extension of the convex hull algorithm to trading with more than one stock is an interesting open problem not addressed in the thesis.

2.1.2 No-Arbitrage and 0-Neutrality

Here we follow Degano et al. [2019] and refer to that paper for details and proofs (as well as Degano et al. [2018] and Ferrando et al. [2019b] for connections with the stochastic literature and references). Restrictions will be imposed into trajectory markets so that no investor has the possibility of generating a profit without the need to incur in a risk of loss. Such investment opportunity is called an arbitrage opportunity.

Definition 4 (Arbitrage opportunity)**.** *Given a trajectory based market $\mathcal{M} = \mathcal{X} \times \mathcal{H}$, $\Phi \in \mathcal{H}$ is an* arbitrage opportunity *if:*

- $\forall \mathbf{X} \in \mathcal{X}$, $V^{\Phi}_{N_\Phi}(\mathbf{X}) \geq V^{\Phi}_0$.
- $\exists \mathbf{X}^* \in \mathcal{X}$ *such that* $V^{\Phi}_{N_\Phi}(\mathbf{X}^*) > V^{\Phi}_0$.

We say that $\mathcal{M}$ is arbitrage-free *if $\mathcal{H}$ does not contain arbitrage opportunities.*

It can be shown (Degano et al. [2019]) that the arbitrage-free condition is sufficient for the model to provide fair option prices (a well known result in the classical financial literature.)

Next we introduce a weaker criteria saying that the largest of the minimum possible gains that can be obtained by means of the strategies available in the market is 0.

Definition 5 (0-neutral market)**.** *Let $\mathcal{M} = \mathcal{X} \times \mathcal{H}$ a trajectory based market. We say that $\mathcal{M}$ is* 0-neutral *if*

$$\sup_{\Phi \in \mathcal{H}} \left\{ \inf_{\mathbf{X} \in \mathcal{X}} \; G^{\Phi}_{N_\Phi}(\mathbf{X}) \right\} = \sup_{\Phi \in \mathcal{H}} \left\{ \inf_{\mathbf{X} \in \mathcal{X}} \left[\sum_{i=0}^{N_\Phi(\mathbf{X})-1} H_i(\mathbf{X}) \cdot \Delta_i X \right] \right\} = 0. \tag{2.6}$$

If the largest minimum profit is zero, the 0-neutral market definition is intuitively saying that for each portfolio there is at least one possibility that the investor loses money, or at best, lose nothing. In Degano et al. [2019] it is shown that this property is also sufficient to obtain a pricing interval for financial derivatives. The next Proposition shows that the property of 0-neutrality is weaker than the arbitrage-free property.

Proposition 2. *Let $\mathcal{M} = \mathcal{X} \times \mathcal{H}$ an arbitrage-free trajectory based market. Then $\mathcal{M}$ is 0-neutral.*

It is clear how to generate simple examples of 0-neutral markets which contain arbitrage (see Degano et al. [2019]).

The following simple characterization of 0-neutral markets will be useful in the next section.

Proposition 3. *A trajectory based market $\mathcal{M} = \mathcal{X} \times \mathcal{H}$ is 0-neutral if and only if, for each $\Phi \in \mathcal{H}$ and $\epsilon > 0$ there exist $\mathbf{X}^\epsilon \in \mathcal{X}$ such that*

$$\sum_{i=0}^{N_\Phi(\mathbf{X}^\epsilon)-1} H_i(\mathbf{X}^\epsilon) \cdot \Delta_i X^\epsilon < \epsilon. \tag{2.7}$$

2.1.3 Local No-Arbitrage, Local 0-Neutrality and Geometric Characterizations

In discrete and finite time one can obtain necessary and sufficient conditions, only involving local properties of the trajectory set, implying trajectorial markets that are arbitrage-free (or 0-neutral). Such characterizations are the analogue of the equivalence of no arbitrage stochastic markets and the possibility to equivalently modify the stochastic process into a martingale process. Here we present the local conditions and indicate their geometric nature. Moreover, in the next section we use one such local condition to establish a useful result for our thesis.

Given $k \geq 0$, $\mathbf{X} \in \mathcal{X}$, define the *conditional set*:

$$\mathcal{X}_{(\mathbf{X},k)} \equiv \{\mathbf{X}' \in \mathcal{X} : \mathbf{X}_i = \mathbf{X}'_i \;\; \forall\, 0 \leq i \leq k\}.$$

Notice that $\mathcal{X}_{(\mathbf{X},0)} = \mathcal{X}$ and that if $\mathbf{X}' \in \mathcal{X}_{(\mathbf{X},k)}$ then $\mathcal{X}_{(\mathbf{X}',k)} = \mathcal{X}_{(\mathbf{X},k)}$. We will refer to the tuple $(\mathbf{X}, k)$ as a *node*.

Define

$$\Delta X(\mathcal{X}_{(\mathbf{X},k)}) \equiv \{\Delta_k X' : \; \mathbf{X}' \in \mathcal{X}_{(\mathbf{X},k)}\} \subseteq \mathbb{R}^d, \tag{2.8}$$

where $\Delta_k X' = (X'_{k+1} - X'_k)$ has been introduced before.

We will refer as *local* to any property relative to a node $(\mathbf{X}, k)$ and only involving elements of $\Delta X(\mathcal{X}_{(\mathbf{X},k)})$.

The definitions below are the local counterpart of those of arbitrage-free and 0-neutral for the whole market. We are going to provide results characterizing the global properties in terms of the local ones. For convenience set: $\mathcal{H}_{\mathcal{X}} \equiv \{H : (H^0, H) \in \mathcal{H}\}$.

Definition 6 (Local notions)**.** *Given a trajectory based market $\mathcal{M} = \mathcal{X} \times \mathcal{H}$, let $\mathbf{X} \in \mathcal{X}$ and $k \geq 0$.*

1. *$(\mathbf{X}, k)$ is called an arbitrage-free node with respect to $\mathcal{H}$ if*

$$[H_k(\mathbf{X}) \cdot \Delta_k X' = 0 \quad \forall \mathbf{X}' \in \mathcal{X}_{(\mathbf{X},k)}] \quad or \quad [\inf_{\mathbf{X}' \in \mathcal{X}_{(\mathbf{X},k)}} H_k(\mathbf{X}) \cdot \Delta_k X < 0],$$

   for all $H \in \mathcal{H}_{\mathcal{X}}$.

2. *$(\mathbf{X}, k)$ is called a 0-neutral node with respect to $\mathcal{H}$ if, for all $H \in \mathcal{H}_{\mathcal{X}}$:*

$$\inf_{\mathbf{X}' \in \mathcal{X}_{(\mathbf{X},k)}} H_k(\mathbf{X}) \cdot \Delta_k X' \leq 0.$$

$\mathcal{M}$ is called locally arbitrage-free (0-neutral) if each $(\mathbf{X}, k)$ is an arbitrage-free (0-neutral) node w.r.t. $\mathcal{H}$. A node that is not arbitrage-free w.r.t. $\mathcal{H}$, will be called an arbitrage node w.r.t. $\mathcal{H}$.

Notice that an arbitrage-free node w.r.t. $\mathcal{H}$ is always 0-neutral w.r.t. $\mathcal{H}$. Clearly, there are natural examples of nodes which are 0-neutral w.r.t. $\mathcal{H}$ but no arbitrage-free w.r.t. $\mathcal{H}$ (hence these are arbitrage nodes). It is then of interest that Degano et al. [2019] obtains results that justify option prices obtained for general 0-neutral markets (in particular these markets may contain 0-neutral nodes which are arbitrage nodes w.r.t. $\mathcal{H}$).

Admittedly, attaching the qualifier "w.r.t. $\mathcal{H}$" is a precise statement but can be unnecessarily distracting. The reader could replace the appearance of $H_k(\mathbf{X})$ and the quantifier $\forall H \in \mathcal{H}_{\mathcal{X}}$ by $\forall h \in \mathbb{R}^d$. In other words, no matter the direction of an investment h there is always a possibility of losing money (in the no-arbitrage case) or at best breaking even (in the 0-neutral case). In fact, we provide below sufficient conditions on trajectory nodes that imply that those nodes are arbitrage-free (0-neutral) w.r.t. $\mathcal{H}$ *for any* $\mathcal{H}$.

Below, the notation $\mathrm{ri}(\mathrm{co}(E))$ for $E \subseteq \mathbb{R}^d$ refers to the relative interior of the convex hull generated by E. Similarly $\mathrm{cl}(\mathrm{co}(E))$ refers to the closure of the convex hull generated by E.

Proposition 4. *Given a trajectory based market $\mathcal{M} = \mathcal{X} \times \mathcal{H}$, consider a node $(\mathbf{X}, k)$.*

1. *If:*
$$0 \in \mathrm{ri}\left(\mathrm{co}\left(\Delta X(\mathcal{X}_{(\mathbf{X},k)})\right)\right). \tag{2.9}$$
then $(\mathbf{X}, k)$ *is an arbitrage-free node w.r.t.* $\mathcal{H}$*.*

2. *If:*
$$0 \in \mathrm{cl}\left(\mathrm{co}\left(\Delta X(\mathcal{X}_{(\mathbf{X},k)})\right)\right). \tag{2.10}$$
then $(\mathbf{X}, k)$ *is a* 0*-neutral node w.r.t.* $\mathcal{H}$*.*

According with these results we introduce the following notions.

Definition 7 ($\mathcal{H}$-Independent local properties)**.** *A node* $(\mathbf{X}, k)$ *is called* arbitrage-free *if (2.9) is satisfied; it is called* 0-neutral *if (2.10) is satisfied. We call* $\mathcal{X}$ *locally arbitrage-free (locally* 0*-neutral), if every node* $(\mathbf{X}, k)$ *is arbitrage-free (*0*-neutral).*

So, if $\mathcal{X}$ is locally arbitrage-free (locally 0-neutral), then $\mathcal{M} = \mathcal{X} \times \mathcal{H}$ is locally arbitrage-free (locally 0-neutral) for any $\mathcal{H}$.

The geometric condition (2.9) is known (with the necessary modifications) in the stochastic literature but may not have been appreciated as a fundamental characterization of the property of arbiytrage-free. As is shown in Degano et al. [2019], it gives an interesting point of view to see the no-arbitrage condition in geometrical terms. Finally, from (2.10), a 0-neutral node $(\mathbf{X}, k)$ which, in turn, is not an arbitrage-free node implies that $0 \in \mathbb{R}^d$ belongs to the boundary of $\mathrm{co}\left(\Delta X(\mathcal{X}_{(\mathbf{X},k)})\right)$. This is a fickle condition which should be rare to find in markets.

The local definitions introduced above allow us to ensure global conditions on a trajectory based market.

Theorem 1 (No arbitrage: local implies global)**.** *If* $\mathcal{M}$ *is locally arbitrage-free (as per Definition 6) and semi-bounded, then* $\mathcal{M}$ *is arbitrage-free.*

Similarly, the following Theorem shows that a trajectory based market will be 0-neutral if it is locally 0-neutral.

Theorem 2 (0-neutral: local implies global)**.** *Let* $\mathcal{M} = \mathcal{X} \times \mathcal{H}$ *be a semi-bounded trajectory market. Then if* $\mathcal{M}$ *is locally* 0*-neutral (as per Definition 6) then,* $\mathcal{M}$ *is* 0*-neutral.*

Converses to both theorems above also hold (proofs and additional statements are in Degano et al. [2019]).

2.1.4 Price Bounds for One Asset Relative to Another Asset

We consider $d = 2$ from now onward and depart slightly from Degano et al. [2019]; in that reference price bounds (for general options) are defined by means of multidimensional trading portfolios. Even though our setting is multidimensional in the sense that $d = 2 > 1$ we will be only trading with one dimensional portfolios. This difference will make it awkward to rely on results from Degano et al. [2019] and so we follow our own independent developments next.

The notation Z will be used for a general function defined on $\mathcal{X}$; it could be thought as the payoff of an option. In the present thesis we will only consider:

$$Z(\mathbf{X}) = X^k_{N(\mathbf{X})} \text{ where } k = 1, 2 \text{ denotes the stock that is being superhedged} \tag{2.11}$$

and $N(\cdot)$ is an integer valued stopping time, that is if $N(\mathbf{X}) = n$ then $N(\mathbf{X}) = N(\mathbf{X}_1, \ldots, \mathbf{X}_n)$.

The quantity $\overline{V}_k(\mathbf{X}, Z, \mathcal{M})$ below will denote the minimum amount of capital required, conditional on a node $(\mathbf{X}, k)$, to superhedge the payoff/function Z. An analogous, dual, interpretation, can be assigned to $\underline{V}_k(\mathbf{X}, Z, \mathcal{M})$. For simplicity in the notation we assume that we trade with asset X^1 in order to superhedge asset X^2, clearly these roles can be reversed.

Definition 8 (Conditional Minmax Bounds)**.** *Given a market $\mathcal{M} = \mathcal{X} \times \mathcal{H}$, $k \geq 0$, and $\mathbf{X} \in \mathcal{X}$. Define:*

$$\overline{V}_k(\mathbf{X}, X^2, \mathcal{M}) \equiv \inf_{H \in \mathcal{H}} \left\{ \sup_{\hat{\mathbf{X}} \in \mathcal{X}_{(\mathbf{X},k)}} \left\{ X^2_{N(\hat{\mathbf{X}})} - \sum_{i=k}^{N_H(\hat{\mathbf{X}})-1} H_i(\hat{\mathbf{X}})\ \Delta_i \hat{X}^1 \right\} \right\}. \tag{2.12}$$

*Also set $\underline{V}_k(\mathbf{X}, X^2, \mathcal{M}) \equiv -\overline{V}_k(\mathbf{X}, -X^2, \mathcal{M})$. We then call these quantities **price bounds** at node $(\mathbf{X}, k)$. For simplicity, we may use the notation $\overline{V}_k \equiv \overline{V}_k(\mathbf{X}, X^2, \mathcal{M})$ and $\underline{V}_k \equiv \underline{V}_k(\mathbf{X}, X^2, \mathcal{M})$ when it is clear what the conditioning node is and the fact that we are superhedging X^2.*

Notice that we have that $\overline{V}_k(\mathbf{X}, X^2, \mathcal{M}) = \overline{V}_k\big((\mathbf{X}_0, \mathbf{X}_1, \ldots, \mathbf{X}_k), X^2, \mathcal{M}\big)$ and so when $k = 0$ we can write the upper price bound simply as $\overline{V}_0(\mathbf{X}_0, X^2, \mathcal{M})$.

Also notice that $H_i(\hat{\mathbf{X}})\ \Delta_i \hat{X}^1 = H_i(\hat{\mathbf{X}})(\hat{X}^1_{i+1} - \hat{X}^1_i)$ and that $H_i(\hat{\mathbf{X}}) \in \mathbb{R}$. That is, we are trading with a single asset and have an ordinary product (as opposed to a dot product). That is, even though $d = 2$ and so we have a market model for two assets X^1, X^2 (in terms of numeraire X^0) we are requiring to trade with a single asset. Notice that previously we used $H_i = (H^1_i, \ldots, H^d_i)$ but hope the reader can differentiate the meaning of the symbol used from context.

We are going to prove $\underline{V}_k(\mathbf{X}, X^2, \mathcal{M}) \leq X_0^2 \leq \underline{V}_k(\mathbf{X}, X^2, \mathcal{M})$, a useful result for our thesis. In the proof we will need to rely on the quantities $\underline{U}_k(\mathbf{X}, X^2, \mathcal{M})$, $\overline{U}_k(\mathbf{X}, X^2, \mathcal{M})$ which we introduce next.

Definition 9 (Dynamic Bounds)**.** *Consider an n-bounded, discrete market $\mathcal{M}$; for a given function Z defined on $\mathcal{X}$, any $\mathbf{X} \in \mathcal{X}$, and $0 \leq i \leq n$ set*

$$\overline{U}_i(\mathbf{X}, Z, \mathcal{M}) = \begin{cases} \inf_{H \in \mathcal{H}} \sup_{\hat{\mathbf{X}} \in \mathcal{X}_{(\mathbf{X},i)}} [\overline{U}_{i+1}(\hat{\mathbf{X}}, Z, \mathcal{M}) - H_i(\mathbf{X})\ \Delta_i \hat{X}^1] & \text{if} \quad 0 \leq i < N(\mathbf{X}) \\ Z(\mathbf{X}) & \text{if} \quad i = N(\mathbf{X}). \end{cases} \tag{2.13}$$

Also define $\underline{U}_i(\mathbf{X}, Z, \mathcal{M}) = -\overline{U}_i(\mathbf{X}, -Z, \mathcal{M})$.

Under a general hypothesis on $\mathcal{H}$ (which we do not discuss so as not to derail our discussion), Corollary 4 from Degano et al. [2018] gives $\overline{U}_i(\mathbf{X}, Z, \mathcal{M}) = \overline{V}_i(\mathbf{X}, Z, \mathcal{M})$ which implies $\underline{U}_i(\mathbf{X}, Z, \mathcal{M}) = \underline{V}_i(\mathbf{X}, Z, \mathcal{M})$. These relationships will be used in the proof of Proposition 5 below.

In stochastic models, the existence of a price interval is given by the notion of an (stochastic) *arbitrage-free* market which leads to a collection of (equivalent) martingale measures. Such collection is used to evaluate upper and lower bounds. The present setting utilizes the notion of *0-neutrality* to obtain $\underline{V}_k \leq \overline{V}_k$, in fact we obtain a more detailed result below.

Proposition 5. *Consider a bounded market $\mathcal{M} = \mathcal{X} \times \mathcal{H}$ that is locally 0-neutral and such that the necessary hypothesis to apply Corollary 4 from Degano et al. [2018] hold. Then for any 0-neutral node $(\mathbf{X}, k)$, $0 \leq k \leq N(\mathbf{X})$:*

$$\underline{V}_k(\mathbf{X}, X^2, \mathcal{M}) \leq X_k^2 \leq \overline{V}_k(\mathbf{X}, X^2, \mathcal{M}). \tag{2.14}$$

Proof. The hypothesis on $(\mathbf{X}, k)$ means that (2.10) holds, in turn this implies (see Degano et al. [2018]) that for each $h \in \mathbb{R}^2$ and $\delta > 0$ there exists $\mathbf{X}' \in \mathcal{X}_{(\mathbf{X},k)}$ satisfying $h \cdot \Delta_k X' \leq \delta$. To simplify notation, sometimes we we may write $\overline{V}_k = \overline{V}_k(\mathbf{X}, X^2, \mathcal{M})$ and a similar shortcut notation for related quantities.

We prove (2.14) by backwards induction on the index k. Notice that (2.14) holds at $k = N(\mathbf{X})$ because $\overline{U}_{N(\mathbf{X})} = X_N^2(\mathbf{X}) = \underline{U}_{N(\mathbf{X})}$ holds by definition according to (2.13). Then (2.14) for $k = N(\mathbf{X})$ follows from $\overline{U}_i = \overline{V}_i$ and $\underline{U}_i = \underline{V}_i$ which are valid for all $0 \leq i \leq N(\mathbf{X})$.

It remains then to assume that (2.14) holds at $k+1$ $(0 < k+1 \leq N(\mathbf{X}))$ for all $\mathbf{X} \in \mathcal{X}$ and then to prove that also holds at k $(0 \leq k)$ for all $\mathbf{X} \in \mathcal{X}$.

We first provide a proof of $X_k^2 \leq \overline{V}_k(X^2)$; assume otherwise, namely $X_k^2 > \overline{V}_k(\mathbf{X}, X^2, \mathcal{M})$

and consider $\epsilon \equiv X_k^2 - \overline{V}_k$. For this $\epsilon > 0$, there exists H_k^1 such that

$$\overline{U}_k(\mathbf{X}, X^2, \mathcal{M}) + \epsilon + H_k^1(\mathbf{X})\ \Delta_k \hat{X}^1 \geq$$

$$\overline{U}_{k+1}(\hat{\mathbf{X}}, X^2, \mathcal{M}) = \overline{V}_{k+1}(\hat{\mathbf{X}}, X^2, \mathcal{M}) \geq \hat{X}_{k+1}^2,\ \forall \hat{\mathbf{X}} \in \mathcal{X}_{(\mathbf{X},k)},$$

where we used the inductive hypothesis to establish the last inequality. Therefore

$$\overline{V}_k(\mathbf{X}, X^2, \mathcal{M}) + \epsilon + H_k(\mathbf{X}) \cdot \Delta_k \hat{X} \geq X_k^2,\ \forall \hat{\mathbf{X}} \in \mathcal{X}_{(\mathbf{X},k)},$$

where $H_k(\mathbf{X}) \cdot \Delta_k \hat{X}$ is the dot product in $\mathbb{R}^2$ of: $h \equiv H_k(\mathbf{X}) = (H_k^1(\mathbf{X}), -1)$ and (this follows notation that we have used before) $\Delta_k \hat{X} = (\Delta_k \hat{X}^1, \Delta_k \hat{X}^2)$. We then obtain

$$h \cdot \Delta_k \hat{X} \geq X_k^2 - \overline{V}_k(\mathbf{X}, X^2, \mathcal{M}) - \epsilon \equiv \delta > 0,\ \forall \hat{\mathbf{X}} \in \mathcal{X}_{(\mathbf{X},k)}, \tag{2.15}$$

this contradicts our assumption of $(\mathbf{X}, k)$ being a 0-neutral node. The proof of $X_k^2 \geq \underline{V}_k(X^2)$ follows a similar argument, but can also be obtained from the following considerations as well. The formalism of trajectorial bounds applies to an arbitrary function Z and we have not used in the derivation of $X_k^2 \leq \overline{V}_k(X^2)$ any particular property of X_i^2 (in particular we have not used $X_i^2 \geq 0$). Therefore our proof implies $-X_k^2 \leq \overline{V}_k(\mathbf{X}, -X^2, \mathcal{M})$ which then implies $\underline{V}_k(\mathbf{X}, X^2, \mathcal{M}) \leq X_k^2$. □

Chapter 3

Charts

Recall that this thesis has two objectives in mind; 1) Construct trajectory market models through an operational framework, and 2) Superhedge one asset with respect to a portfolio of another. In this chapter we visit the framework and assumptions used to employ an operational approach when constructing a trajectory market model. Throughout this thesis we often refer to *charts*. This term is used to represent a multidimensional time series of values of a set of *undiscounted* or *discounted* assets, where a *discounted* asset is the price of an asset that has been discounted by some numeraire of choice. To relate the notation given in Chapter 2 and the framework proposed in the present chapter, we refer to $s(t) = \big(s^0(t), s^1(t), \ldots, s^d(t)\big)$ as a time series of a set of $d+1$ *undiscounted asset values* and $x(t) = \big(x^1(t), \ldots, x^d(t)\big)$ as a time series of a set of d *discounted asset values*. We note that we use $s(t)$ in this chapter in order to relate the operational framework to the background material, and sections and chapters following Section 3.2 will only require availability of charts $x(t)$. The notation $s(t)$ also allows us to introduce a general notation for a choice of numeraire and numeraire change. In following chapters we refer to *charts* as the discounted chart values $x(t)$, and always refer to $s(t)$ as *undiscounted charts*.

Beginning with an general overview of the setting used to observe historical chart values, we move on to introduce the assumptions used for two different operational approaches. The difference between these two models being very minimal; changing the way we define a δ-escape. As previously mentioned, one way allows the two assets of interest to move in any given direction, as long as the inequality (1.1) is satisfied. Conversely, the second model only allows the assets to move in the same direction; up in price, or down in price, which obeys the inequalities (1.2).

The chapter is laid out as follows. Section 3.1 informally introduces the notions the reader will encounter throughout the chapter. Section 3.2 provides a generalized overview of how an investor may use an aribtrary selection of assets to observe charts. This is related to how we provide a general multidimensional notation for trajectory markets in Chapter 1. Section 3.3 introduces the operational framework, in which Section 3.3.1 asset prices may move in any direction, while Section 3.3.2 introduces the framework in which asset prices move together. To distinguish between the two we refer to the former as δ*-uncorrelated* market models, and the latter as δ*-correlated* market models. Following this, we discuss the discretization of observed quantities, which is required to create a discrete trajectorial market model, and a brief discussion of calibrating model parameters. We then formally define historical estimates an investor may utilize to construct market models in Sections 3.3.4, 3.4.

We note that the main purpose of this chapter is to discuss the operations that an investor performs to observe charts and introduce some parameters which are concerned with observing said charts. None of the parameters introduced in this chapter are used to construct market models until Chapter 5.

3.1 Operational Setting

We refer to *undiscounted charts* as the market quoted price $s(t) = \big(s^0(t), s^1(t), ..., s^d(t)\big)$ of risky assets at time t. Similarly, the term *discounted charts* will represent the value of our assets $s(t)$ discounted by an investor chosen numeraire, $s^k(t) \in s(t)$, $k \in \{0, ...d\}$. As this thesis is concerned with obtaining relative prices, after choosing a numeraire we will obtain discounted prices $x(t)$. Since an investor will perform operations on discounted charts $x(t)$, for simplicity we refer to the values of $x(t)$ as *charts*.

The paper Ferrando et al. [2019a] considers a market created with $d = 1$, with a simple zero interest rate bank account. The difference in our multidimensional case is that we first provide a general notation for observing an arbitrary d, and later we will concern ourselves with $d = 2$. Although we limit ourselves to $d = 2$ in this paper, the generalized notation goes to show that trajectory market models are not limited as we provide the framework for an arbtirary number of assets.

This is then formalized by considering a chart to be a map $x : \mathcal{T} \to \mathbb{R} \times \mathbb{R}$, where $\mathcal{T}$ is a *time interval* (more specifically, $\mathcal{T}$ is the time interval that our investor has access to chart data). We note that when referring to time intervals, we use the following continuous interval notation to actually mean the following: $[a, b] \equiv [a, b] \cap \Delta\mathbb{Z}$, where Δ is the smallest time resolution at which the investor will observe the market. So, $\mathcal{T} \subseteq \Delta\mathbb{Z}$. Although this is an abuse of notation, it is required to simplify the writing. We will use T to denote the maximum amount of time for our model trajectories to unfold. Given T, the investor will perform operations on charts within specific time intervals $[t_0, t_0 + T] \subseteq \mathcal{T}$.

We construct market models with two different operational approaches. For the first, our investors will only react to the market when they observe a normed vector change in value greater than some value $\delta \in \mathbb{Q}_+$ (for δ-uncorrelated models). For another, when one chart component $x^1(t)$ moves up δ_{up} times more than $x^2(t)$ or moves down δ_{down} times more than $x^2(t)$ (for δ-correlated models). Any time when an investor reacts to market changes is referred to as a δ-escape, δ-move, or δ-movement (for both uncorrelated and correlated models).

The parameter $\delta_0 \leq \delta$ provides the investor a set of sampling times which is used to determine the state of a financial observable $w(x, [t_0, t_0+t])$. The investor will also calibrate parameters $\hat{\delta}^1$, $\hat{\delta}^2$, and $\hat{\nu}_0$ which will be used to discretize observed chart quantities and eventually construct our trajectory model's coordinates X_i^1, X_i^2, and W_i, respectively.

Then, for a fixed δ (or $\delta_{up},\ \delta_{down}$), δ_0, and $\Delta > 0$:

1. Charts $x(t)$ are sampled at dynamic times r_l which depend on investor prescribed δ_0. The set of sampling times is given as $\{r_l\} \subseteq [t_0, t_0 + T] \subseteq \mathcal{T}$.

2. Time intervals have a lower resolution bound $\Delta > 0$, so that $(r_{l+1} - r_l) \geq \Delta$.

3. Times t_i, for the i-th δ-movement, satisfy $t_i \in \{r_l\}$, which are given by an investor prescribed δ, and have a lower time resolution bound Δ so that $(t_{i+1} - t_i) \geq \Delta$. These are the times at which an investor would rebalance their portfolio. The set of rebalancing times is given as $\{t_i\} \subseteq \{r_l\}$.

4. A sampled financial observable $w(x,t)$ is accumulated for the samples $x(r_l)$, which will allow us to restrict possible future events.

5. There is a dynamical number of portfolio rebalances $N(x, [t_0, t_0 + \rho])$ that take place in time interval $[t_0, t_0 + \rho]$, where $0 \leq \rho \leq T$. Given that there are $N(x, [t_0, t_0 + T])$ δ-movements in the time interval $[t_0, t_0 + T]$, then for this interval we have that $i \in \{0, 1, ..., N(x, [t_0, t_0 + T])\}$.

For each operational approach, δ-correlated and δ-uncorrelated, we will incorporate observable constraints allowing us to construct trajectory paths which incorporate states which may appear as rebalancing stock values. This is in contrast to stochastic processes which may grow unboundedly (i.e. Brownian motion has infinite speed) and include paths which may never appear as a possible rebalancing stock values. Our observable constraints may be constructed in many ways, in the sense that an investor is not limited to the observable constraints proposed in this chapter. For example, this thesis is concerned with an investor who will operate on chart values and keep track of *worst-case* scenarios of any observable quantity. The term *worst-case* may refer to the extreme values of the investor's observable quantities for the entire dataset our investor has access to. Then, for the purpose of this thesis, possible worst-case constraints will be formed through the observation of chart values $x(t)$. Chart values will yield measureable quantities such as the sampled financial observable $w(x, [t_0, t_0 + t])$, portfolio rebalancing times t_i, and number of portfolio rebalances $N(x, [t_0, t_0 + T])$. Thus, for a chart $x(t)$ and a given set of data $\mathcal{T}$, the investor will be able to observe the following worst-cases:

1. $X^*(x, \mathcal{T}, i)$ and $X_*(x, \mathcal{T}, i)$; the maximum and minimum amount of vector change after the i'th portfolio rebalance occurs,

2. $N^*(x, \mathcal{T}, \rho)$ and $N_*(x, \mathcal{T}, \rho)$; the maximum and minimum number of portfolio rebalances that occur after ρ time elapses,

3. $N^*(x, \mathcal{T}, w)$ and $N_*(x, \mathcal{T}, w)$; the maximum and minimum number of portfolio rebalances that occur after $w = w(x, [t_0, t_0 + t])$ vector variation is accumulated,

4. $T^*(x, \mathcal{T}, i)$ and $T_*(x, \mathcal{T}, i)$; the maximum and minimum amount of time elapsed after the i'th portfolio rebalance,

5. $T^*(x, \mathcal{T}, w)$ and $T_*(x, \mathcal{T}, w)$; the maximum and minimum amount of time elapsed after $w = w(x, [t_0, t_0 + t])$ vector variation is accumulated,

6. $W^*(x, \mathcal{T}, i)$ and $W_*(x, \mathcal{T}, i)$; the maximum and minimum amount of accumulated sample financial observable $w(x, [t_0, t_i])$ after the i'th portfolio rebalance occurs,

7. $W^*(x, \mathcal{T}, \rho)$ and $W_*(x, \mathcal{T}, \rho)$; the maximum and minimum amount of accumulated sample financial observable $w(x, [t_0, t_0 + \rho])$ after the i'th portfolio rebalance occurs.

In Ferrando et al. [2019a], the objective was to determine a fair price for a european call option which expires at maturity time T. However, we are concerned with super hedging one stock with respect to another and therefore the need for our trajectories to expire at some specific future time is dispensed with. We do however require some stopping criteria in order to dictate when the historical observation (and later in the production of future nodes) is complete. Such stopping criteria could be an investor prescribed future time T, or maximum number of portfolio rebalances i^*. To be precise, a time interval $[t_0, t_0 + T] \subseteq \mathcal{T}$ will have $N(x, [t_0, t_0 + T]) \leq i^*$ δ-movements and we consider the trajectory within this interval to be complete at the time $t_N \leq T$ or at the maximum rebalancing i^*. We allow trajectories beginning at time t_0 to end at any time before the expiration time T and do not force them to continue to time $t_0 + T$.

In this setting we then have an investor who is interested in comparing market prices of one asset relative to their operational portfolio rebalancing. Rebalancing times, determined by δ, will end at some time $t_N \leq T$, which allows for trajectories to naturally finish as close as possible to T. This contrasts what is done in Ferrando et al. [2019a], where trajectories are forced to finish at time T. This is due to the fact that Ferrando et al. [2019a] deals with the valuation of options, which must reach their maturity time T.

Given this setup, the operations of sampling and portfolio rebalancing performed on chart samples will create the following associations:

$$x(t_i) \rightarrow X_i,\ t_i \rightarrow t_0 + T_i,\ w(x, [t_0, t_i]) \rightarrow W_i$$

which we write compactly as $\mathbf{x}(t_i) \rightarrow \mathbf{X}_i$, where $\mathbf{x}(t_i) \equiv (x^1(t_i), x^2(t_i), i, t_i, w(x, [t_0, t_i]))$ and $\mathbf{X}_i \equiv (X_i^1, X_i^2, i, T_i, W_i)$. Historical worst case estimates of t_i and $w(x, [t_0, t_i])$ will limit the possible future states of our trajectories. Such worst case estimates will serve as necessary constraints restricting the possible future values of $\mathbf{X}_{i+1}$ for a given $\mathbf{X}_0,\ ..., \mathbf{X}_i$.

Note at this point that any δ-escape will be referred to as a $\delta - move$, $\delta - increment$, or $\delta - movement$, which will also be the investor's protfolio rebalancing times.

3.2 Chart Values

Now that we have briefly described the operational setting let us begin with deploying an operational approach as generally as possible. Starting with a mathematical framework using $d+1$ assets, we eventually limit ourselves to $d = 2$. At the moment our historical data starts with *undiscounted* asset prices $s(t) = \big(s^0(t), s^1(t), \ldots, s^d(t)\big)$, $t \in \mathcal{T}$, with all data in currency units and we obtain discounted prices with respect to a numeraire $s^0(t)$, given by:

$$x^k(t) = \frac{s^k(t)}{s^0(t)}, \quad k = 0, ..., d.$$

For example, we superhedge $s^2(t)$ with $s^1(t)$ and use $s^0(t)$ as numeraire. This then gives $x^0(t) = 1$, $\forall t \in \mathcal{T}$. This means we hold the numeraire from t_i to t_{i+1} and it does not change because is expressed in units of itself. Say we have 6 shares at t_i and then we also have 6 shares at t_{i+1}. This is exactly a more general case of a bank account with 0 interest rates.

To provide a more general framework, we could also start by selecting our numeraire to be $s^p(t)$, $p \in \{0, 1, ..., d\}$, and define

$$x^k(t) = \frac{s^k(t)}{s^p(t)} \quad k = 0, ..., d,$$

which now gives $x^p(t) = 1$, $\forall t \in \mathcal{T}$. Notice that we do not have a general notation for this as in Chapter 2 as we select S^0 as numeraire, however, Chapter 2 could be changed accordingly but we do not want to introduce confusing generality.

Thus, the above is just a preamble for the following important comment: our setting allows for more general relative pricing (in contrast to what is portrayed later in the thesis: $x^2(t)$ price in terms of $x^1(t)$ price all in terms of $x^0(t)$).

Let $A : \mathbb{R}^{d+1} \to \mathbb{R}$ be a linear transformation (i.e. a matrix) and $B : \mathbb{R}^{d+1} \to \mathbb{R}_+$ be another linear transformation. Then, we let $y^0(t) = Bs(t)$, such that $B_k > 0$ so $y^0(t) > 0$ for all $t \in \mathcal{T}$. We could then take this linear combination (i.e. $y^0(t)$) as a numeraire.

Then, we let

$$y^b(t) = \frac{As^b(t)}{y^0(t)},\ b \in \{0, 1, ..., d\},$$

and $y^b(t)$ is stock b in terms of the $y^0(t)$ numeraire. Notice that we have $y^0(t) > 0$ and that the coordinates $As(t)$ could be negative.

If we were to limit ourselves to $d = 2$, we could then price $y^2(t)$ in terms of $y^1(t)$ with numeraire $y^0(t)$. In other words, we create an index portfolio $y^2(t) = As^2(t)/y^0(t)$ and use another linear combination as numeraire (this second linear combination guarantees $y^0(t) > 0$). This keeps trading asset $y^1(t)$ simple (i.e. it is just $s^1(t)$ in different units) as it would be impractical (but possible) to trade with another linear combination.

The above framework is indicative that the investor may be free to change numeraire given the same set of assets $s(t)$, and that the change of numeraire may be given by a linear combination of assets in $s(t)$.

3.3 Charts and Investors' Operations

We now concern ourselves with $d = 2$ (i.e. $s(t) = \big(s^0(t), s^1(t), s^2(t)\big)$) and simply choose $s^0(t)$ to be our numeraire. This setting provides us with $x^0(t) = 1$ for all $t \in \mathcal{T}$, and both $x^1(t)$ and $x^2(t)$ in units of $s^0(t)$. In our setting this will allow the investor to obtain a relative pricing: $x^2(t)$ in terms of $x^1(t)$ all in units of $s^0(t)$.

The next subsections lay out the operations an investor will perform on charts. We introduce two different operational frameworks: one where the investor observes charts as is done in Ferrando et al. [2019a], a second where discounted assets $x^1(t)$ and $x^2(t)$ must move in the same direction.

3.3.1 δ-Uncorrelated Models

Set

$$T = M_T\Delta;\ \delta,\ \delta_0,\ \hat{\delta}^1,\ \hat{\delta}^2,\ \hat{\nu}_0 \in \mathbb{Q}_+;\ \delta \geq \delta_0$$

where M_T is some positive integer. Δ is the smallest time resolution at which the investor may observe charts. The investor will fix δ, δ_0, $\hat{\delta}^1$, $\hat{\delta}^2$, and $\hat{\nu}_0$ by a process of calibration, where one method of calibrating and selecting meaningful values of δ and δ_0 will be discussed in Section 5.2. The investor will only rebalance their portfolio for normed vector changes greater or equal than δ while δ_0 will provide the sampling times.

As in Ferrando et al. [2019a], to define estimates for model variables X_i^1, X_i^2, T_i, and W_i one needs to evaluate first the notions introduced in this section on the historical data $x(t)$ within a fixed time interval $[t_0, t_0 + T] \subseteq \mathcal{T}$. These estimates are updated by shifting the time interval through the historical dataset $\mathcal{T}$. The method for shifting the time interval through the data is not explicitly stated as this is investor dependent. We do discusss the method which is used in this thesis in Section 6.1.

Definition 10 (Dynamic Sampling Times)**.** *Given $\delta_0 > 0$, a chart $x = (x^1(t), x^2(t))$, and an interval $[t_0, t_0 + T]$, where $t \in [t_0, t_0 + T] \subseteq \mathcal{T}$; a sequence of increasing dynamic sampling times is given by $r = r(x, [t_0, t_0 + T]) = \{r_l\}_{l=0}^{L} \subseteq [t_0, t_0 + T]$ (so $r_l \in \Delta\mathbb{Z}$), where $L = (x, [t_0, t_0 + T])$, and $r_0 = t_0$ satisfying:*

$$\delta_0 \leq ||x(r_{l+1}) - x(r_l)||, \;\; 0 \leq l \leq L - 1, \;\; r_L \leq t_0 + T. \tag{3.1}$$

Definition 11 (Dynamic Rebalancing Times)**.** *Given $\delta > 0$, $T \in \Delta\mathbb{N}$, a chart $x = (x^1(t), x^2(t))$ and an interval $[t_0, t_0 + T]$, where $t \in [t_0, t_0 + T] \subseteq \mathcal{T}$; a sequence of increasing dynamic portfolio rebalancing times is given by $t = t(x, [t_0, t_0 + T]) = \{t_i\}_{i=0}^{N} \subseteq \{r_l\}_{l=0}^{L}$, $N = N(x, [t_0, t_0 + T])$, $t_0(x) = t_0$, $T \in \Delta\mathbb{N}$, satisfying:*

$$\delta \leq ||x(t_{i+1}) - x(t_i)||, \;\; 0 \leq i \leq N - 1, \;\; t_N \leq t_0 + T. \tag{3.2}$$

Definition 12 (Number of Portfolio Rebalances)**.** *Given a chart x, portfolio rebalancing times $\{t_i\}$ given by Definition 11, and length of time $\rho \in [0, T] \subseteq \Delta\mathbb{N}_+$, the number of times a portfolio is rebalanced in an interval $[t_0, t_0 + \rho]$ is a function and is denoted $N(x, [t_0, t_0 + \rho])$ given by:*

$$N(x, [t_0, t_0 + \rho]) = i; \;\; t_0 + \rho \in [t_i, t_{i+1}) \tag{3.3}$$

The collection $\{r_l\}$, given in Definition 10, is referred to as sampling times. Similar to Definition 10, we have that the collection $\{t_i\}$ (given in Definition 11) is the collection of times where the investor would have rebalanced their portfolio in the interval $[t_0, t_0 + T]$. A 2D representation of possible chart sampling and portfolio rebalancing times is given in Figure 3.1, where it is seen there can be many δ_0-movements between each δ-movement.

We have that $N(x, [t_0, t_0 + \rho])$ is the number of times a portfolio is rebalanced in the interval $[t_0, t_0 + \rho]$ for some duration of time $\rho \in \{0, \Delta, ..., M_T\Delta\}$. We also set the notation $N \equiv N(x, [t_0, t_0 + T])$. It follows that $N \leq L \equiv L(x, [t_0, t_0 + T])$, where L represents the number of samples taken in interval $[t_0, t_0 + T]$. We also make mention that the notation

$N \equiv N(x, [t_0, t_0+T])$ should not be confused with the notation $N(\mathbf{X})$ used in later chapters.

Here we define some critical items that are used to estimate parameters used for building a trajectory markets. The items are defined consecutively and described following the definitions. Given a segment $[t_0, t_0+T] \subseteq \mathcal{T}$, if s_i is the number of chart samples between two portfolio rebalances, then we can write for $0 \leq i \leq N-1$:

$$\Delta_{t_i} x^j \equiv x^j(t_{i+1}) - x^j(t_i),\ j = 1, 2.$$

$$\Delta_{t_i} w \equiv w(t_{i+1}) - w(t_i) = \sum_{j=0}^{s_i - 1} ||x(r_{l_i+j+1}) - x(r_{l_i+j})||$$

$$\Delta_i t \equiv t_{i+1} - t_i = r_{l_i+s_i} - r_l$$

where $t_i = r_{l_i} < ... < r_{l_i+s_i} = t_{i+1}$, i.e. $l_{i+1} = l_i + s_i$ and $0 \leq s_i$. Then, by setting $w(x, [t_0, t_0]) = 0$, then the historical variation can be written as follows:

$$w(x, [t_0, t_i]) = \sum_{l=0}^{l_i+1} ||x(r_{l+1}) - x(r_l)||$$

3.3.2 δ-Correlated Models

The operational framework proposed in Section 3.3.1 relies on portfolio rebalances given by a prescribed parameter δ. Notice that historical δ-movements of the chart $x(t) = (x^1(t), x^2(t))$ can correspond to the following observations:

- $\Delta_{t_i} x^1 \geq 0, \Delta_{t_i} x^2 \geq 0$
- $\Delta_{t_i} x^1 \geq 0, \Delta_{t_i} x^2 \leq 0$
- $\Delta_{t_i} x^1 \leq 0, \Delta_{t_i} x^2 \leq 0$
- $\Delta_{t_i} x^1 \leq 0, \Delta_{t_i} x^2 \geq 0$

That is, the chart components $x^1(t)$ and $x^2(t)$ can move in the same direction or in different directions (in their values). The new operational framework proposed in this section limits the movement of the assets to moving in the same direction; both $x^1(t)$ and $x^2(t)$ increase in value, or both decrease in value.

The investor will set

$$T = M_T \Delta;\ \delta_{up},\ \delta_{down},\ \delta_0,\ \hat{\delta}^1,\ \hat{\delta}^2,\ \hat{\nu}_0 \in \mathbb{Q}_+,$$

where M_T is a positive integer. Similar to the investor prescribing the parameter δ in Section 3.3.1, the investor will prescribe mulitple values $\delta_{up}, \delta_{down} \geq 0$. Then, the investor will only

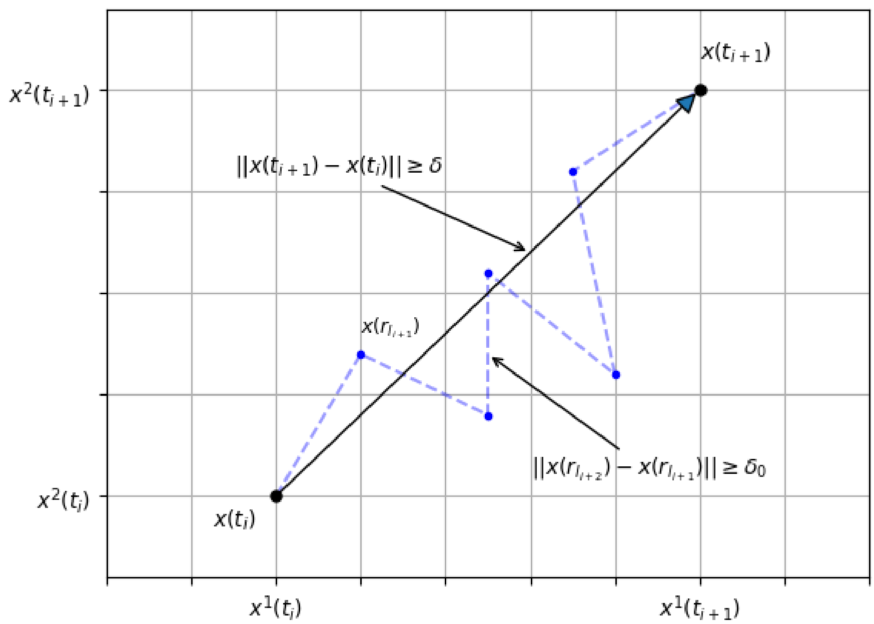


Figure 3.1: Portfolio rebalances and sampling times are given by normed vector changes in $x(t) = (x^1(t), x^2(t))$ which satisfy Equations 3.1 and 3.2. Here we see a 2D representation of a possible portfolio rebalance. The vector represents the change in value between portfolio rebalances ($\geq \delta$), while the dotted blue line represents the sampled path of the chart.

rebalance their portfolio when the following is satisfied:

$$0 \leq \Delta_{t_i} x^2 = (x^2(t_{i+1}) - x^2(t_i)) \leq \delta_{up} \Delta_{t_i} x^1 = \delta_{up}(x^1(t_{i+1}) - x^1(t_i))$$

or

$$\delta_{down} \Delta_{t_i} x^1 = \delta_{down}(x^1(t_{i+1}) - x^1(t_i)) \leq \Delta_{t_i} x^2 = (x^2(t_{i+1}) - x^2(t_i)) \leq 0$$

That is, we are looking for moments in time when the historical price of $x^2(t)$ increases a ratio δ_{up} less than $x^1(t)$, or likewise, $x^2(t)$ decreases by a ratio δ_{down} less than $x^1(t)$.

To create a market model in this manner the Definition 11 in Section 3.3 will be replaced with the definition below. We still use charts $x(t) \equiv (x^1(t), x^2(t))$ and observe historical trajectories which occur in a fixed window $[t_0, t_0 + T] \subseteq \mathcal{T}$ and is updated by rolling the window through $\mathcal{T}$. We also rely on the same Definitions 10 and 12 as introduced in Section 3.3.1.

Definition 13 (Correlated Dynamic Rebalancing Times)**.** *Given* $\delta_{up}, \delta_{down} > 0$*, a chart* $x = (x^1, x^2)$ *and an interval* $[t_0, t_0 + T]$*; a sequence of increasing dynamic portfolio rebalancing times is given by is given by* $t = t(x) = \{t_i\}_{i=0}^N \subseteq \{r_l\}_{l=0}^L$*,* $N = N(x, [t_0, t_0 + T])$*,* $t_0(x) = t_0$*,* $t' \in \Delta\mathbb{N}$*, satisfying:*

$$0 \leq (x^2(t_{i+1}) - x^2(t_i)) \leq \delta_{up}(x^1(t_{i+1}) - x^1(t_i)) \tag{3.4}$$

or

$$\delta_{down}(x^1(t_{i+1}) - x^1(t_i)) \leq (x^2(t_{i+1}) - x^2(t_i)) \leq 0 \tag{3.5}$$

where we have that $0 \leq i < N - 1,\ t_N \leq t_0 + T$*.*

We note that the operational framework for a δ-correlated market model will still incorporate parameter δ_0 in order to obtain sampling times given by Definition 10. One could go about incorporating new sampling parameters (i.e. $\delta_{0,up} \leq \delta_{up},\ \delta_{0,down} \leq \delta_{down}$) which would be used to satisfy similar inequalities as those in Definition 13 (but given similar to Definition 10 for sampling times), however for simplicity we only change the definition of our rebalancing times.

Thus, the only difference between the operational framework porposed in Section 3.3.1 and that proposed in this section is in how the investor obtains their historical rebalancing times. The financial meaning of all values such as $\{r_l\}$, $\{t_i\}$, $N(x, [t_0, t_0 + T])$, as well as all parameters to be introduced in following sections remains the same. The operational framework proposed in both this section as well as Section 3.3.1 both apply to parameters introduced in following sections in this chapter.

One might question why we bother to define rebalancing times in the way we defined in this section. Performing observations in this manner may allow the investor to create a market model where X^2 can be more efficiently superhedged by a portfolio consisting of X^1. Observing the past in such a way will allow the investor to create a set of future nodes which will allow them to superhedge (similarly for underhedge)

$$X_{i+1}^2 \leq \delta_{up}(X_{i+1}^1 - X_i^1) + X_i^2.$$

3.3.3 Discretization of Observed Quantities

We are concerned with observing the past in an operational manner, and prescribing possible future times at which the investor might rebalance a portfolio. In doing so, observed chart quantities $\Delta_{t_i}x^1$, $\Delta_{t_i}x^2$ and $\Delta_{t_i}w$ will be rounded to the nearest multiples of $\hat{\delta}^1$, $\hat{\delta}^2$, and $\hat{\nu}_0$, respectively, and used to force $X_i^1 \in \hat{\delta}^1\mathbb{Z}$, $X_i^2 \in \hat{\delta}^2\mathbb{Z}$, and $W_i \in \hat{\nu}_0\mathbb{N}$ when constructing the trajectory set $\mathcal{X}$. As will be seen during model construction in Section 4.2, this creates a discrete grid of coordinates: $\mathbf{X}_i = (X_i^1, X_i^2, i, T_i, W_i) \in \hat{\delta}^1\mathbb{Z} \times \hat{\delta}^2\mathbb{Z} \times \mathbb{N} \times \Delta\mathbb{N} \times \hat{\nu}_0\mathbb{N}$. This section formalizes the rounding of $\Delta_{t_i}x^1$, $\Delta_{t_i}x^2$, $\Delta_{t_i}t$ and $\Delta_{t_i}w$ and how these quantities are associated to model values $\mathbf{X}_i$.

We mention that the above usage of the hat notation '$\hat{\ }$' is different than in Ferrando et al. [2019a]. In Ferrando et al. [2019a] the notation is used to distinguish the smallest value a certain parameter can be. For example, in Ferrando et al. [2019a] the parameter $\hat{\delta}_0 = 0.01$ is the smallest unit of monetary currency used to trade an asset. This is practical in Ferrando et al. [2019a] since there is no usage of numeraire, and all assets are in units of currency, rather than units of numeraire. In our paper we incorporate relative pricing and lose the simplicity of prices being in terms of currency. Thus, the '$\hat{\ }$' notation given to $\hat{\delta}^1$, $\hat{\delta}^2$, and $\hat{\nu}_0$ is simply to distinguish the fact that these parameters are used to round observed chart changes and are termed *discretization parameters*. The rounding of a quantity may also be referred to as the *discretization* of a quantity.

To indicate this discretization, we will introduce the notation $\lfloor \cdot \rceil_{\hat{\delta}^k}$ to distinguish the values which are rounded to $\hat{\delta}^k$, and we will then have $\lfloor \Delta_i x^k \rceil_{\hat{\delta}^k} = \lfloor x^k(t_{i+1}) - x^k(t_i) \rceil_{\hat{\delta}^k} \in \hat{\delta}^k\mathbb{Z}$. Similarly, we will round the changes of the observed financial observable $\Delta_{t_i}w$ to the nearest value of $\hat{\nu}_0$, so that we have $\left\lfloor \Delta_{t_i}w \right\rceil_{\hat{\nu}_0} = \left\lfloor \sum_{j=0}^{s_i-1} ||x(r_{l_i+j+1}) - x(r_{l_i+j})|| \right\rceil_{\hat{\nu}_0} \in \hat{\nu}_0\mathbb{N}$, where s_i is the number of chart samples between two portfolio rebalances. Then, given a segment $[t_0, t_0 + T] \subseteq \mathcal{T}$ and s_i, we can write the following for $0 \leq i \leq N-1$:

$$\lfloor \Delta_{t_i}x^j \rceil_{\hat{\delta}^j} \equiv m_i^j \hat{\delta}^j, \ j = 1, 2, \tag{3.6}$$

$$\lfloor \Delta_{t_i}w \rceil_{\hat{\nu}_0} \equiv \hat{\nu}_0 \sum_{j=0}^{s_i-1} |p_j| \equiv \hat{\nu}_0 P_i, \tag{3.7}$$

$$\Delta_{t_i} t \equiv (n_{i+1} - n_i)\Delta \equiv q_i \Delta, \tag{3.8}$$

where $t_i = r_{l_i} < ... < r_{l_i+s_i} = t_{i+1}$, i.e. $l_{i+1} = l_i + s_i$, $0 \leq s_i$, $m_i^j \in \mathbb{Z}$, and $j_i, P_i, q_i \in \mathbb{N}$. By setting $w(x, [t_0, t_0]) = 0$, $P_0 = 0$, the historical variation can be written as follows:

$$w(x, [t_0, t_i]) = \Big\lfloor \sum_{l=0}^{l_i+1} ||x(r_{l+1}) - x(r_l)|| \Big\rfloor_{\hat{\nu}_0} \equiv \hat{\nu}_0 \sum_{j=0}^{i} P_j \tag{3.9}$$

3.3.4 Unfolding Chart Parameters

In this section we formally introduce a possible empirically measured set $\mathcal{N}_E(x, [t_0, t_0 + T])$ which can be used to create the trajectorial market models described in Section 4.2. The reason for introducing the definition now is that it deals with a chart $x(t)$ and time interval $[t_0, t_0 + T]$. Familiarizing the reader with this parameter here also helps to clarify the meaning of the models introduced in Chapter 4.

Definition 14 (Empirically Measured Chart Changes)**.** *Given a segment* $[t_0, t_0 + T] \subseteq \mathcal{T}$*, sampling rimes* $\{r_l\}_{l=0}^{L}$ *and portfolio rebalancing times* $\{t_i\}_{i=0}^{N}$*, the set of empirically measured chart changes is denoted:*

$$\mathcal{N}_E(x, [t_0, t_0+T]) = \Big\{ (\frac{\lfloor \Delta_{t_i} x^1 \rfloor_{\hat{\delta}^1}}{\hat{\delta}^1}, \frac{\lfloor \Delta_{t_i} x^2 \rfloor_{\hat{\delta}^2}}{\hat{\delta}^2}, \frac{t_{i+1} - t_i}{\Delta}, \frac{\lfloor \Delta_{t_i} w \rfloor_{\hat{\nu}_0}}{\hat{\nu}_0}) : 0 \leq i \leq N-1 \Big\} \tag{3.10}$$

where the $\{r_l\}$ are defined as in Definition 10, the $\{t_i\}$ are defined as in Definition 11 or Definition 13, $N \equiv N(x, [t_0, t_0 + T])$ as defined in Definition 12, $\lfloor \Delta_{t_i} x^1 \rfloor_{\hat{\delta}^1}$ and $\lfloor \Delta_{t_i} x^2 \rfloor_{\hat{\delta}^2}$ as given in Equation (3.6), and $\lfloor \Delta_{t_i} w \rfloor_{\hat{\nu}_0}$ as given in Equation (3.7). Notice that the set $\mathcal{N}_E(x, [t_0, t_0 + T])$ is a set of (m_i^1, m_i^2, q_i, P_i) values as seen in Equations (3.6), (3.8), and (3.7). Also, recall that $m_i^1, m_i^2 \in \mathbb{Z}$ and $q_i, P_i \in \mathbb{N}$.

To summarize the topics introduced in this section we have the following:

- the investor samples charts $x(t)$ at the times r_l, given by a chart's δ_0-movement; $\{r_l\}_{l=0}^{L}$ is the set of sampling times within a given interval $[t_0, t_0 + T]$.
- s_i is the number of chart samples between portfolio rebalances i and $i + 1$.
- Rebalancing times $t_i \in \{r_l\}$ occur at a chart's δ-movements; $\{t_i\}_{i=0}^{N}$ is the set of portfolio rebalancing times within a given interval $[t_0, t_0 + T]$.

- $N(x, [t_0, t_0 + \rho])$ is the total number of portfolio rebalances after $\rho \in \{0, \Delta, ..., M_T\Delta\}$ duration of time has passed. N is the total number of δ-movements which occur in time interval $[t_0, t_0 + T]$.

- m_i^1 and m_i^2 are the number of $\hat{\delta}^1$ and $\hat{\delta}^2$ value changes of assets $x^1(t)$ and $x^2(t)$, respectively, between portfolio rebalances i and $i + 1$.

- q_i is the number of time intervals of size Δ between two consecutive portfolio rebalances i and $i + 1$.

- P_i is the number of $\hat{\nu}_0$ changes of the accumulated vector variation between two consecutive portfolio rebalances i and $i + 1$.

- $\mathcal{N}_E(x, [t_0, t_0 + T])$ is a collection of all (m_i^1, m_i^2, q_i, P_i) values that occur in the interval $[t_0, t_0 + T]$.

3.4 Observable Worst-Case Pruning Constraints

The build up of notation introduced in Sections 3.1 and 3.3 have been for the operations an investor will perform on historical charts $x(t)$, primarily to obtain an empirically measured set $\mathcal{N}_E(x, [t_0, t_0 + T])$. As will be discussed in Chapter 4, the notation $\mathcal{N}_E$ differs from the previously introduced $\mathcal{N}_E(x, [t_0, t_0 + T])$; $\mathcal{N}_E$ will be used to indicate a general set of empirical changes used to create trajectory models. In this sense we leave $\mathcal{N}_E$ to be specified by the investor when creating models, where $\mathcal{N}_E$ may be instantiated to be $\mathcal{N}_E(x, [t_0, t_0+T])$ or some other observed estimate. In Chapter 5 we discuss how to formulate a *worst-case estimate* for $\mathcal{N}_E$.

Our models will be created by beginning in a specified initial state $\mathbf{X}_0$ and moving forward recursively. Possible future events $\mathbf{X}_{i+1},\ i \geq 0$ will be constructed by a state $\mathbf{X}_i$ and elements of the empirical set $\mathcal{N}_E$. For example, the elements in our empirical sets $(m^1, m^2, q, P) \in \mathcal{N}_E$ will provide possible future states $\mathbf{X}_{i+1}$ where we will have $X_{i+1}^1 = X_i^1 + m^1\hat{\delta}^1,\ X_{i+1}^2 = X_i^2 + m^2\hat{\delta}^2,\ T_{i+1} = T_i + q\Delta$, and $W_{i+1} = W_i + P\hat{\nu}_0$. Trajectory sets constructed in this recursive and combinatorial manner will cause the trajectories grow unboundedly and not resemble historical worst-case events observed in the charts. This reason is why we incorporate what we term *worst-case pruning constraints* into our models; enabling us an operational way to restrict our future nodes from growing unboundedly.

In this section we define the worst-case pruning constraints used to construct trajectory sets. The definitions are not used until model creation in Section 4.2, however, similar to

$\mathcal{N}_E(x, [t_0, t_0 + T])$, these definitions are concerned with a chart $x(t)$. In constrast to the parameter $\mathcal{N}_E(x, [t_0, t_0 + T])$, the definitions given in this section are concerned with the whole dataset $\mathcal{T}$ rather than a single interval $[t_0, t_0 + T] \subseteq \mathcal{T}$. We could reserve these worst-case definitions for Chapter 5, where we introduce worst-case definitions of $\mathcal{N}_E$, and not lose any source of clarity in the paper.

As previously stated, the investor will set T, the historical data interval $\mathcal{T}$, and calibrate both a sampling parameter δ_0 and portfolio rebalancing parameter δ (or δ_{up} and δ_{down}). The investor will then observe the times at which δ_0 and δ-movements occured historically while keeping track of the worst-cases which occur throughout our dataset $\mathcal{T}$. For example, we may keep track of the maximum and minimum number of δ-movements which occur over the time interval $[t_0, t_0 + \rho] \subseteq \mathcal{T}$, $\rho \in \{0, \Delta, ..., M_T\Delta\}$ (denoted $N^*(\rho)$ and $N_*(\rho)$ in Definition 17). Then, $\mathbf{X}_{i+1}$ will be restricted by only allowing states which reside within our observable bounds. For example, in Type I models we only allow states $\mathbf{X}_{i+1}$ which satisfy $i+1 \in [N_*(\rho), N^*(\rho)]$. This will be further clarified in the next subsections, the topic is only introduced here to familiarize the reader with notions used later in model construction. We call this process of omitting possible future states *worst-case pruning.*

Note that in the following sections we refer to a given interval $[t_0, t_0+T] \subseteq \mathcal{T}$, historical portfolio rebalancing times $\{t_i\}_{i=0}^N$, and portfolio sampling times $\{r_l\}_{l=0}^L$ where we will always have that: $\{t_i\}_{i=0}^N \subseteq \{r_l\}_{l=0}^L \subseteq [t_0, t_0 + T]$.

Our investor will begin by observing historical δ-movements that have occured in the chart $x(t) \equiv (x^1(t), x^2(t))$ where $t \in [t_0, t_0+T]$. Given the segment $[t_0, t_0+T]$ and rebalancing times $\{t_i\}_{i=0}^N$, we create the set $\mathcal{N}_E(x, [t_0, t_0 + T])$ as described in Definition 14. Each interval $[t_0, t_0 + T]$ will provide a unique set of sampling and rebalancing times which in turn provide the investor with $\mathcal{N}_E(x, [t_0, t_0 + T])$ unique to the given interval. The interval $[t_0, t_0+T]$ will then be moved and observable parameters are updated for each new window.

We begin the series of pruning constraint definitions by first defining the following:

Definition 15 (Maximum Number of Portfolio Rebalances)**.**

$$i^* \equiv i^*(x, \mathcal{T}) \equiv \max_{\forall [t_0, t_0+T] \in \mathcal{T}} N(x, [t_0, t_0 + T]). \tag{3.11}$$

$i^*(x, \mathcal{T})$ represents the maximum number of δ-movements that occur within each interval $[t_0, t_0 + T]$ in our whole set of data $\mathcal{T}$. It is also not used until model creation in Section 4.2, but we remark at this time that it is used to terminate trajectories in models which do not incorporate rebalancing time T_i as well as serve as an upperbound to i used in the following subsections.

3.4.1 Type 0 Pruning Constraint

While the interval $[t_0, t_0 + T]$ is moved through $\mathcal{T}$ and the investor observes historical δ-movements within each interval he moniters the maximum and minimum normed vector percent change that occurs between time t_0 and each δ-movement. This maximum and minimum percent change is called our Type 0 worst-case pruning constraint. This is now formalized:

Definition 16 (Historical Maximum and Minimum Normed Vector Percent Change)**.** *Given a chart x over time length $\mathcal{T}$, an interval $[t_0, t_0 + T] \subseteq \mathcal{T}$, and portfolio rebalancing times $\{t_i\}_{i=0}^{N}$, the maximum and minimum normed vector percent change that occurs in $\mathcal{T}$ is denoted by $X^*(x, \mathcal{T}, i)$ and $X_*(x, \mathcal{T}, i)$, respectively, and are defined as following:*

$$X^*(x, \mathcal{T}, i) = \max_{\forall [t_0, t_0+T] \subseteq \mathcal{T}} \frac{||x(t_i) - x(t_0)||}{||x(t_0)||}, \quad X_*(x, \mathcal{T}, i) = \min_{\forall [t_0, t_0+T] \subseteq \mathcal{T}} \frac{||x(t_i) - x(t_0)||}{||x(t_0)||} \tag{3.12}$$

where $0 \leq i \leq i^*$

3.4.2 Type I Pruning Constraints

Just as is indicated for the type 0 pruning constraint, the time interval $[t_0, t_0 + T]$ will be updated by rolling through the data, and our investor observes historical δ-movements. As the moving interval is updated the investor keeps track of the maximum and minimum number of portfolio rebalances that have occured after a certain time $\rho \in \Delta\mathbb{N}$ has elapsed since time t_0. This is formalized here.

Definition 17 (Historical Maximum and Minimum Number of δ-movements (at time ρ))**.** *Given a chart x over time length $\mathcal{T}$, time interval $[t_0, t_0 + T] \subseteq \mathcal{T}$, and portfolio rebalancing times $\{t_i\}_{i=0}^{N}$, the maximum and minimum number of δ-movements (portfolio rebalances) that occurs an interval of length $\rho \in \Delta\mathbb{N}$ is denoted by $N^*(x, \mathcal{T}, \rho)$ and $N_*(x, \mathcal{T}, \rho)$, respectively, and are defined as following:*

$$N^*(x, \mathcal{T}, \rho) = \max_{\forall [t_0, t_0+T] \subseteq \mathcal{T}} N(x, [t_0, t_0 + \rho]), \quad N_*(x, \mathcal{T}, \rho) = \min_{\forall [t_0, t_0+T] \subseteq \mathcal{T}} N(x, [t_0, t_0 + \rho]) \tag{3.13}$$

and we then have that $N^*(x, \mathcal{T}, \rho), N_*(x, \mathcal{T}, \rho) \in \mathbb{N},\ \forall \rho$.

We also keep track of the maximum and minimum amount of time elapsed at the i'th rebalancing. This is formalized in the defintion below.

Definition 18 (Historical Maximum and Minimum Elapsed Time (for variation))**.** *Given a chart x over time length $\mathcal{T}$, time intervals $[t_0, t_0 + T] \subseteq \mathcal{T}$, and portfolio rebalancing times*

$\{t_i\}_{i=0}^{N}$*, the maximum and minimum time that elapses since* t_0 *is denoted by* $T^*(x, \mathcal{T}, i)$ *and* $T_*(x, \mathcal{T}, i)$*, respectively, and are defined as following:*

$$T^*(x, \mathcal{T}, i) = \max_{\forall [t_0, t_0+T] \subseteq \mathcal{T}} t_i - t_0, \tag{3.14}$$
$$T_*(x, \mathcal{T}, i) = \min_{\forall [t_0, t_0+T] \subseteq \mathcal{T}} t_i - t_0$$

where $0 \le i \le i^*$ *and we have that* $T^*(x, \mathcal{T}, i)$*,* $T_*(x, \mathcal{T}, i) \in \mathbb{N}$*.*

3.4.3 Type II Pruning Constraints

We repeat that given an interval $[t_0, t_0 + T]$, the investor will observe historical δ_0 and δ-movements to the chart $x(t)$. While the interval is updated the investor will keep track of the maximum and minimum amount of vector variation that $x(t)$ has accumulated after the i'th portfolio rebalance has occurred will also be observed.

Definition 19 (Historical Maximum and Minimum Vector Variation (at rebalances))**.** *Given a chart* x *over time length* $\mathcal{T}$*, time intervals* $[t_0, t_0 + T] \subseteq \mathcal{T}$*, portfolio rebalancing times* $\{t_i\}_{i=0}^{N}$*, and accumulated vector variation* $w(x, [t_0, t_i])$*, the maximum and minimum amount of vector variation accumulated after the i'th portfolio rebalance is denoted as* $W^*(x, \mathcal{T}, i)$ *and* $W_*(x, \mathcal{T}, i)$*, respectively, and defined as the following:*

$$W^*(x, \mathcal{T}, i) = \max_{\forall [t_0, t_0+T] \subseteq \mathcal{T}} \Big\lfloor w(x, [t_0, t_i]) \Big\rfloor_{\hat{\nu}_0}, \quad W_*(x, \mathcal{T}, i) = \min_{\forall [t_0, t_0+T] \subseteq \mathcal{T}} \Big\lfloor w(x, [t_0, t_i]) \Big\rfloor_{\hat{\nu}_0} \tag{3.15}$$

where $0 \le i \le i^*$ *and then we have that* $W^*(x, \mathcal{T}, i)$*,* $W_*(x, \mathcal{T}, i) \in \hat{\nu}_0\mathbb{N}$*.*

With the incorporation of many observable quantities (namely, the number of portfolio rebalances $N(x, [t_0, t_0 + \rho])$, the times t_i, and the vector variation $w(x, [t_0, t_i])$) we are now able to incorporate many observable pruning constraints. We can observe the effect of one variable (say the number of rebalances) with respect to many other variables (an elapsed time ρ or accumulated vector variation $w = w(x, [t_0, t_0 + \rho])$). We now define the rest of these observable pruning constraints consecutively below.

Definition 20 (Historical Maximum and Minimum Vector Variation (at time ρ))**.** *Given a chart* x *over time length* $\mathcal{T}$*, time intervals* $[t_0, t_0 + T] \subseteq \mathcal{T}$*, portfolio rebalancing times* $\{t_i\}_{i=0}^{N}$*, and accumulated vector variation* $w(x, [t_0, t_i])$*, the maximum and minimum amount of vector variation accumulated after* $\rho \in [0, T]$ *time has elapsed is denoted* $W^*(x, \mathcal{T}, \rho)$ *and* $W_*(x, \mathcal{T}, \rho)$*, respectively, and defined as the following:*

$$W^*(x, \mathcal{T}, \rho) = \max_{\forall [t_0, t_0+T] \subseteq \mathcal{T}} \Big\lfloor w(x, \rho) \Big\rfloor_{\hat{\nu}_0}, \quad W_*(x, \mathcal{T}, \rho) = \min_{\forall [t_0, t_0+T] \subseteq \mathcal{T}} \Big\lfloor w(x, \rho) \Big\rfloor_{\hat{\nu}_0} \tag{3.16}$$

and then we have that $W^*(x,\mathcal{T},\rho)$, $W_*(x,\mathcal{T},\rho) \in \hat{\nu}_0\mathbb{N}$.

Definition 21 (Historical Maximum and Minimum Number of δ-movements (at accumulated vector variation $w(x,[t_0,t_i])$)**.** *Given a chart* x *over time length* $\mathcal{T}$*, time intervals* $[t_0, t_0+T] \subseteq \mathcal{T}$*, portfolio rebalancing times* $\{t_i\}_{i=0}^N$*, and accumulated vector variation* $w(x,[t_0,t_i])$*, the maximum and minimum number of* δ*-movements (portfolio rebalances) that occurs after the chart accumulates* $w(x,[t_0,t_i])$ *amount of vector variation is denoted by* $N^*(x,\mathcal{T},w)$ *and* $N_*(x,\mathcal{T},w)$*, respectively, and are defined as following:*

$$\begin{aligned} N^*(x,\mathcal{T},w) &= \max_{\forall [t_0,t_0+T]\subseteq \mathcal{T},\ w(x,[t_0,t_0+\rho])=w} N(x,[t_0,t_0+\rho]), \\ N_*(x,\mathcal{T},w) &= \min_{\forall [t_0,t_0+T]\subseteq \mathcal{T},\ w(x,[t_0,t_0+\rho])=w} N(x,[t_0,t_0+\rho]) \end{aligned} \tag{3.17}$$

and we then have that $N^*(x,\mathcal{T},w), N_*(x,\mathcal{T},w) \in \mathbb{N}$.

Definition 22 (Historical Maximum and Minimum Elapsed Time (for variation))**.** *Given a chart* x *over time length* $\mathcal{T}$*, time intervals* $[t_0,t_0+T] \subseteq \mathcal{T}$*, portfolio rebalancing times* $\{t_i\}_{i=0}^N$*, and accumulated vector variation* $w(x,[t_0,t_i])$*, the maximum and minimum time that elapses since* t_0 *is denoted by* $T^*(x,\mathcal{T},w)$ *and* $T_*(x,\mathcal{T},w)$*, respectively, and are defined as following:*

$$\begin{aligned} T^*(x,\mathcal{T},w) &= \max_{\forall [t_0,t_0+T]\subseteq \mathcal{T},\ w(x,[t_0,t_0+\rho])=w} t_i - t_0, \\ T_*(x,\mathcal{T},w) &= \min_{\forall [t_0,t_0+T]\subseteq \mathcal{T},\ w(x,[t_0,t_0+\rho])=w} t_i - t_0 \end{aligned} \tag{3.18}$$

where $0 \le i \le i^*$ *and we have that* $T^*(x,\mathcal{T},w)$*,* $T_*(x,\mathcal{T},w) \in \mathbb{N}$.

We summarize all definitions introduced in this section below. Note that when constructing trajectory sets the investor is not limited to the worst-case pruning constraints mentioned in this paper. Note that each of the constraints are dependent on the investor's choice of δ (or δ_{up} and δ_{down}).

- i^* is the maximum number of possible portfolio rebalances that occur historically in the interval $[t_0, t_0+T]$. This is only used in Type 0 models (first introduced in 4.2) to terminate the recursive creation of trajectory paths.
- $X^*(x,\mathcal{T},i)$ and $X_*(x,\mathcal{T},i)$ for $i \ge 0$ represent the maximum and minimum ratio of normed vector changes that occurs at the i'th δ-movement within the charts $x(t)$, respectively. This constraint will limit the amount our trajectory asset values may fluctuate since an initial portfolio rebalancing ($i = 0$).

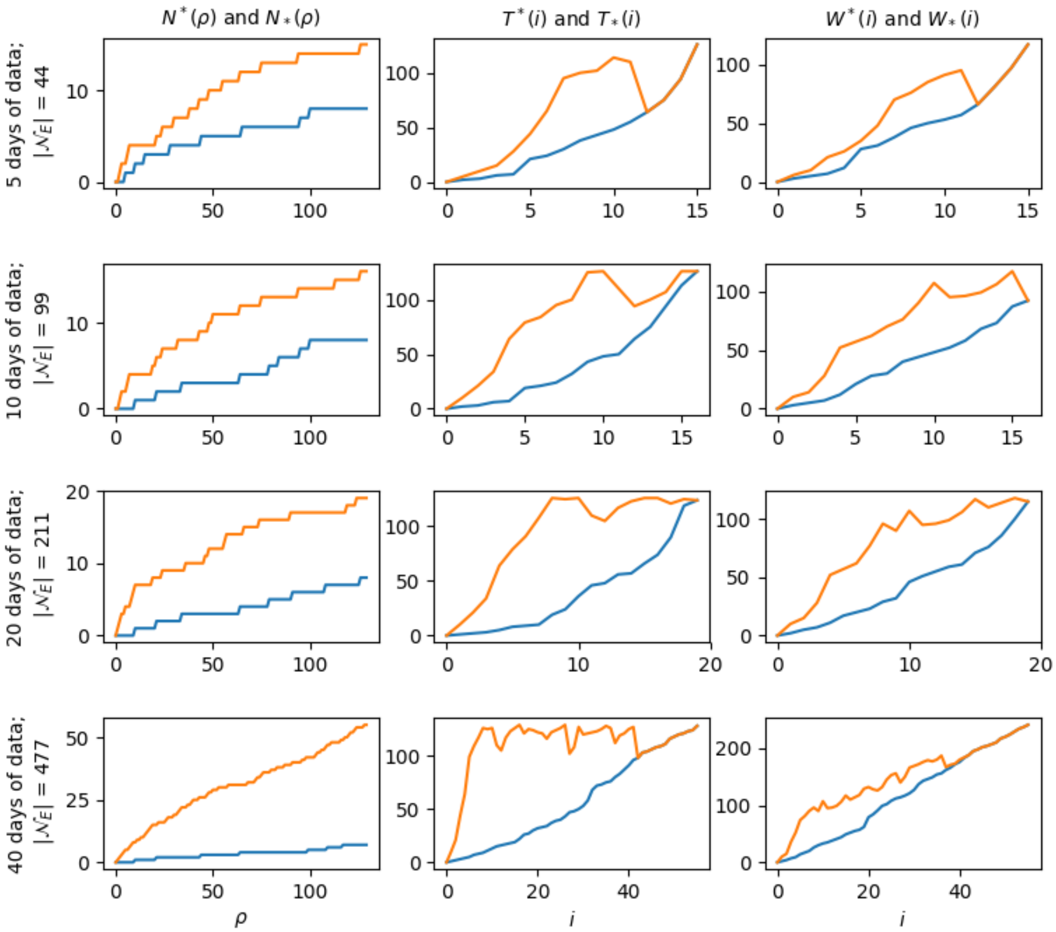


Figure 3.2: Using historical time interval $\mathcal{T} = \mathcal{T}^1$ and data described in [1.] in the enumeration in Section 6.1 (currency as numeraire), we show how the pruning constraints widen as more data is used in our historical estimation process. Here we select $\delta_0 = 0.5$ and $\delta = 1.0$.

- $N^*(x, \mathcal{T}, \rho)$ and $N_*(x, \mathcal{T}, \rho)$ represent the maximum and minimum portfolio rebalances that occur within $x(t)$. This is used to limit the number of portfolio rebalances that occur after time $\rho \in \Delta\mathbb{N}$ has elapsed. The investor will then not rebalance a portfolio more (and less) often than they would have historically.

- $N^*(x, \mathcal{T}, w)$ and $N_*(x, \mathcal{T}, w)$ for $w = w(x, [t_0, t_i]) \geq 0$ represent the maximum and minimum portfolio rebalances that occur within $x(t)$ after a chart has accumulated $w(x, [t_0, t_i])$ amount of variation at the i'th rebalancing.

- $T^*(x, \mathcal{T}, i)$ and $T_*(x, \mathcal{T}, i)$ represent the maximum and minimum amount of time elapsed after the i'th portfolio rebalancing. This restricts the investor to perform the i'th portfolio rebalancing at times which they would have done so historically.

- $T^*(x, \mathcal{T}, w)$ and $T_*(x, \mathcal{T}, w)$ represent the maximum and minimum amount of time elapsed after $w = w(x, [t_0, t_i])$ amount of variation is accumulated after the i'th portfolio rebalancing. This restricts the investor to perform the i'th portfolio rebalancing at times which they would have done so historically.

- $W^*(x, \mathcal{T}, i)$ and $W_*(x, \mathcal{T}, i)$ for $i \geq 0$ represent the maximum and minimum amount of accumulated variation after the i'th portfolio rebalancing time. This is used to limit the amount that model asset values X^1, X^2 can vary up to the i'th portfolio rebalance.

- $W^*(x, \mathcal{T}, \rho)$ and $W_*(x, \mathcal{T}, \rho)$ for $\rho \in [0, T]$ represent the maximum and minimum amount of accumulated variation between historical portfolio rebalancing times. This is used to limit the amount that model asset values X^1, X^2 can vary after time ρ has elapsed.

Chapter 4

Models

Once again, we reiterate the fact that this thesis is concerned with two objectives; 1) Construct trajectory market models through an operational framework, and 2) Superhedge one asset with respect to a portfolio of another. The previous chapter completely dealt with the operational framework that an investor will follow to obtain the parameters required to create a trajectory model. In this chapter we delve into the construction of trajectory sets $\mathcal{X}$ using the operational assumptions introduced previously. The chapter is organized as follows. We begin with an informal description of the recursive framework used to construct trajectory sets. Following this, a formal definition of the trajectory set is given alongside the methods used to construct a worst-case trajectory market model. We then discuss how the investor may go about incorporating risk into their trajectory sets.

We will be building different model types, each consisting of a distinct set of coordinates; either (X_i^1, X_i^2, i), (X_i^1, X_i^2, i, T_i), or $(X_i^1, X_i^2, i, T_i, W_i)$, which we refer to as Types 0, I, and II, respectively. That is, we omit the time component (T_i) from type 0 models and the variation component (W_i) from both type 0 and I models. To shorten the notation at times we will refer to coordinates for Type II models since they contain all variables mentioned in previous chapters. So, when constructing a trajectory based market model we will only utilize the components of $(m^1, m^2, q, P) \in \mathcal{N}_E$ which correspond to the coordinates used in a specific model type.

In each model, we utilize *pruning constraints* to restrict possible future states $\mathbf{X}_{i+1}$ through a method we refer to as *pruning*. We incorporate many of these pruning constraints which inhibit the growth of trajectory sets with regards to each model's specific choice of coordinates. For example, since Type 0 models will utilize the coordinates (X_i^1, X_i^2, i) we are only able to bound future states with respect to the behaviour of X_i^1 and X_i^2. However, when including T_i into models (as is done in Type I and II models), future states can be restricted even further by pruning nodes with respect to the times possible portfolio rebalances occur. This is the reason why many *observable* worst-case pruning constraints were defined in the previous chapter.

The process of pruning possible $\mathbf{X}_{i+1}$ yields future states which we admit into our trajectory set $\mathcal{X}$. Since we only allow states into our trajectory sets which are bounded by the pruning constraints incorporated into a specific model, we require a way to distinguish these states from those states not included in our sets $\mathcal{X}$. We refer to the states included in $\mathcal{X}$ as *admissible states* or *admissible nodes*.

We note that we wish the model building described in this chapter is supposed to stand as its own entity, not *relying* on notions described in Chapter 3, but allowing the investor the choice of incorporating different instantiations for the model's parameters in model construction. For that reason we utilize a different set of notation in this chapter which is not the same as, but related to the meanings introduced in Chapter 3. For example, in Chapter 5 the set $\mathcal{N}_E(x, \mathcal{T})$ represents a *worst-case estimate* of chart changes. In this chapter we will rely on the notation $\mathcal{N}_E$ to represent the set of chart changes used for model building, where an investor is free to set $\mathcal{N}_E = \mathcal{N}_E(x, \mathcal{T})$ or any other preferred instantiation. The new notation for model building will further be clarified in following sections.

4.1 Trajectory Sets: General Properties and Recursive Definition

This section deals with the constructions of the trajectory sets associated with each of the model coordinates described in the beginning of this chapter. Similar to Ferrando et al. [2019a], the previous section assumed the availability of charts $x(t)$, and introduced sequences of integers $s_i,\ m_i^1,\ m_i^2, q_i, P_i$, and $N(x, [t_0, t_0 + \rho]),\ \rho \in [0, T]$. In this thesis the observation of historical δ_0 and δ-movements will also provide us with pruning constraints which will be used as parameters to create a worst-case trajectory based market model. Some examples of such pruning constraints are more precisely defined in Section 3.4. We will now describe the process through which we will construct trajectorial market models by allowing our models to be specified by the introduced constraints. The end result being the definition of a possible set of chart values, portfolio rebalance times, and accumulated variation values.

The values $r_l, t_i, m_i^1, m_i^2, q_i, p_j$ and $N(x, [t_0, t_0{+}T])$ are either observable or operationally prescribed and are a result of acting on the chart $x(t)$. Similar to Ferrando et al. [2019a], to avoid confusion in creating our trajectory models we will adopt the same symbols and meaning for our modelling variables which are used to create the trajectories. Trajectory models will be described with a mathematical construction of capitalized variables while we have already used lower case variables to describe the mathematical construction of the empirically observed variables. Model variable meanings will still share similar meaning as their empirical counterparts; X represents model asset values, T_i represents model time values, and W_i will represent the accumulated variation of vector X in our models. The empirical counterparts of X, T_i, and W_i will be x, t_i, and $w(x,[t_0,t_i])$, respectively. In this same manner, when referring to empirical changes (m_i^1, m_i^2, q_i, P_i) used to construct $\mathbf{X}_{i+1}$, we will drop reference to the subscript i. This allows us to refer to any arbitrary element within the empirically observed set of vector changes. So, when referring to trajectory model construction, we replace m_i^1 with m^1, m_i^2 with m^2, q_i with q, P_i with P, and $N(x,[t_0,t_0+T])$ with $N(X,[t_0,t_0+T])$. These associations will allow our trajectories to lie within a discrete grid of points based on historical data (specifically Δ) and the investor calibrated parameters $\hat{\delta}^1$, $\hat{\delta}^2$, and $\hat{\nu}_0$: $\mathbf{X}_i \equiv (X_i^1, X_i^2, i, T_i, W_i) \in (\hat{\delta}^1\mathbb{Z} \times \hat{\delta}^2\mathbb{Z} \times \mathbb{N} \times \Delta\mathbb{N} \times \hat{\nu}_0\mathbb{N})$.

We now state a general recursive formulation of the trajectory set $\mathcal{X}$. To describe the models generally we utilize the coordinates used in Type II models as it contains all coordinates used in each other model type. $\mathcal{N}_E$ represents a given set of quadruples (m^1, m^2, q, p), $m^1, m^2 \in \mathbb{Z}$, $q, p \in \mathbb{N}$, obtained through the observation of historical trajectories. This empirical set ($\mathcal{N}_E$) will be used to produce possible future states $\mathbf{X}_{i+1}$ which will reside on some point within our discrete grid, which we call nodes. We will often refer to states corresponding to $\mathbf{X}_i$ as parent nodes and their future counterparts $\mathbf{X}_{i+1}$ will often be refered to as children nodes. Each node will correspond to a tuple $(k_i^1, k_i^2, i, n_i, j_i)$ where we set the notation $\mathbf{X} = \{\mathbf{X}_i = (X_i^1, X_i^2, i, T_i, W_i)\} = \{(k_i^1\hat{\delta}^1, k_i^2\hat{\delta}^2, i, n_i\Delta, j_i\hat{\nu}_0)\}$, $k_i^1, k_i^2 \in \mathbb{Z}$, $n_i < n_{i+1}$, $0 \le j_i < j_{i+1}$, $n_i, j_i \in \mathbb{N}$.

Trajectory coordinates of $\mathcal{X}$ will be specified by beginning with initial values $X_0^1 \equiv x_0^1$, $X_0^2 \equiv x_0^2$, $T_0 \equiv t_0$, $W_0 \equiv w(x(t_0), t_0) = 0$ and proceeding recursively: given a node $(k_i^1, k_i^2, i, n_i, j_i)$ we will provide a set of *admissible* values of $(k_{i+1}^1, k_{i+1}^2, i+1, n_{i+1}, j_{i+1})$; for each admissible tuple we set:

$$\Delta_i X^d \equiv (X_{i+1}^d - X_i^d) = (k_{i+1}^d - k_i^d)\hat{\delta}^d = m_i^d\hat{\delta}^d,\ d = 1, 2,$$

$$\Delta_i T \equiv (T_{i+1} - T_i) = (n_{i+1} - n_i)\Delta = q_i\Delta,$$

$$\Delta_i W \equiv (W_{i+1} - W_i) = (j_{i+1} - j_i)\hat{\nu}_0 = P_i\hat{\nu}_0,$$

What dictates whether a construced node is included in our models, otherwise called *admissible*, depends on whether or not the model values obey the pruning constraints which an investor wishes to incorporate into their models. Then, given $(X_i^1, X_i^2, i, T_i, W_i)$, the admissible set is given by the convenient notation, each for Types 0, I, and II models, respectively:

$$\begin{aligned}\mathcal{N}_A(X_i^1, X_i^2, i) =& \{(X_{i+1}^1 = x_0^1 + k_{i+1}^1\hat{\delta}^1,\ X_{i+1}^2 = x_0^2 + k_{i+1}^2\hat{\delta}^2,\\ & i+1),\ with\ (k_{i+1}^1, k_{i+1}^2, i+1)\ admissible\}\end{aligned}$$

$$\begin{aligned}\mathcal{N}_A(X_i^1, X_i^2, i, T_i) =& \{(X_{i+1}^1 = x_0^1 + k_{i+1}^1\hat{\delta}^1,\ X_{i+1}^2 = x_0^2 + k_{i+1}^2\hat{\delta}^2,\ i+1,\\ & T_{i+1} = n_{i+1}\Delta),\ with\ (k_{i+1}^1, k_{i+1}^2, i+1, n_{i+1})\ admissible\}\end{aligned} \tag{4.1}$$

$$\begin{aligned}\mathcal{N}_A(X_i^1, X_i^2, i, T_i, W_i) = & \{(X_{i+1}^1 = x_0^1 + k_{i+1}^1\hat{\delta}^1,\ X_{i+1}^2 = x_0^2 + k_{i+1}^2\hat{\delta}^2,\\ & i+1,\ T_{i+1} = n_{i+1}\Delta,\ W_{i+1} = j_{i+1}\hat{\nu}_0),\ with\\ & (k_{i+1}^1, k_{i+1}^2, i+1, n_{i+1}, j_{i+1})\ admissible\}\end{aligned}$$

The notation $\mathcal{N}_A(X_i^1, X_i^2, i, T_i, W_i)$ may be written compactly as $\mathcal{N}_A(\mathbf{X}_i)$. The next section discusses the specification of each model type; Type 0, I and II by specifying the set of admissible tuples for each model type.

4.2 Trajectory Model Specification

This section completes the general recursive framework introduced in Section 4.1, completing the introduction of the trajectory sets introduced in Equation (4.1). The investor will have prescribed some values of $\hat{\delta}^1$, $\hat{\delta}^2$, and $\hat{\nu}_0$ by a means of calibration as is discussed in Section 5.2. $\hat{\delta}^1$ and $\hat{\delta}^2$ will simply be used to provide X_i^1 and X_i^2 values while $\hat{\nu}_0$ provides variation W_i values. The models assume a fixed Δ and T.

$\mathcal{N}_E$ represents a set of the changes in chart observables $\{(m^1, m^2, q, P)\}$ with $m^1,\ m^2 \in \mathbb{Z}$, and $q,\ P \in \mathbb{N}$. Future states $\mathbf{X}_{i+1}$ will be restricted by bounding them within incorporated pruning constraints $\overline{X}(i)$, $\underline{X}(i)$, $\overline{N}(\rho)$, $\underline{N}(\rho)$, $\overline{N}(W_i)$, $\underline{N}(W_i)$, $\overline{T}(i)$, $\underline{T}(i)$, $\overline{T}(W_i)$, $\underline{T}(W_i)$, $\overline{W}(i)$, $\underline{W}(i)$, $\overline{W}(T_i)$, and $\underline{W}(T_i)$. Given that each model type only contains certain information with its given set of coordinates, specific models will be able to incorporate specific pruning constraints. For example, a model with coordinates (X_i^1, X_i^2, i) is not able to restrict the future with $\underline{W}(i)$ since variation information is not available to the investor. We also note that these constraints could be instantiated to be the worst-case estimates defined in Chapter 3, or any investor prescribed value (which is regarded as a *risk-taking estimate*).

Each model assumes the availability of the state $(X_i^1, X_i^2, i, t_0{+}T_i, W_i)$; with initialization

$X_0^1 = x_0^1$, $X_0^2 = x_0^2$, $T_0 = 0$, $W_0 = 0$. We also remark that the following model constructions may deal with both δ-correlated and δ-uncorrelated models, as the descriptions below are only concerned with building a trajectory set with a given set of coordinates.

These models each have a defined set of coordinates, as mentioned in the beginning of this chapter. Beginning with Type 0 models we change the coordinate system in Type I and II models by adding an extra dimension each time. Doing so allows us to nest each pruning constraint in the next model; i.e. nest the Type 0 pruning constraint into Type I and II models, and nest the Type I pruning constraints into Type II models. Nesting then allows us to compare the effect of pruning future nodes with respect to each model by evaluating the price bounds given by each trajectory set.

We expect that models with more pruning constraints will exhibit more pruning at each proposed future δ-movement. This increased pruning in certain models will likely prune more nodes (and have less admissible X_i^1 and X_i^2 values) and therefore produce tighter price bounds $[\underline{V}_0(\mathbf{X}_0, X^2, \mathcal{M}), \overline{V}_0(\mathbf{X}_0, X^2, \mathcal{M})]$.

4.2.1 Trajectory Termination

Before defining each model type, we briefly discuss the means used to terminate a trajectory at a certain node. The models assume a fixed time T, yet we only incorporate the time coordinate T_i into two model types. We also mentioned that in this thesis, trajectories do not need to be forced to terminate exactly at time T. While this allows our model trajectories to unfold more naturally, this might cause some confusion to the reader on *how* these trajectories reach their end.

When creating our trajectory models, the investor should have some freedom in choosing when trajectories are to stop. Thus we allow models to incorporate an upperbound to the number of portfolio rebalances that may occur. As will be seen, in models with coordinates (X_i^1, X_i^2, i), the only appropriate information to use when terminating trajectories lies within the portfolio rebalancing number i.

For the construction of our trajectory models we assume a given quantity $N(\mathbf{X})$ is available and it will represent the maximum number of rebalances allowed to occur in any trajectory in our models. This then restricts $0 \leq i \leq N(\mathbf{X})$ and a trajectory will terminate at rebalance number $N(\mathbf{X})$. Within this chapter, and upon model construction, we assume this quantity $N(\mathbf{X})$ is given such that $T_{N(\mathbf{X})} \leq T$. We may define $N(\mathbf{X})$ in such a way that allows each trajectory to unfold naturally, which is given generally by the following denotion:

$$N(\mathbf{X}) \equiv i \text{ such that } \mathcal{N}_A(\mathbf{X}_i) = \emptyset$$

Then let us call G the superhedging (or subhedging) optimal investor portfolio. Let

$N_G(\mathbf{X})-1$ be the last time she performs a portfolio rebalance. This means that the investor will liquidate her portfolio when $i = N_G(\mathbf{X})$. The investor is able to to select $N_G(\mathbf{X}) \leq N(\mathbf{X})$, something not yet explored.

When extending the models to use the coordinates (X_i^1, X_i^2, i, T_i) and $(X_i^1, X_i^2, i, T_i, W_i)$, there could be many appropriate criteria used to terminate trajectories, however, the most natural would be to stop a trajectory once we have reached a node with $T_i = t_0 + T$. This then restricts rebalancing times to the interval $[t_0, t_0 + T]$. The investor is still able to prescribe $N(\mathbf{X})$ to terminate trajectories at earlier rebalancing times.

One might then ask, if we are restricting model states to lie within incorporated pruning bounds then what happens when we encounter a parent node with no admissible children nodes? Then this parent node is regarded as a terminal node, where trajectories naturally complete themselves at this point.

In the next few subsections we describe a process which creates a set $\mathcal{N}_A(X_i^1, X_i^2, i, T_i, W_i)$ for three different model types. Following this we address some issues that may be encountered when constructing $\mathcal{N}_A(X_i^1, X_i^2, i, T_i, W_i)$.

4.2.2 Type 0 Model

The most basic model will be limited to the coordinates (X_i^1, X_i^2, i), which will act as the base model, as it contains the least amount of infomation for the investor. It will also provide the investor with a smaller discrete grid which will lower the overall computation time of creating and valuing trajectory sets.

Coordinates: (X_i^1, X_i^2, i)

Input Parameters:

1. $\hat{\delta}^1, \hat{\delta}^2 \in \mathbb{Q}_+$
2. A set $\mathcal{N}_E$.
3. Maximum number of portfolio rebalances $N(\mathbf{X}) \in \mathbb{N}$.
4. Historical worst case bounds $\overline{X}(i)$, $\underline{X}(i)$, where $i \in \mathbb{N}$, $\overline{X}(i)$, $\underline{X}(i) \in \mathbb{R}$, $0 \leq i \leq N(\mathbf{X})$, and $\overline{X}(0) = \underline{X}(0) = 0$.

Beginning in $\mathbf{X}_0 = (X_0^1, X_0^2, 0)$ and elements $m^1, m^2 \in \mathcal{N}_E$, the next set of future nodes $\mathbf{X}_{i+1}$ will be produced (briefly discussed in Section 4.2). The trajectory market model will only allow future nodes $\mathbf{X}_{i+1} = (X_i^1 + m^1\hat{\delta}^1, X_i^2 + m^2\hat{\delta}^2, i+1)$ which correspond to having a normed vector percent change which is both not greater than our maximum observed normed vector percent change, and not less than our minimum observed vector change; or rather:

$$\frac{||(X_{i+1}^1, X_{i+1}^2) - (X_0^1, X_0^2)||}{||(X_0^1, X_0^2)||} \in \left[\underline{X}(i+1), \overline{X}(i+1)\right] \subseteq \mathbb{R}. \tag{4.2}$$

$$\begin{aligned}\mathcal{N}_A(X_i^1, X_i^2, i) \equiv \Big\{ &\big(X_i^1 + \Delta_i X^1,\ X_i^2 + \Delta_i X^2,\ i+1\big)\ :\ \exists (m^1, m^2) \in \mathcal{N}_E,\\ &0 \le i \le N(\mathbf{X}),\ (\Delta_i X^1, \Delta_i X^2) = (m^1\hat{\delta}^1, m^2\hat{\delta}^2)\\ &\frac{||(X_{i+1}^1, X_{i+1}^2) - (X_0^1, X_0^2)||}{||(X_0^1, X_0^2)||} \in [\underline{X}(i+1), \overline{X}(i+1)]\Big\}\end{aligned} \tag{4.3}$$

Note that the set of coordinates does not contain information about how near terminal time T the trajectories finish, as we omit the time coordinate altogether. This may be a major issue to some investors as knowing how far into the future the trajectories may travel might be crucial. This model is designed to act as a base model before adding more coordinates to incorporate more information. It should provide the investor with the widest price bounds $\underline{V}_0(\mathbf{X}_0, X^2, \mathcal{M}) \le \overline{V}_0(\mathbf{X}_0, X^2, \mathcal{M})$. We also note that with less dimensions the number of computations required to construct a trajectory set is reduced, as with more dimensions a trajectory model's discrete grid of nodes increases in size. For example, searching for specific nodes in Type II models $\mathbf{X}_{i+1} \in (\hat{\delta}^1\mathbb{Z} \times \hat{\delta}^2\mathbb{Z} \times \mathbb{N} \times \Delta\mathbb{N} \times \hat{\nu}_0\mathbb{N})$ requires many more computations than in Type 0 models. This is due to the fact that our Type 0 model grids are much smaller than a grid produced for a Type II model.

4.2.3 Type I Model

Coordinates: (X_i^1, X_i^2, i, T_i)

Input Parameters:

1. $\hat{\delta}^1, \hat{\delta}^2 \in \mathbb{Q}_+$
2. A set $\mathcal{N}_E$.
3. Pruning Constraints $\overline{X}(i)$ and $\underline{X}(i)$, where $i \in \mathbb{N}$, $\overline{X}(i)$, $\underline{X}(i) \in \mathbb{R}$, $0 \le i \le N(\mathbf{X})$, and $\overline{X}(0) = \underline{X}(0) = 0$.

4. Pruning Constraints $\overline{N}(\rho)$ and $\underline{N}(\rho)$, where $\rho \in \Delta\mathbb{N}$, $\rho \leq T$, $\overline{N}(\rho), \underline{N}(\rho) \in \mathbb{N}$, and $\overline{N}(0) = \underline{N}(0) = 0$.
5. Pruning constraints $\overline{T}(i)$ and $\underline{T}(i)$, where $i \in \mathbb{N}$, $\overline{T}(i)$, $\underline{T}(i) \in \Delta\mathbb{N}$, $0 \leq i \leq N(\mathbf{X})$, and $\overline{T}(0) = \underline{T}(0) = 0$.

We begin creating this model with initial state $\mathbf{X}_0 = (X_0^1, X_0^2, 0, T_0)$. The elements $(m^1, m^2, q) \in \mathcal{N}_E$ will generate the $(i+1)$'th possible rebalancing values $\mathbf{X}_{i+1}$, while only allowing the trajectory set $\mathcal{X}$ to contain admissible nodes. We classify nodes as admissible only if they have coordinate values which lie within all indicated pruning constraints.

How we utilize pruning constraints $\overline{X}(i)$ and $\underline{X}(i)$ is indicated in the Type 0 models. Type I models incorporates new pruning constraints where at the $(i+1)$'th portfolio rebalance we only allow future states $\mathbf{X}_{i+1}$ which satisfy:

$$\begin{aligned} i+1 &\in \left[\underline{N}(T_{i+1}), \overline{N}(T_{i+1})\right] \cap \mathbb{N}, \\ T_{i+1} &\in \left[\underline{T}(i+1), \overline{T}(i+1)\right] \cap \Delta\mathbb{N}. \end{aligned} \tag{4.4}$$

Then, the following defines the admissible set of nodes, which is used to construct $\mathcal{X}$:

$$\begin{aligned} \mathcal{N}_A(X_i^1, X_i^2, i, t_0 + T_i) \equiv \Big\{ \big(&X_i^1 + \Delta_i X^1,\ X_i^2 + \Delta_i X^2,\ t_0 + T_i + \Delta_i T,\ i+1 \big) \ : \\ &\exists (m^1, m^2, q) \in \mathcal{N}_E,\ q\Delta \leq (T - T_i),\ i \geq 0, \\ &(\Delta_i X^1, \Delta_i X^2, \Delta_i T) = (m^1 \hat{\delta}^1, m^2 \hat{\delta}^2, q\Delta), \\ &\frac{||(X_{i+1}^1, X_{i+1}^2) - (X_0^1, X_0^2)||}{||(X_0^1, X_0^2)||} \in [\underline{X}(i), \overline{X}(i)] \\ (i+1) \in [\underline{N}(X, [t_0 + T_i &+ q\Delta]), \overline{N}(X, [t_0 + T_i + q\Delta])] \\ &t_0 + T_i + q\Delta \in \left[\underline{T}(i+1), \overline{T}(i+1)\right] \Big\} \end{aligned} \tag{4.5}$$

This model construction allows our trajectories to get as close to the terminal time T, something which might be important to know for the investor. Future nodes are pruned to ensure there are not too many or too few number of portfolio rebalancing moments at a specific moment of time $\rho \in [0, T]$. Our worst-case pruned trajectories will then have a number of portfolio rebalances which reflects what has occured historically in $\mathcal{T}$. As we will discuss in future sections, the amount of pruning done within a model is directly related to the choice of δ and the amount of data used in the historical observation of trajectories.

4.2.4 Type II Model

Coordinates: $(X_i^1, X_i^2, i, T_i, W_i)$

Input Parameters:

1. $\hat{\delta}^1, \hat{\delta}^2, \hat{\nu}_0 \in \mathbb{Q}_+$
2. A set $\mathcal{N}_E$.
3. Pruning Constraints $\overline{X}(i)$ and $\underline{X}(i)$, where $i \in \mathbb{N}$, $\overline{X}(i)$, $\underline{X}(i) \in \mathbb{R}$, $0 \leq i \leq N(\mathbf{X})$, and $\overline{X}(0) = \underline{X}(0) = 0$.
4. Pruning Constraints $\overline{N}(\rho)$ and $\underline{N}(\rho)$, where $\rho \in \Delta\mathbb{N}$, $\rho \leq T$, $\overline{N}(\rho), \underline{N}(\rho) \in \mathbb{N}$, and $\overline{N}(0) = \underline{N}(0) = 0$.
5. Pruning Constraints $\overline{N}(W_i)$ and $\underline{N}(W_i)$, where $W_i \in \hat{\nu}_0\mathbb{N}$, $\overline{N}(W_i), \underline{N}(W_i) \in \mathbb{N}$, and $\overline{N}(0) = \underline{N}(0) = 0$.
6. Pruning constraints $\overline{T}(i)$ and $\underline{T}(i)$, where $i \in \mathbb{N}$, $\overline{T}(i)$, $\underline{T}(i) \in \Delta\mathbb{N}$, $0 \leq i \leq N(\mathbf{X})$, and $\overline{T}(0) = \underline{T}(0) = 0$.
7. Pruning constraints $\overline{T}(W_i)$ and $\underline{T}(W_i)$, where $W_i \in \Delta\mathbb{N}$, $\overline{T}(W_i)$, $\underline{T}(W_i) \in \Delta\mathbb{N}$, and $\overline{T}(0) = \underline{T}(0) = 0$.
8. Pruning constraints $\overline{W}(i)$ and $\underline{W}(i)$, where $i \in \mathbb{N}$, $\overline{W}(i)$, $\underline{W}(i) \in \hat{\nu}_0\mathbb{N}$, $0 \leq i \leq N(\mathbf{X})$, and $\overline{W}(0) = \underline{W}(0) = 0$.
9. Pruning constraints $\overline{W}(T_i)$ and $\underline{W}(T_i)$, where $T_i \in \Delta\mathbb{N}$, $\overline{W}(i)$, $\underline{W}(i) \in \hat{\nu}_0\mathbb{N}$, and $\overline{W}(0) = \underline{W}(0) = 0$.

Similar to the previous models, we begin in the inital state $\mathbf{X}_0 = (X_0^1, X_0^2, 0, T_0, W_0)$ and progress recursively to create the trajectory set $\mathcal{X}$. We will utilize elements $(m^1, m^2, q, P) \in \mathcal{N}_E$ to create the possible future rebalancing states $\mathbf{X}_{i+1}$. We prune trajectory states that do not lie within our historical worst-case bounds.

The use of pruning constraints $\overline{X}(i)$ and $\underline{X}(i)$, $\overline{N}(\rho)$ and $\underline{N}(\rho)$, and $\overline{T}(i)$ and $\underline{T}(i)$ are given in the description for the previous model types. We use these previously mentioned constraints as well as some new pruning constraints and this model will only allow future state $\mathbf{X}_{i+1}$ to satisfy the following constraints:

$$i+1 \in [\underline{N}(W_{i+1}), \overline{N}(W_{i+1})] \cap \mathbb{N}$$

$$T_{i+1} \in [\underline{T}(W_{i+1}), \overline{T}(W_{i+1})] \cap \Delta\mathbb{N}$$

$$W_{i+1} \in [\underline{W}(i+1), \overline{W}(i+1)] \cap \hat{\nu}_0\mathbb{N}$$

$$W_{i+1} \in [\underline{W}(T_{i+1}), \overline{W}(T_{i+1})] \cap \hat{\nu}_0\mathbb{N}$$

Then, the following defines the admissible set of nodes, which is used to construct $\mathcal{X}$:

$$\begin{aligned}
\mathcal{N}_A(X_i^1, X_i^2, i, t_0+T_i, W_i) \equiv \Big\{ &\big(X_i^1 + \Delta_i X^1,\ X_i^2 + \Delta_i X^2,\ t_0 + T_i + \Delta_i T, \\
&W_i + \Delta_i W,\ i+1\big)\ : \\
&\exists (m^1, m^2, q, P) \in \mathcal{N}_E,\ q\Delta \leq (T - T_i),\ i \geq 0, \\
&(\Delta_i X^1, \Delta_i X^2, \Delta_i T, \Delta_i W) = (m^1\hat{\delta}^1, m^2\hat{\delta}^2, q\Delta, p\hat{\nu}_0), \\
&\frac{||(X_{i+1}^1, X_{i+1}^2) - (X_0^1, X_0^2)||}{||(X_0^1, X_0^2)||} \in [\underline{X}(i), \overline{X}(i)] \\
(i+1) \in [&\underline{N}(X, [t_0 + T_i + q\Delta]), \overline{N}(X, [t_0 + T_i + q\Delta])] \\
(i+1) \in [&\underline{N}(W_i + \Delta_i W), \overline{N}(W_i + \Delta_i W)] \\
&t_0 + T_i + q\Delta \in \big[\underline{T}(i+1), \overline{T}(i+1)\big] \\
&t_0 + T_i + q\Delta \in \big[\underline{T}(W_{i+1}), \overline{T}(W_{i+1})\big] \\
&W_i + \Delta_i W \in [\underline{W}(i+1), \overline{W}(i+1)] \\
&W_i + \Delta_i W \in [\underline{W}(T_{i+1}), \overline{W}(T_{i+1})] \Big\}
\end{aligned} \tag{4.6}$$

Just as in Type I models, we allow our trajectories to unfold as naturally as possible; progressing through the interval $[t_0, t_0 + T]$, approaching time T. It is possible that a trajectory $\mathbf{X} \in \mathcal{X}$ will stop at rebalancing number $N(\mathbf{X})$. We also incorporate the variation component into our coordinate system. This added dimension allows us to prune trajectories with values which move excessively and accumulate too much variation, or move not enough and do not accumulate enough variation.

Notice that, for each model type, the incorporation of new dimensions in our coordinates allows us to prune with respect to this new dimension. Especially apparent in the Type II models, these pruning constraints may prune with respect to one coordinate while regarding many other coordinates as an independent variable. For example, we prune nodes with regards to W_i, while we may regard i or T_i as the independent variable.

We also remark that the only dimensions which we require in the valuation process are X_i^1 and X_i^2. Including the other variables i, T_i, W_i is solely for the purpose of pruning potential future nodes.

4.2.5 Issues with Pruning of Possible $\mathbf{X}_{i+1}$: Dealing with Arbitrage Nodes

We just described a process which an investor may construct $\mathcal{N}_A(\mathbf{X}_i)$ for a given set of coordinates. There will be instances where the investor will encounter issues with $\mathcal{N}_A(\mathbf{X}_i)$. It is possible that $\mathcal{N}_A(\mathbf{X}_i)$ with produce an arbitrage opportunity at $\mathbf{X}_i$, or (since we concern ourselves with a 2-dimensional convex hull) $\mathcal{N}_A(\mathbf{X}_i)$ will not have enough children nodes $\mathbf{X}_{i+1}$. In this section we describe how an investor will deal with these issues to force $\mathbf{X}_i$ to be 0-neutral (a concept introduced in Definition 6 and Proposition 4 of Chapter 2). Once each node is constructed to be 0-neutral, our trajectory set $\mathcal{X}$ will be globally 0-neutral, and we will be able to obtain super and sub-hedging values $\overline{V}_K(\mathbf{X}, X^2, \mathcal{M})$ and $\underline{V}_K(\mathbf{X}, X^2, \mathcal{M})$ for all stages $0 \leq k \leq N_H(\mathbf{X}) \leq N(\mathbf{X})$, $\forall \mathbf{X} \in \mathcal{X}$, $H \in \mathcal{H}$.

Let us begin with the issue of encountering a child node which is out of our time of interest $[t_0, t_0 + T]$. Suppose $\exists T_{i+1} \in \mathbf{X}_{i+1} \in \mathcal{N}_A(\mathbf{X}_i)$ such that $T_{i+1} = T_i + q\Delta > T$. One may incorporate assumptions to deal with children nodes which reside past $t_0 + T$, however to reduce the complexity of our models we refrain from doing so. Since we do not include a way of dealing with such cases we regard the parent node $\mathbf{X}_i$ as a final node, or rather, for this $\mathbf{X} \in \mathcal{X}$ we have that $N_H(\mathbf{X}) = i$. That is, in our market model $\mathcal{M} = \mathcal{X} \times \mathcal{H}$ for each $(\mathbf{X}, \Phi) \in \mathcal{M}$ (i.e. $\mathbf{X} \in \mathcal{X}$ and $\Phi \in \mathcal{H}$) we then have that $\Phi_k(\mathbf{X}) = 0$ for all $k > N(\mathbf{X})$. Note that this case only arises in the Type I and Type II models introduced in the previous sections.

If we do not encounter any issues with T_{i+1}, there still might be issues enountered with regards to a node's 0-neutrality. Recall that we are constructing market models which incorporate nodes which obey a 2-dimensional 0-neutrality. Our market $\mathcal{M}$ is globally 0-neutral if each parent node $\mathbf{X}_i$ is locally 0-neutral. Since we construct $\mathcal{X}$ recursively we ensure each $\mathbf{X}_i$ is 0-neutral before moving to the next rebalancing $\mathbf{X}_{i+1}$. This concept is discussed in Chapter 2, Proposition 4, where it dicusses that a given parent node $\mathbf{X}_i$ is locally 0-neutral if its given children nodes $\mathbf{X}_{i+1} \in \mathcal{N}_A(\mathbf{X}_i)$ satisfy the 0-neutrality condition given in Definition 6. This definition states that, $(\mathbf{X}, \Phi)$ is called a 0-nuetral node with respect to $\mathcal{H}$ ($H \in \mathbb{R}^2$) if, for all $H \in \mathcal{H}_{\mathcal{X}}$ and $k \geq 0$ the following is satisfied:

$$\inf_{\mathbf{X}' \in \mathcal{X}_{(\mathbf{X}, \Phi)}} H_k(\mathbf{X}) \cdot \Delta_k X' \leq 0.$$

That is, there must be one scenario at rebalancing $i+1$ where the investor makes at least no profit since rebalancing i. We will be utilizing the geometric properties of the *convex hull* to ensure the above condition holds. In doing so, we determine whether or not a 2-dimensional point cloud of children node's asset values (X^1_{i+1}, X^2_{i+1}) includes the parent node's asset values (X^1_i, X^2_i). Then a parent node $\mathbf{X}_i$ is called 0-neutral if for the following set:

$$E = \{(X^1_{i+1}X^2_{i+1}) : \ \exists \mathbf{X}_{i+1} = ((X^1_{i+1}X^2_{i+1}, t_0 + T_i.W_i) \in \mathcal{N}_A(\mathbf{X}_i)\}$$

the following is satisfied:

$$(X^1_i, X^2_i) \in cl\big(co(E)\big).$$

Notice that, since we concern ourselves with 2-dimensional 0-neutrality, in order to determine if $(X^1_i, X^2_i) \in cl\big(co(E)\big)$ holds we require that there are more than 1 distinct tuples in E. When there is only 1 distinct tuple in E and $0 \in cl\big(co(E)\big)$, then $(X^1_{i+1}, X^2_{i+1}) = (X^1_i, X^2_i)$ and we do not gain any new information from this trajectory, however it allows the trajectory to carry into the future (perhaps it will then terminate at this $i+1$ or continues by still being a 0-neutral node). We supply a visual representation of the above condition to show the reader when $cl\big(co(E)\big)$ is satisfied or not in Figure 4.1.

Suppose along some $\mathbf{X} \in \mathcal{X}$ we have that $(X^1_i, X^2_i) \notin cl\big(co(E)\big)$. This means that at $\mathbf{X}_i$ the investor will encounter an arbitrage opportunity. As discussed in Chapter 2, Proposition 5, we will always find that $\underline{V}_0(\mathbf{X}_0, X^2, \mathcal{M}) \leq X^2_0 \leq \overline{V}_0(\mathbf{X}_0, X^2, \mathcal{M})$. In order to bring more information to our trajectory set, while still maintaining meaningful price bounds we incorporate these events in our models by adding an artificial child node which is zero neutral, and set $N_H(\mathbf{X}) = i+1$. Thus, the following node would be included in our admissible set:

$$\mathbf{X}_{i+1} = (X^1_i, X^2_i, i+1, t_0 + T, W_i)$$

We summarize this section with the following:

Given a parent node $\mathbf{X}_i = (X^1_i, X^2_i, i, t_0 + T_i, W_i)$ and $\mathcal{N}_A(\mathbf{X}_i)$, if:

1. $\big[\mathcal{N}_A(\mathbf{X}_i) = \emptyset\big]$:

   Portfolio rebalancing for $\mathbf{X} \in \mathcal{X}$ naturally ends at rebalancing i due to the pruning constraints; that is, we have $N_H(\mathbf{X}) = i$ and $H_i(\mathbf{X}) = 0$ for all $N_H(\mathbf{X}) \leq i \leq N(\mathbf{X})$.

2. $\big[\exists T_{i+1} \in \mathbf{X}_{i+1} \in \mathcal{N}_A(\mathbf{X}_i)$ such that $t_0 + T_{i+1} > T\big]$:

   Force portfolio rebalancing for $\mathbf{X} \in \mathcal{X}$ to end at rebalancing i by setting $\mathcal{N}_A(\mathbf{X}) \equiv \emptyset$; that is, we have $N_H(\mathbf{X}) = i$ and $H_i(\mathbf{X}) = 0$ for all $N_H(\mathbf{X}) \leq i \leq N(\mathbf{X})$.

3. $\left[\forall T_{i+1} \in \mathbf{X}_{i+1} \in \mathcal{N}_A(\mathbf{X}_i)\colon\ t_0 + T_{i+1} \leq T\right]$ AND $\left[(X_i^1, X_i^2) \notin cl(co(E))\right]$:

   Although it is an abuse of notation, we set $\mathcal{N}_A(\mathbf{X}_i) \equiv \mathcal{N}_A(\mathbf{X}_i) \cup (X_i^1, X_i^2, i+1, t_0 + T, W_i)$. That is, we add an artificial 0-neutral node to the pruned set of children $\mathcal{N}_A(\mathbf{X}_i)$ and force the rebalancing to end at this rebalancing: $N_H(\mathbf{X}) = i$ and $H_i(\mathbf{X}) = 0$ for all $N_H(\mathbf{X}) \leq i \leq N(\mathbf{X})$.

4. $\left[\forall T_{i+1} \in \mathbf{X}_{i+1} \in \mathcal{N}_A^*(\mathbf{X}_i)\colon\ t_0 + T_{i+1} \leq T\right]$ AND $\left[(X_i^1, X_i^2) \in cl(co(E))\right]$:

   Then we set $\mathcal{N}_A(\mathbf{X}_i) \equiv \mathcal{N}_A(\mathbf{X}_i)$. $\mathbf{X}_i$ is 0-neutral, and we continue to construct trajectory paths $\mathcal{X}_{(\mathbf{X},i)} \subseteq \mathcal{X}$.

The summary above holds for all models introduced in this section.

4.2.6 Nested Model Values

As indicated, the pruning constraints used in Type 0 Models are nested into Type I and II models, and similarly, we nest the pruning constraints used in Type I models into Type II models. We also mentioned that the extra dimensions incorporated into Type I and II models play no role in the valuation process of $\mathcal{X}$. The valuation process only concerns itself with the value of our assets at specific rebalancing; i.e. it only requires the values X_i^1 and X_i^2.

Due to the nesting of pruning constraints, we expect this to reflect a nesting of price intervals obtained from each model. We explain this here.

Recall that the price bounds for an asset X^2 at the k'th rebalancing are denoted $\underline{V}_k(\mathbf{X}, X^2, \mathcal{M}) \leq \overline{V}_k(\mathbf{X}, X^2, \mathcal{M})$. Let us denote the the markets $\mathcal{M}^0$, $\mathcal{M}^I$, $\mathcal{M}^{II}$ as the market produced for a Type 0, I, or II model, respectively. For simplicity in the next statement let us denote the superhedging value for X^2 at the k'th rebalancing given by Type 0, I or II models as the following: $\overline{V}_k(\mathbf{X}, X^2, \mathcal{M}^0) \equiv \overline{V}_k(\mathcal{M}^0)$, $\overline{V}_k(\mathbf{X}, X^2, \mathcal{M}^I) \equiv \overline{V}_k(\mathcal{M}^I)$, $\overline{V}_k(\mathbf{X}, X^2, \mathcal{M}^{II}) \equiv \overline{V}_k(\mathcal{M}^{II})$, respectively. The same notation will be given to the subhedging value. Then, we expect to see the following behaviour:

$$\underline{V}(\mathcal{M}^0) \leq \underline{V}(\mathcal{M}^I) \leq \underline{V}(\mathcal{M}^{II}) \leq X_i^2 \leq \overline{V}(\mathcal{M}^{II}) \leq \overline{V}(\mathcal{M}^I) \leq \overline{V}(\mathcal{M}^0) \tag{4.7}$$

That is, the price bounds given by a Type II model should be tighter than (or at least equivalent to) the price bounds given by a Type I or Type 0 model. This sensible claim may be informally proven by the following argument: if we create Type 0, I and II models all with the same parameters (i.e. $\hat{\delta}^1$, $\hat{\delta}^2$, $\hat{\nu}_0$, $\mathcal{N}_E$ and pruninig constraints), then we will

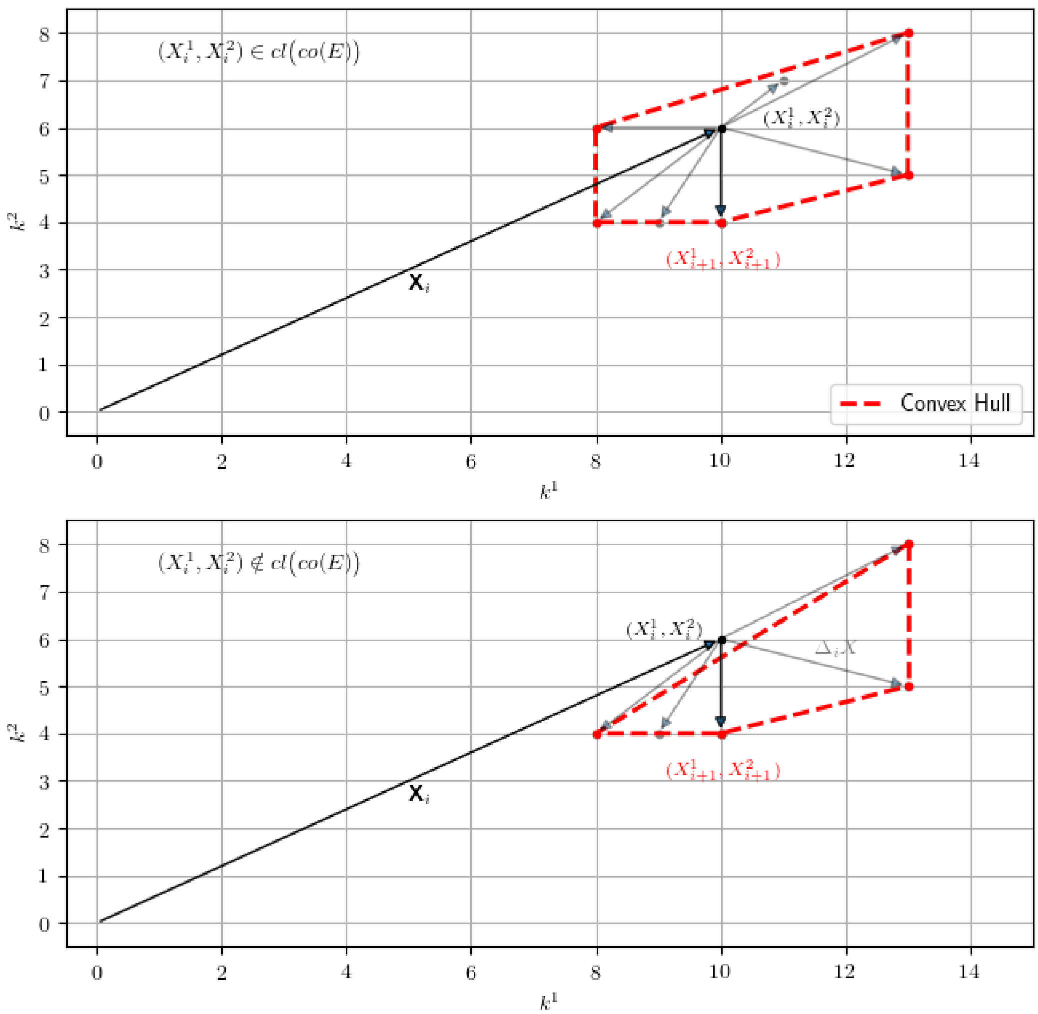


Figure 4.1: A visual representation of when $(X_i^1, X_i^2) \in cl\big(co(E)\big)$ (in the top subplot) and when $(X_i^1, X_i^2) \notin cl\big(co(E)\big)$ (lower subplot).

always have $\mathcal{M}^{II} \subseteq \mathcal{M}^{I} \subseteq \mathcal{M}^{0}$. Let us first concern ourselves with the simplest set of coordinates: (X_i^1, X_i^2, i). An investor will produce $\mathbf{X}_{i+1}$ by using the set $\mathcal{N}_E$ to obtain $\mathbf{X}_{i+1} = (X_i^1 + m^1\hat{\delta}^1, X_i^2 + m^2\hat{\delta}^2, i+1)$, where $m^1, m^2 \in \mathcal{N}_E$. If the pruning constraint in Equation (4.2) is satisfied then $\mathbf{X}_i$ is admissible. It is very possible that, if we were to extend the coordinates to the Type I or II coordinates, increasing the possibilities for pruning will cause the node $\mathbf{X}_{i+1}$ to be pruned. For example, that same node might now have some value for T_{i+1} which does not satisfy Equation (4.4). Then $\exists \mathbf{X} \in \mathcal{M}^0$ such that $\mathbf{X} \notin \mathcal{M}^I$. This same logic may then be applied to Type II models and the pruning constraints they utilize.

Note that through the reasoning above, it is never possible to have $\mathcal{M}^0 \subseteq \mathcal{M}^I \subseteq \mathcal{M}^{II}$. Each subsequent model has the same if not more pruning occuring at each parent node $\mathbf{X}_i$. Thus, there will be more nodes in Type 0 models than there is in Type II models. This is indicative that we will always see $\underline{V}_i(\mathcal{M}^0) \leq \underline{V}(\mathcal{M}^I) \leq \underline{V}(\mathcal{M}^{II}) \leq X_i^2 \leq \overline{V}(\mathcal{M}^{II}) \leq \overline{V}(\mathcal{M}^I) \leq \overline{V}(\mathcal{M}^0)$.

Note that this only pertains to Type 0, I, and II models created with the same values for the input parameters $\hat{\delta}^1$, $\hat{\delta}^2$, and $\hat{\nu}_0$, and the same set $\mathcal{N}_E$.

4.3 Risk Taking in Trajectory Models

Up until now, this paper purely deals with the mathematical framework to observing charts operationally and constructing a trajectory set. There is now enough foundation to create a trajectory set, use the valuation process as described in Chapter 2, Definition 8, to yield a trajectory market model $\mathcal{M} = \mathcal{X} \times \mathcal{H}$ with price bounds $\underline{V}_0(\mathbf{X}_0, X^2, \mathcal{M}) \leq \overline{V}_0(\mathbf{X}_0, X^2, \mathcal{M})$. We proved in Proposition 2.14 in Chapter 2 that we will always have $\underline{V}_0(\mathbf{X}_0, X^2, \mathcal{M}) \leq X_0^2 \leq \overline{V}_0(\mathbf{X}_0, X^2, \mathcal{M})$ and since we know our market models will never provide price bounds which indicate a possible market mispricing, we wish to extract further information from our models.

We have constructed a market $\mathcal{M}$, in which with initial capital $\overline{V}_0(\mathbf{X}_0, X^2, \mathcal{M})$ an investor is able to form some portfolio $H \in \mathcal{H}$ of X^1 and superhedge X^2 at any state in trajectories in $\mathcal{X}$. That is, according to our model, an investor will be able to superhedge X^2 with no risk if they begin with an initial capital of $\overline{V}_0(\mathbf{X}_0, X^2, \mathcal{M})$. What if the investor was interested in taking on some amount of risk by beginning with initial capital $v < \overline{V}_0(\mathbf{X}_0, X^2, \mathcal{M})$? In this section we address this question and discuss the process of risk taking within our trajectorial market models.

Let us first describe the issue at hand. Without our models signalling a market misprice, our investor would never be inclined to trade the assets of interest. Therefore, there should be some mechanism in place to indicate that with a certain level of risk, the investor should profit from trading X^1 with initial capital $v < \overline{V}_0(\mathbf{X}_0, X^2, \mathcal{M})$. This might then interest our investor to use insight obtained from our market models to actually place trades.

The mechanism used to indicate a certain level of risk taken with initial capital v is our profit and losses when simulating trajectory paths within $\mathcal{X}$. Given some $\mathbf{X} \in \mathcal{X}$ and $H \in \mathcal{H}$, trading X^1 while shorting X^2 will result in our investor profitting or losing money within the future time interval $[0, T]$.

Suppose that we short X^2, while forming a portfolio of asset X^1 to superhedge X^2 over $[0, T]$, then the value of the portfolio of asset X^1 at the i'th rebalancing in $\mathbf{X} \in \mathcal{X}$ is given by

$$V_H(i, X^1) = v + \sum_{k=0}^{i-1} H_k(S)(X^1_{k+1} - X^1_k).$$

Our investor will then profit at any stage i if we have that $V_H(i, X^1) > X^2_i$. Then, to quantify the level of risk the investor is taking on by creating a portfolio of value v to superhedge X^2 we will sample trajectories from the trajectory set $\mathcal{X}$, and determine the proportion of trajectories which profit in our model. As a further step, the investor may also have a preference for a certain probability distribution for the sampled trajectories in $\mathcal{X}$, such probability distribution will then provide a probability of gains. We do not pursue this further step in the thesis.

Given that the amount of relative capital required to superhedge X^2 in all states is $\overline{V}_0(\mathbf{X}_0, X^2, \mathcal{M})$, if the investor begins with an initial investment of size $v \geq \overline{V}_0(\mathbf{X}_0, X^2, \mathcal{M})$ they will always superhedge X^2.

The trajectory set $\mathcal{X}$ may be regarded as an object which is synonymous to a stochastic process's support, which consists of all path where the process may lie. This then forces our trajectories to move recursively from our initial state $i = 0$ and end at rebalancing number $i = N_H(\mathbf{X}) \leq N(\mathbf{X}),\ \mathbf{X} \in \mathcal{X}$. This process for simulating trajectory paths is analogous to simulating stock prices with a discrete Geometric Brownian Motion, where the stochastic process must lie within some level of support.

Chapter 5

Measurements on Charts for Estimation and Criteria for Calibration

In this section we discuss the process for estimating parameters used in model instantiation. We mention once again that the process for obtaining model parameters is divided into two parts. The first being a *calibration* process, where the investor sets parameters to individual preferences which allow for model construction. The calibration process is crucial to constructing our models as the choice of calibrated parameters (δ, δ_0, δ_{up}, δ_{down}, $\hat{\delta}^1$, $\hat{\delta}^2$, $\hat{\nu}_0$) reflect the outcome of the obtained *estimated* parameters. In Section 5.2 we discuss criteria for an optimal calibration of these parameters, with a main focus on the calibration of δ, δ_{up}, and δ_{down}. The second part of obtaining model parameters is an *estimation* process, where the word *estimation* refers to the computations performed on historical data. Given the investor's calibration of δ (or δ_{up} and δ_{down}), δ_0, $\hat{\delta}^1$, $\hat{\delta}^2$, and $\hat{\nu}_0$, he operates on data through the operational framework described throughout Chapter 3 which yields the estimated parameters used for model construction. Recall that the operational framework described in Chapter 3 which relies on a *worst-case* persepctive, which allows trajectory models to be objective.

We assume availability of charts $x(t)$ where $t \in \mathcal{T}$, $\mathcal{T} \subseteq \Delta\mathbb{Z}$ and $\mathcal{T}$ denotes the set of historical times for which the investor has access to values $x(t)$. The investor will prescribe $T = M_T\Delta$ and perform operations described in Section 3.3 on intervals $[t_0, t_0 + T] \subseteq \mathcal{T}$. We utilize definitions of observable worst-case pruning constraints as introduced in Section 3.4. In this section we discuss how to go about obtaining a worst-case estimate for $\mathcal{N}_E$. This involves performing the operations described in Section 3.3 on intervals $[t_0, t_0+T] \subseteq \mathcal{T}$. We also mention here that treating $\mathcal{T}$ as an interval would incorporate overnight effects when updating $[t_0, t_0+T]$. To avoid this we enforce T to be one trading day of data, and only allow $t_0 = k(M_T+1)\Delta,\ k \in \mathbb{Z}$.

Given the historical data $x(t)$, $t \in \mathcal{T}$, calibrated parameters δ_0, δ (or δ_{up} and δ_{down}), we obtain chart sampling times $\{r_l\}_{l=0}^{L}$ and historical portfolio rebalancing times $\{t_i\}_{i=0}^{N}$, both dependent on δ_0, δ, and an interval $[t_0, t_0+T] \subseteq \mathcal{T}$. The interval is updated by shifting it by one day and starting again, omitting overnight effects. Each set of sampling times $\{r_l\}_{l=0}^{L}$, one for each such window, will have a variable length of $L(x, [t, t+T]) + 1$. Likewise, each set of rebalancing times $\{t_i\}_{i=0}^{N}$, one for each such window, will have a variable length of $N(x, [t, t+T]) + 1$.

The following sections rely on notation introduced in Section 3.3.

5.1 Rounded Chart Values

Given an interval $[t_0, t_0+T]$, the investor *first* evaluates the times $\{r_l\}_{l=0}^{L}$, and $\{t_i\}_{i=0}^{N}$ using exact values. The rounding, mentioned in section 3.3, occurs after and reversing this order would give different results. When rounding, the case of ambiguity would rise when distinguishing how to round values (with regards to a floor or ceiling rounding).

To create a disambiguation the reader is turned to Section 9.1 of the paper Ferrando et al. [2019a], which shows that after rounding we will always have:

$$\delta \le ||(\lfloor \Delta_{t_i} x^{*1} \rfloor_{\hat{\delta}^1}, \lfloor \Delta_{t_i} x^{*2} \rfloor_{\hat{\delta}^2})||$$

for all $0 \le i \le N$. It clarifies that, regardless of choice of floor or ceiling rounding, the above inequality will always hold. The same will apply for similar arguments pertaining to sampling times.

5.2 Calibration of δ, δ_0, δ_{up}, δ_{down}, $\hat{\delta}^1$, $\hat{\delta}^2$, $\hat{\nu}_0$

Before an investor can begin to perform the historical estimations required to build a trajectory based market model, they go about selecting appropriate values for the parameters δ, δ_0, δ_{up}, and δ_{down} as well as the discretization parameters $\hat{\delta}^1$, $\hat{\delta}^2$, and $\hat{\nu}_0$. The process of selecting values for these parameters is referred to as calibration. It is these calibrated parameters that the investor is free to select which will have a direct effect on the outcome of our trajectory models. All other model parameters are given as a result of the choice of the calibrated parameters. These parameters must be set to guarantee that the investor may observe charts as described in Chapter 3, and will be able to construct a trajectory based market as described in Chapter 4. In this section we discuss the concerns of calibrating δ, δ_0, δ_{up}, δ_{down}, $\hat{\delta}^1$, $\hat{\delta}^2$, and $\hat{\nu}_0$ to ensure a model's construction.

5.2.1 δ

Our δ-uncorrelated models are constructed through portfolio rebalances which are dependent on chart δ-movements, and thus it should be clear that an investor's choice of δ has the greatest effect on their ability to create a trajectory based market model. The investor must choose δ so that there are observable δ-movements within the charts $x(t)$. We denote the maximum and minimum chart changes as following:

$$\delta_{max} = \max_{\forall [t_0, t_0+T] \subseteq \mathcal{T}} \Big(||x(t') - x(t)|| \Big), \quad \delta_{min} = \min_{\forall [t_0, t_0+T] \subseteq \mathcal{T}} \Big(||x(t') - x(t)|| \Big)$$
$$for\ t < t';\ t, t' \in [t_0, t_0 + T];\ 0 < t' - t \leq T$$

Then, for any $\delta \geq \delta_{max}$ the investor will not have rebalanced their portfolio once historically and we will then have $\mathcal{N}_E(x, [t_0, t_0 + T]) = \emptyset$. Thus to obtain meaningful results, the investor must choose some values of $\delta, \in [\delta_{min}, \delta_{max}] \subset \mathbb{R}_+$. Notice that when $\delta = \delta_{min}$ the investor will rebalance their portfolio at every time step, or rather, $q_i = 1$ or equivalently $t_{i+1} = t_i + \Delta$ for each element in our empirical set of nodes $\mathcal{N}_E(x, [t_0, t_0 + T])$. Therefore we limit our choices of δ to the interval $[\delta_{min}, \delta_{max}]$.

We first mention that δ should be selected such that we observe at least $N(x, [t_0, t_0+T]) \geq 1$ for some $[t_0, t_0+T] \subseteq \mathcal{T}$. In fact, δ should be chosen so that we are able to create a market which is at least 0-neutral (*at least* meaning it must be 0-neutral however arbitrage free is preferred). In order to do so we determine if $0 \in ri\big(co(M)\big)$ where $M \equiv \{(m^1, m^2) : \exists (m^1, m^2, q, P) \in \mathcal{N}_E)\}$. Although market models can be constructed when $0 \in cl\big(co(M)\big)$ is satisfied, we aim to have that $0 \in ri\big(co(M)\big)$ (the notation given here was first introduced in Chapter 2, Proposition 4). Recall the way we construct market models is given in Section 4.2.5, where we said we stop the recursive formulation of a trajectory once an arbitrage opportunity arises. If $0 \notin cl\big(co(M)\big)$ then our market would become a one step arbitrage market, i.e. $N_H(\mathbf{X}) = 1$. We show some sets M, described above, in Chapter 6 where we show model output for chosen δ values.

Note that the investor's choice of δ has an effect on $|\mathcal{N}_E|$, which affects the overall computation time when constructing $\mathcal{X}$. We also emphasize that the calibration of δ (and actually, all other parameters discussed in this section) and the estimation of models parameters go hand in hand. The investor is required to perform an estimation process for many values of δ in order to determine which reflects their own trading behaviour.

We provide output for various pruning constraints which contribute to an investor calibrating their choice of δ. These values can be seen in Figures 5.1, 5.2, 5.3, and 5.4. Figure 5.1 is helpful to observe the amount our vector X may change from initial state $\mathbf{X}_0$. Figure 5.2 portrays the maximum and minimum number of portfolio rebalances that occur at time T in the intervals $[t_0, t_0 + T] \subseteq \mathcal{T}$. This figure is particularly helpful in calibrating δ since the investor may select δ so that $N^*(x, \mathcal{T}, T)$ and $N_*(x, \mathcal{T}, T)$ are within bounds which represents an investor's trading behaviour. Figure 5.3 helps the investor determine which δ values allow trajectories to reach terminal time T while Figure 5.4 shows how wide the bounds $T^*(x, \mathcal{T}, i)$ and $T_*(x, \mathcal{T}, i)$ are (the wider these bounds, the less number of nodes pruned at each stage when constructing $\mathcal{X}$). Similar plots to Figure 5.4 showing $\big(W^*(x, \mathcal{T}, i) - W_*(x, \mathcal{T}, i)\big)$ are given in the Appendix. Also, similar plots to Figures 5.1, 5.2, 5.3, and 5.4 which utilize Twitter as a numeraire will be provided in the Appendix.

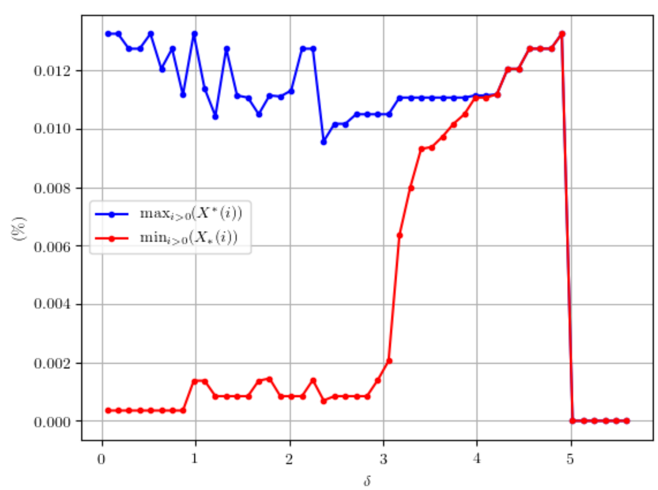


Figure 5.1: $\max_i \big(X^*(x, \mathcal{T}, i)\big)$ and $\min_i \big(X_*(x, \mathcal{T}, i)\big)$ vs. δ for δ-uncorrelated models. Here we have that $\mathcal{T} = \mathcal{T}^2$ as given in Chapter 6. This represents the maximum and minimum amount X_i may vary from X_0. Notice that there are instances of δ where we have that $\max_i \big(X^*(x, \mathcal{T}, i)\big) = \min_i \big(X_*(x, \mathcal{T}, i)\big)$. This indicates that if δ is calibrated to any of these values, there is only one $[t_0, t_0 + T] \in \mathcal{T}$ where we observe a δ-movement.

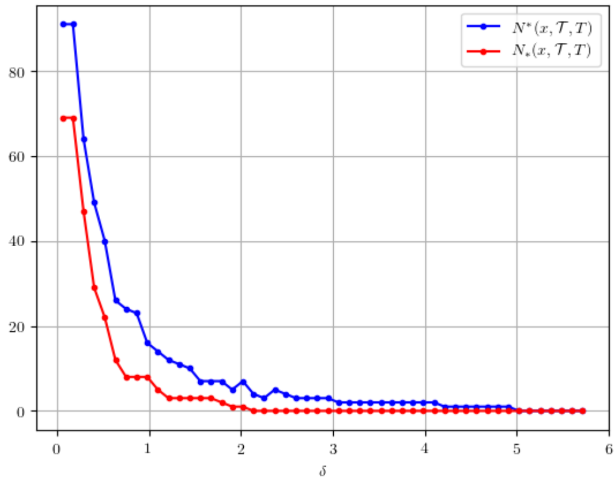


Figure 5.2: $N^*(x, \mathcal{T}, T)$ and $N_*(x, \mathcal{T}, T)$ vs. δ for δ-uncorrelated models. Here we have that $\mathcal{T} = \mathcal{T}^2$. Greater the value of $N^*(x, \mathcal{T}, T) - N_*(x, \mathcal{T}, T)$ in this figure provide more stable output as it does not restrain the future entirely. i.e. choosing $\delta = 1.0$ allows trajectories to reach time T with at least 8 rebalances but less than 18 rebalances.

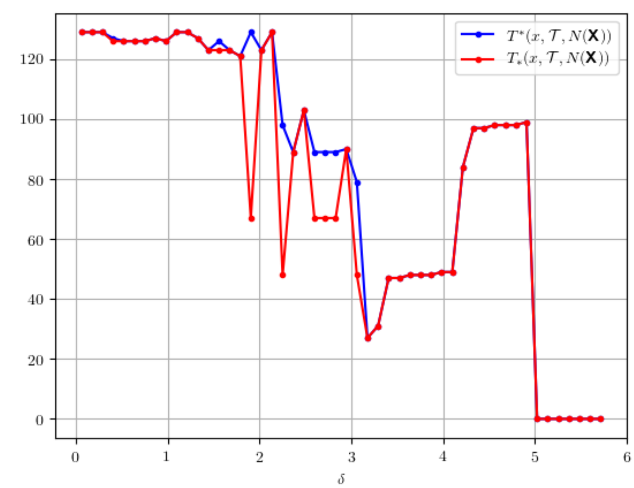


Figure 5.3: $T^*(x, \mathcal{T}, N(\mathbf{X}))$ and $T_*(x, \mathcal{T}, N(\mathbf{X}))$ vs. δ for δ-uncorrelated models. Here we have that $\mathcal{T} = \mathcal{T}^2$ as given in Chapter 6.

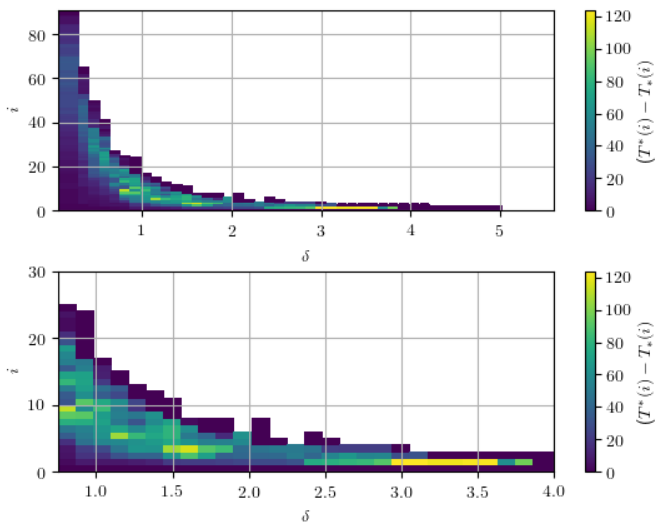


Figure 5.4: $\big(T^*(x, \mathcal{T}, i) - T_*(x, \mathcal{T}, i)\big)$ vs. δ for δ-uncorrelated models. Here we have that $\mathcal{T} = \mathcal{T}^2$ as given in Chapter 6. This plot shows how wide the pruning constraints $T^*(x, \mathcal{T}, i)$ and $T_*(x, \mathcal{T}, i)$ are at each δ-movement.

5.2.2 δ_{up} and δ_{down}

Given that we define our δ-movements entirely different in δ-correlated models, limiting our choices of δ_{up} and δ_{down} less intuitive. As can be seen in Figures 5.5 and 5.6, an increase in δ_{up} and δ_{down} causes an increase in the number of observed δ-movements. This is contrasted by what occurs when increasing δ in δ-uncorrelated models, which causes a decrease in the number of observed δ-movements. The reasoning for this is due to how δ-movements are defined in δ-correlated models. For example, if we were to increase $\delta_{up} >> 1$ it is *more likely* that the following will hold: $0 \le x^2(t) - x^2(t_i) \le \delta_{up}(x^1(t) - x^1(t_i))$ for some $t > t_i$, $t, t_i \in [t_0, t_0 + T]$.

In order to construct a trajectory market model an investor must prescribe values for δ_{up} and δ_{down} such that they obtain stable parameters for model building. This is done by ensuring the same arguments in the previous section for calibrating δ are satistied. We repeat these here. We require that $N(x, [t_0, t_0 + T]) \ge 1$ for some $[t_0, t_0 + T] \subseteq \mathcal{T}$ and must have at least that $0 \in cl\big(co(M)\big)$. Although market models can be constructed when $0 \in cl\big(co(M)\big)$ is satisfied, we aim to have that $0 \in ri\big(co(M)\big)$.

In the same respect for δ, the investor's choice of δ_{up} and δ_{down} has an effect on $|\mathcal{N}_E|$, which affects the overall computation time when constructing $\mathcal{X}$. Similarly, the calibration of δ_{up} and δ_{down} and the estimation of models parameters go hand in hand. The investor is required to perform an estimation process for many values of δ_{up} and δ_{down} in order to determine which reflects their own trading behaviour.

We provide output for the same pruning constraints shown in the previous section. These figures show values for these constraints which contribute to an investor calibrating their choice of δ_{up} and δ_{down}. Note that while the values for δ_{up} and δ_{down} may be different (i.e. $\delta_{up} \neq \delta_{down}$), we provide output by setting $\delta_{up} = \delta_{down}$. These values can be seen in Figures 5.5, 5.6, 5.7, and 5.8. Rather than being very repititive, we mention that these plots help with the calibration of δ_{up} and δ_{down} by showing the same characteristics described in the previous section. There are similar plots to those given here in the Appendix, just as is stated in the section above for δ calibration.

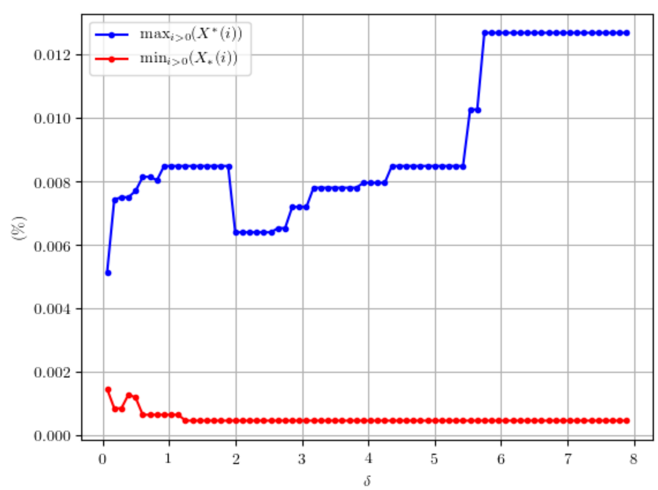


Figure 5.5: $\max_i \left(X^*(x, \mathcal{T}, i)\right)$ and $\min_i \left(X_*(x, \mathcal{T}, i)\right)$ vs. δ for δ-correlated models. we have that $\mathcal{T} = \mathcal{T}^2$ as given in Chapter 6, $\delta_{up} = \delta_{down} = \delta$ in the above plot. Here we use δ as a reference to the value along the x-axis in the above plot.

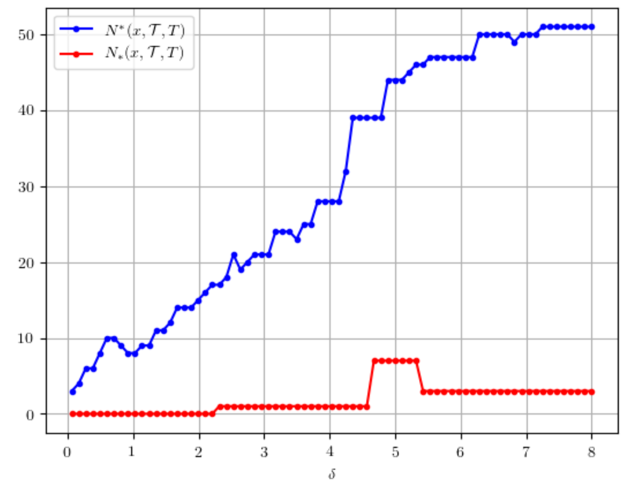


Figure 5.6: $N^*(x, \mathcal{T}, T)$ and $N_*(x, \mathcal{T}, T)$ vs. δ for δ-correlated models. Here we have that $\mathcal{T} = \mathcal{T}^2$ as given in Chapter 6, $\delta_{up} = \delta_{down} = \delta$ in the above plot. Here we just use δ as a reference to the value along the x-axis in the above plot.

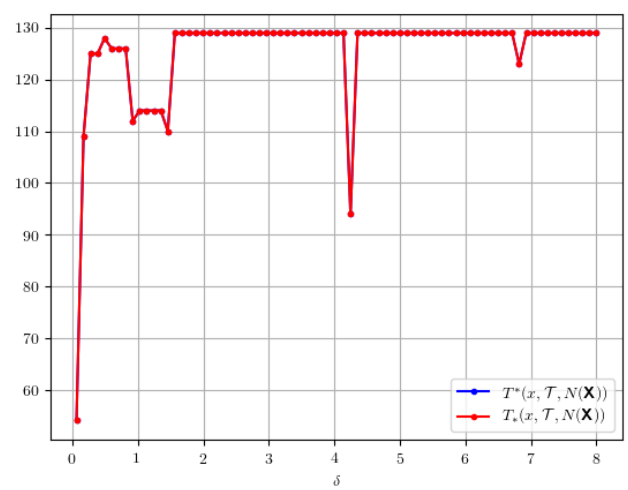


Figure 5.7: $T^*(x, \mathcal{T}, N(\mathbf{X}))$ and $T_*(x, \mathcal{T}, N(\mathbf{X}))$ vs. δ for δ-correlated models. Here we have that $\mathcal{T} = \mathcal{T}^2$ as given in Chapter 6, and $\delta_{up} = \delta_{down} = \delta$ in the above plot. Here we just use δ as a reference to the value along the x-axis in the above plot.

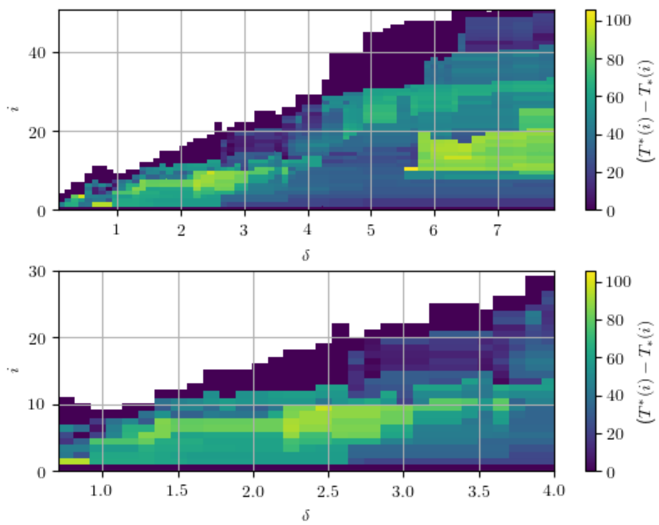


Figure 5.8: $\big(T^*(x, \mathcal{T}, i) - T_*(x, \mathcal{T}, i)\big)$ vs. δ for δ-correlated models. Here we have that $\mathcal{T} = \mathcal{T}^2$ as given in Chapter 6. This plot shows how wide the pruning constraints $T^*(x, \mathcal{T}, i)$ and $T_*(x, \mathcal{T}, i)$ are at each δ-movement.

5.2.3 δ_0

The choice of δ_0 is not so crucial to ensure a trajectory model's construction as it is only used to determine the times $\{r_l\}_{l=0}^{L}$ to sample the charts $x(t)$ between δ-movements. These sampling times are then used to determine the amount of vector variation the chart accumulates between δ-movements in a trajectory interval $[t_0, T]$. As the choice of δ_0 only affects the accumulated vector variation we note that it would be perfectly acceptable to set $\delta_0 = \delta$ in trajectory models that do not incorporate the vector variation coordinate W_i.

In models which incorporate the vector variation, the investor is limited to $\delta_0 \in [\delta_{min}, \delta]$. The effect of δ_0 on model construction is not covered in this thesis, and thus we have limited our choice of δ_0 to arbitrary values which allow for model construction in a reasonable time. A reasonable computation time would of course be dependent on the investor's choice of T and Δ. In this thesis we limit ourselves to $T = 130\Delta$, with $\Delta = 3$ minutes which is one trading day. Thus the investor should be able to build and value a trajectory model at the beginning of each trading day in a matter of minutes.

5.2.4 $\hat{\delta}^1$, $\hat{\delta}^2$ and $\hat{\nu}_0$

The parameters $\hat{\delta}^1$, $\hat{\delta}^2$, and $\hat{\nu}_0$ provide the discretization of the observable charts and limit the possible values of our coordinates. The selection of these parameters also helps regulate the computation time for constructing models; the smaller the values of $\hat{\delta}^1$, $\hat{\delta}^2$, and $\hat{\nu}_0$ corresponds to a larger discretization. There is also a tradeoff presented here when calibrating these parameters. With a larger discretization the investor is able to construct models with a higher degree of precision, yet increases the computation time of constructing market models.

Once again, the effect of these parameters on model construction is not explored in this thesis so we limit ourselves to $\hat{\delta}^1$, $\hat{\delta}^2$, $\hat{\nu}_0$ values which allow us to build traecjtory models with a reasonable amount of computation time. We do mention that if the investor chooses to use a bank account with zero-interest rate as their numeraire, the smallest values for $\hat{\delta}^1$, $\hat{\delta}^2$ and $\hat{\nu}_0$ would be $\hat{\delta}^1 = \hat{\delta}^2 = \hat{\nu}_0 = 0.01$. This is exactly the same case as that in Ferrando et al. [2019a].

5.3 Comments on Worst-Case Calibration and Estimation

The calibrated parameters should be set so that the estimated parameters have stable ranges. That is, their ranges should have informative values in the sense of restricting the offspring of children via pruning and reflect the investor's own trading behaviour. There are many different criteria an investor may use to select δ, and to some extent T.

We mentioned in the previous section various criteria an investor will use to calibrate δ and that doing so requires the investor to test estimated parameters over a variety of δ values. Given the choice of historical data and δ, the estimated parameters should also reflect an investor's own trading behaviour. However, to obtain estimated parameters that reflect a investor's preference one must select begin by selecting δ; where choosing δ will involve an estimation process performed several times over many δ values.

After observing the effect δ has on estimated parameters, criteria for choosing a value of δ becomes purely investor dependent. For example, the investor might wish to rebalance a portfolio only 5 times in a trading day to avoid accruing too many transaction costs. This investor would then select a δ value corresponding to $i^* \geq 5$ (or $N^*(x, \mathcal{T}, T) \geq 5$ if using type I or II models) and setting $N(\mathbf{X}) = 5$. This same logic can be applied to any observable quantity which the investor measures.

The parameters associated with an operational setting must be chosen so that they satisfy the topics discussed in the above section. In this section, we discuss the implications of selecting a δ value.

Choosing δ too large (or δ_{up} and δ_{down} too small) will degenerate down to binomial models and we will obtain the price bounds $\underline{V}_0(\mathbf{X}_0, X^2, \mathcal{M}) = \overline{V}_0(\mathbf{X}_0, X^2, \mathcal{M})$. Choosing insufficient δ (or δ_{up} and δ_{down}) will also cause the set $\mathcal{N}_E$ to be empty. Then for the interval of historical data, $\mathcal{T}$, $|\mathcal{N}_E|$ is controlled by the value of δ (δ_{up} and δ_{down}). Also for a fixed δ (δ_{up} and δ_{down}), increasing $\mathcal{T}$ changes the shape of the set $|\mathcal{N}_E|$. The investor should determine if the shape of $|\mathcal{N}_E|$ stabilizes with an accumulation of data. If the shape of $|\mathcal{N}_E|$ stabilizes over time, new data introduced to our models will only contribute to the relative interior of the convex hull of $|\mathcal{N}_E|$. We show this stabilization in various figures such as Figure 5.11 and figures throughout the appendix. We also mention that since large $|\mathcal{N}_E|$ result in increased computation times when creating $\mathcal{X}$.

5.4 Worst-Case Estimate for $\mathcal{N}_E$

Define,

$$\mathcal{N}_E(x, \mathcal{T}) \equiv \cup_{[t_0, t_0+T] \subseteq \mathcal{T}} \mathcal{N}_E(x, [t_0, t_0 + T]),$$

where $\mathcal{N}_E(x, [t_0, t_0 + T])$ was introduced in Definition 14. This new set, $\mathcal{N}_E(x, \mathcal{T})$, is what we refer to as a *worst case* estimate of this parameter. If the investor chooses to remove any arbitrary subset from the worst-case estimates then this would be called a *risk-taking estimate.*

For the purposes of this paper we go ahead and set the model variable $\mathcal{N}_E$ to be the observed worst-case historical estimate:

$$\mathcal{N}_E = \mathcal{N}_E(x, \mathcal{T})$$

However, one might be interested in recasting models with a direct product of worst-case values observed in $\mathcal{N}_E(x, \mathcal{T})$ by defining $\mathcal{N}_E$ in the following manner. Define:

$$\begin{aligned}
\underline{m}^1 &= \min\{m^1 : \exists(m^1, m^2, q, P) \in \mathcal{N}_E(x, \mathcal{T})\}, \\
\overline{m}^1 &= \max\{m^1 : \exists(m^1, m^2, q, P) \in \mathcal{N}_E(x, \mathcal{T})\}, \\
\underline{m}^2 &= \min\{m^2 : \exists(m^1, m^2, q, P) \in \mathcal{N}_E(x, \mathcal{T})\}, \\
\overline{m}^2 &= \max\{m^2 : \exists(m^1, m^2, q, P) \in \mathcal{N}_E(x, \mathcal{T})\}, \\
\underline{q} &= \min\{q : \exists(m^1, m^2, q, P) \in \mathcal{N}_E(x, \mathcal{T})\}, \\
\overline{q} &= \max\{q : \exists(m^1, m^2, q, P) \in \mathcal{N}_E(x, \mathcal{T})\}, \\
\underline{P} &= \min\{P : \exists(m^1, m^2, q, P) \in \mathcal{N}_E(x, \mathcal{T})\}, \\
\overline{P} &= \max\{P : \exists(m^1, m^2, q, P) \in \mathcal{N}_E(x, \mathcal{T})\},
\end{aligned}$$

where $m^1, m^2, q, P \in \mathcal{N}_E(x, \mathcal{T})$. Then we would construct $\mathcal{N}_E$ as:

$$\begin{aligned}\mathcal{N}_E = \mathcal{N}_E^{DP} \equiv &\{m^1 \in \mathbb{Z} : \underline{m}^1 \leq m^1 \leq \overline{m}^1\} \times \{m^2 \in \mathbb{Z} : \underline{m}^2 \leq m^2 \leq \overline{m}^2\} \times \\ &\{q \in \mathbb{N} : \underline{q} \leq q \leq \overline{q}\} \times \{P \in \mathbb{N} : \underline{P} \leq P \leq \overline{P}\}\end{aligned}$$

where the DP in $\mathcal{N}_E^{DP}$ indicates 'Direct Product'. Notice that $\mathcal{N}_E(x, \mathcal{T}) \subseteq \mathcal{N}_E^{DP}$. Although $\mathcal{N}_E^{DP}$ is used in Ferrando et al. [2019a], the use of this set is not explored in this thesis. We concentrate our efforts on exploring the use of introducing various pruning constraints in our models.

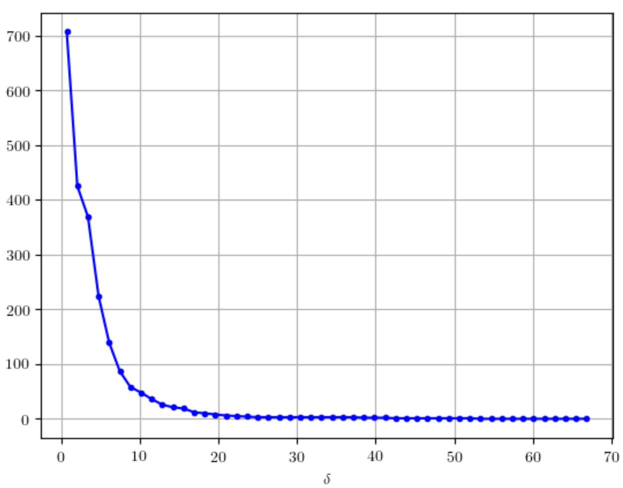
Figures 5.9, 18, 19, 20, 21, show how the structure of each of the extreme values $\overline{m}^1$, $\underline{m}^1$, $\overline{m}^2$, $\underline{m}^2$, $\overline{q}$, $\underline{q}$, $\overline{P}$, and $\underline{P}$ behave with respect to an investor's choice of δ. This behaviour is also affected by the investor's choice of $\hat{\delta}^1$, $\hat{\delta}^2$, and $\hat{\nu}_0$. Although given in each plot, for the estimation process we set $\hat{\delta}^1 = \hat{\delta}^2 = \delta/2$ and $\hat{\nu}_0 = \delta$ for δ-uncorrelated models, and $\hat{\delta}^1 = \hat{\delta}^2 = 0.5$ and $\hat{\nu}_0 = 1.0$ for δ-correlated models. We also set $\delta_0 = \delta_{min}$ for δ-uncorrelated models while we set $\delta_0 = 0.01$ for δ-correlated models for every δ in these figures. We also mention that we provide analogous output for the structure of $\mathcal{N}_E(x, \mathcal{T})$ while using a numeraire in the Appendix.

We mentioned that we are concerned in utilizing properties of the convex hull to construct our trajectory market models (outlined in Proposition 4 of Chapter 2), and in particular, we are concerned with the convex hull of the set defined in Equation (2.8) of Section 2.1.3. To ensure the conditions in Proposition 4 will be satisfied *before* constructing our models we may observe the convex hull of a set of possible $(m^1\hat{\delta}^1, m^2\hat{\delta}^2)$ pairs. Then, for the set $M = \{(m^1\hat{\delta}^1, m^2\hat{\delta}^2) : \exists (m^1, m^2, q, P) \in \mathcal{N}_E(x, \mathcal{T})\}$ we observe the convex hull.

For our trajectory market models to be objective, we note that the convex hull of M should stabilize with an accumulation of data for an investor's choice of calibrated parameters. Otherwise if the convex hull of M grows indefinitely our models will not remain to be objective as the price bounds $[\underline{V}_0, \overline{V}_0]$ will grow indefinitely as well. Therefore, we observe the effect that an accumulation of data has on the shape of the convex hull. This is seen in Figure 5.11 below and Figures A.3-A.7 in the appendix. As can be seen in the figures is that the shape of the convex hull does stabilize after an accumulation of data, which is expected.

5.5 Worst-Case Estimate for Pruning Constraints

The *historical* pruning constraints are formally defined in Section 3.4. We repeat for clarity that these parameters are obtained by iterating through different intervals in our historical data, $[t_0, t_0 + T] \subseteq \mathcal{T}$, and updated at each interval. Thus, the definitions given in Section



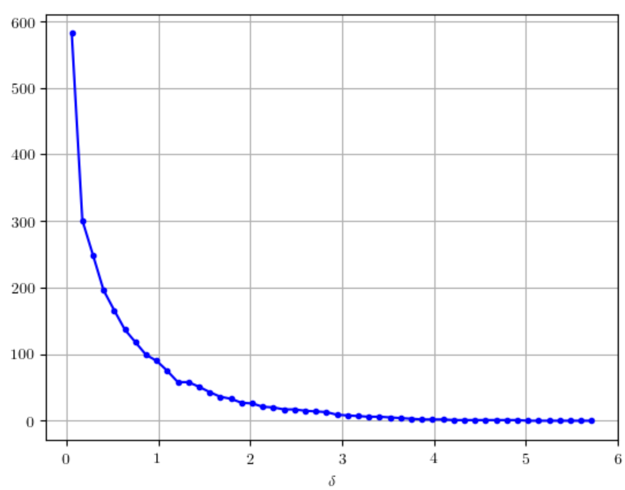


Figure 5.9: $|\mathcal{N}_E(x, \mathcal{T})|$ vs. δ for δ-uncorrelated models. We use two different time intervals of historical data in the plots provided. The first uses $\mathcal{T} = \mathcal{T}^1$ while the second plot uses $\mathcal{T} = \mathcal{T}^2$, where $\mathcal{T}^1$ and $\mathcal{T}^2$ are given in Chapter 6. Here we have $\delta \in [\delta_{min}, \delta_{max}]$ for each time interval used.

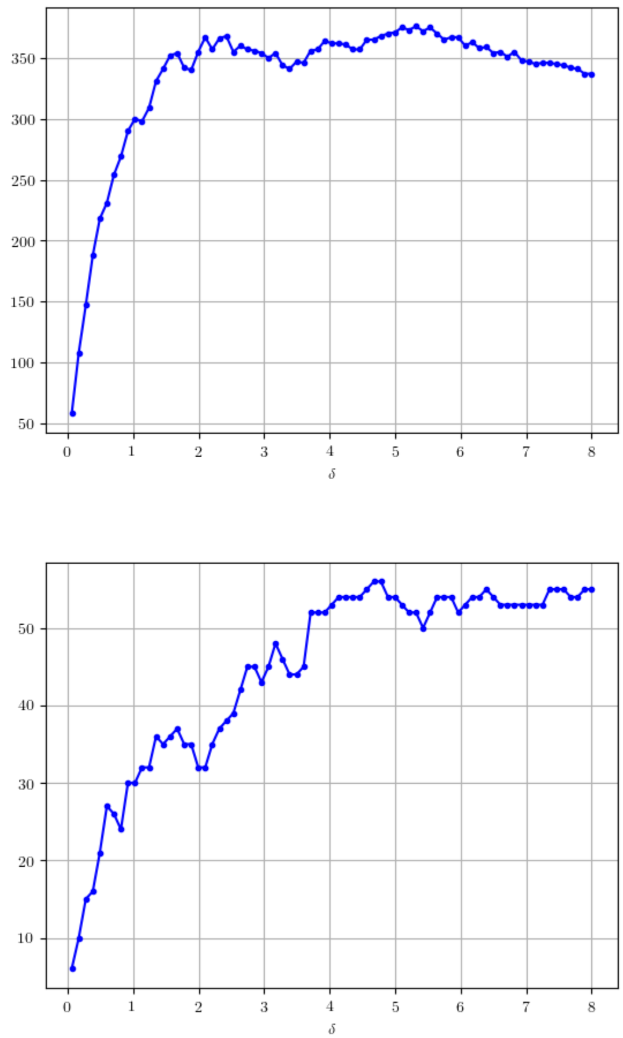


Figure 5.10: $|\mathcal{N}_E(x, \mathcal{T})|$ vs. δ for δ-correlated models. We use two different time intervals of historical data in the plots provided. The first uses $\mathcal{T} = \mathcal{T}^1$ while the second plot uses $\mathcal{T} = \mathcal{T}^2$, where $\mathcal{T}^1$ and $\mathcal{T}^2$ are given in Chapter 6. Here we have $\delta \in [0.01, 8.0]$ for each time interval used. It is shown in a figure in the appendix that the size $|\mathcal{N}_E(x, \mathcal{T})|$ stabilizes after a certain δ. Note that we have $\delta_{up} = \delta_{down} = \delta$ where δ represents the value along the x-axis in the figure.

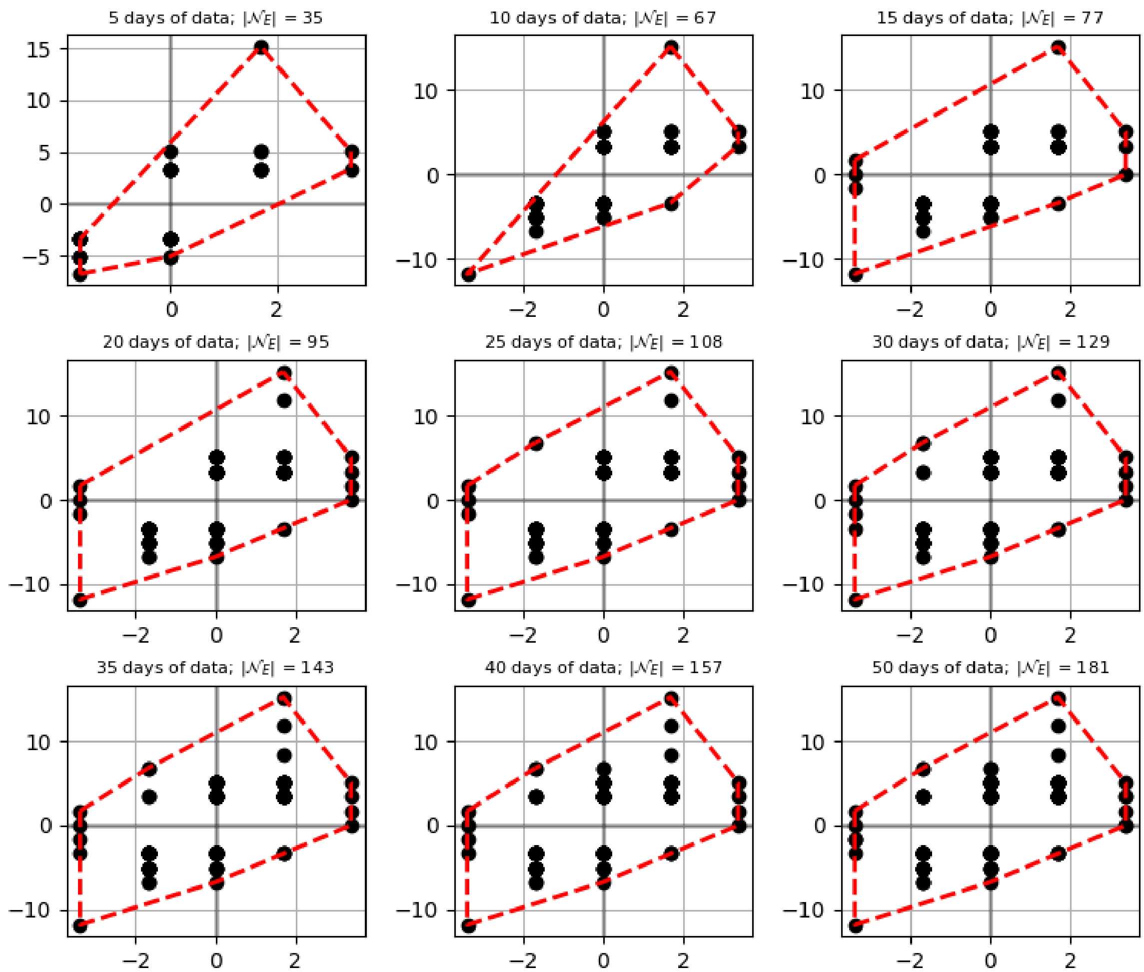


Figure 5.11: For the following set $M = \{(m^1\hat{\delta}^1, m^2\hat{\delta}^2) : \exists(m^1, m^2, q, P) \in \mathcal{N}_E(x, \mathcal{T})\}$ we show how $co(M)$ grows with the increase of data. Here we use historical time interval $\mathcal{T} = \mathcal{T}^1$ and data described in [1.] in the enumeration in Section 6.1. Each subplot has values for $m^1\hat{\delta}^1$ along the x-axis and $m^2\hat{\delta}^2$ along the y-axis. Notice the convex hull's stability after incorporating about 15 days of data.

3.4 gives the investor *worst-case* estimates of these pruning constraints.

Unless otherwise indicated, we will set our model's pruning constraints to be the following:

$$\begin{aligned}
\underline{X}(i) &= X_*(x, \mathcal{T}, i), & \overline{X}(i) &= X^*(x, \mathcal{T}, i) \\
\underline{N}(T_i) &= N_*(x, \mathcal{T}, \rho), & \overline{N}(T_i) &= N^*(x, \mathcal{T}, \rho) \\
\underline{N}(W_i) &= N_*(x, \mathcal{T}, w), & \overline{N}(W_i) &= N^*(x, \mathcal{T}, w) \\
\underline{T}(i) &= T_*(x, \mathcal{T}, i), & \overline{T}(i) &= T^*(x, \mathcal{T}, i) \\
\underline{T}(W_i) &= T_*(x, \mathcal{T}, w), & \overline{T}(W_i) &= T^*(x, \mathcal{T}, w) \\
\underline{W}(i) &= W_*(x, \mathcal{T}, i), & \overline{W}(i) &= W^*(x, \mathcal{T}, i) \\
\underline{W}(T_i) &= W_*(x, \mathcal{T}, t_i), & \overline{W}(T_i) &= W^*(x, \mathcal{T}, t_i)
\end{aligned}$$

The behaviour of some of these pruning constraints is seen in Figures 5.1, 5.5, 5.2, 5.6, 5.3, 5.7, 5.4 and 5.8. Figures 5.1, 5.5, 5.2, 5.6, 5.3, 5.7 are particularly helpful to the investor when calibrating δ. The investor is able to determine which values of δ allow the most change in value, how close trajectories get to time T, and the number of portfolio rebalances performed historically at termination time T.

Note that in Figures 5.3 and 5.7, we observe that $T^*(x, \mathcal{T}, N(\mathbf{X})) = T_*(x, \mathcal{T}, N(\mathbf{X}))$ for many δ. This behaviour indicates that - although there are multiple rebalances occuring - there is only *one* trajectory that occurs historically which exhibits $N(\mathbf{X})$ δ-movements. As is seen in Figures 5.4 and 5.8, it is *not* the case that $T^*(x, \mathcal{T}, i) = T_*(x, \mathcal{T}, i)$ for all $i > 0$.

5.6 Worst-Case Estimate for Stopping Number

For the models defined in 4.2, the investor must prescribe some maximum rebalancing number for the trajectories to take. This is mostly important to the construction of any Type 0 model. To closely compare each of the model types we set $N(\mathbf{X}) = i^*(x, \mathcal{T})$, where i^* is defined in Definition 15. Then, i^* is a worst-case estimate of the maximum rebalancing number.

As will be illustrated in the next chapter, supplying $N(\mathbf{X})$ *does not* mean that each trajectory in our models will definitely result in having $N(\mathbf{X})$ rebalances. The pruning constraints used in our models will still have the ability to restrict the models further, and result in trajectories paths where an investor would liquidate their portfolio when reaching the $N_H(\mathbf{X})$'th rebalancing. Then we would have $H_i(\mathbf{X}) = 0$ for all $N_H(\mathbf{X}) \leq i \leq N(\mathbf{X})$.

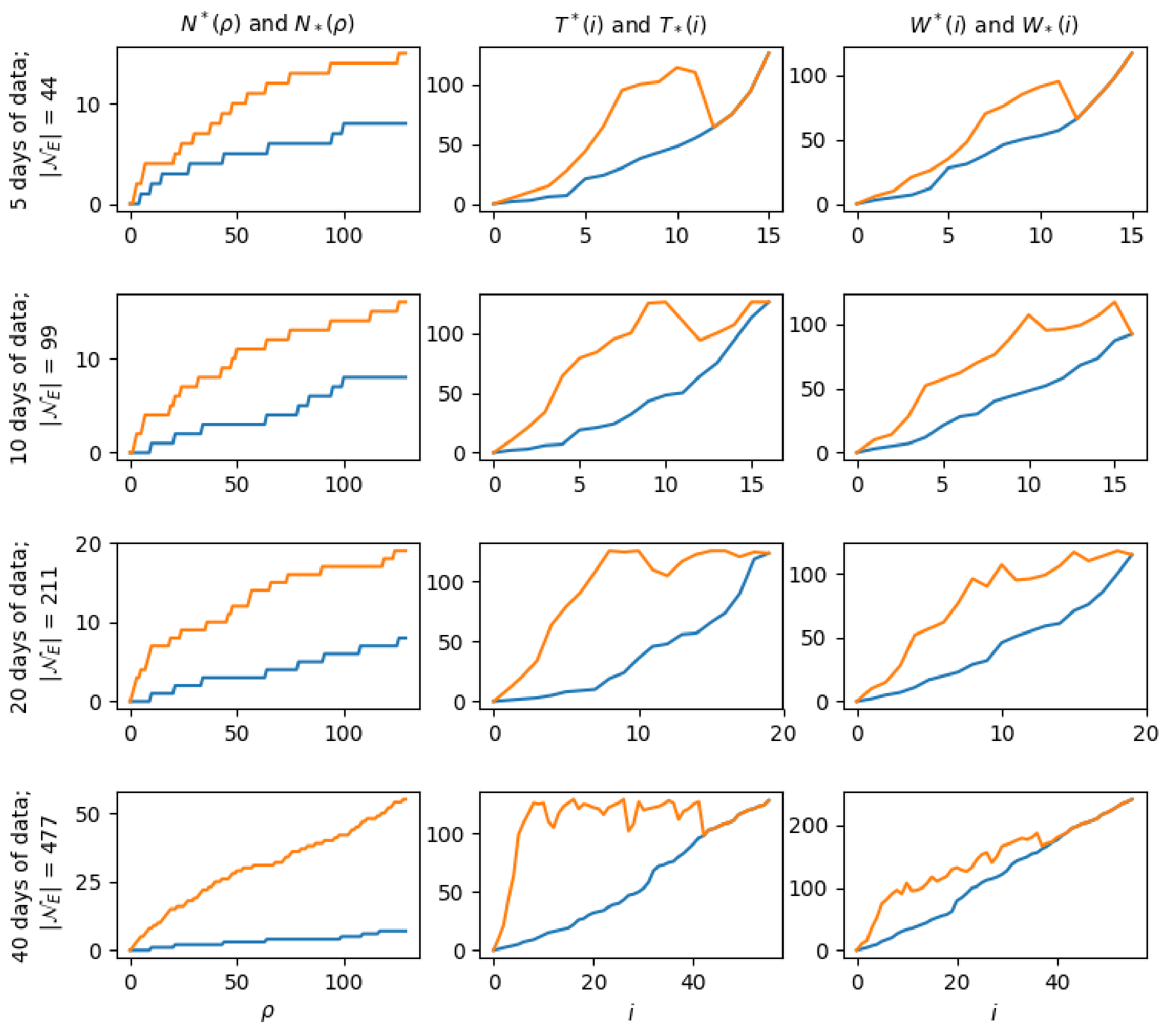


Figure 5.12: Using historical time interval $\mathcal{T} = \mathcal{T}^1$ and data described in [1.] in the enumeration in Section 6.1 (currency as numeraire), we show how the pruning constraints widen as more data is used in our historical estimation process. Here we select $\delta_0 = 0.5$, $\delta = 1.0$, $\hat{\delta}^1 = \hat{\delta}^2 = \delta/2 = 0.5$ and $\hat{\nu}_0 = \delta = 1.0$.

Chapter 6

Output for Trajectory Models

This chapter completes the specification of the trajectory models introduced throughout this thesis by discussing the data used for estimation and showing some output for the worst-case estimated parameters. The estimation methodology is discussed in Section 3.3 and constructions of the worst-case pruning constraints are introduced in Section 3.4. Model construction is described in Section 4.2.

The word *calibration* refers to the process which an investor fixes values to model parameters. The parameters δ_0, δ as well the discretiaztion parameters $\hat{\delta}^1,\ \hat{\delta}^2,\ \hat{\nu}_0$ are investor dependent and require the investor to fix values (calibrate) to these in order to begin the estimation process. Model parameters, such as $\mathcal{N}_E$ and worst-case pruning constraints, must be estimated are dependent on the calibrated parameters and the historical data $x(t),\ t \in \mathcal{T}$. While the only role the parameters $\hat{\delta}^1,\ \hat{\delta}^2,\ \hat{\nu}_0$ mentioned in this thesis is the rounding (discretization) of observable quantities introduced in Section 3.3.4, the parameters also help regulate the total computation time required to perform the estimation and model construction.

The output in this section is obtained by setting $T = M_T\Delta = 130\Delta$ with $\Delta = 3\ min$. We provide output for arbitrarily calibrated parameters which provide stable pruning constraints and construct a trajectory model in a reasonable computation time. Since we have access to 3 minute intraday data and are forcasting our prices at the end of a trading day, a *reasonable* computation time would be any duration less than a few hours. This way an investor could perform their computations following market close or early before a day of trading.

6.1 Data Employed

Our data consists of 3 minute intraday data over the course of approximately 6 months; from 2018-05-09 9:30 AM to 2018-10-15 3:57:00 PM. A trading day begins at 9:30 AM, and ends at 4:30 PM, for a total of 6.5 trading hours (390 minutes) in one day since we are going by 3 minute ticks there are 130 Δ intervals in one day. In this thesis we set the following:

[1.] $x^0(t)$ to be a currency (US \$ to be exact), $x^1(t)$ to be the price of Facebook and $x^2(t)$ to be the price of Netflix;

[2.] $x^0(t)$ is the price of Twitter, $x^1(t)$ to be the price of Facebook and $x^2(t)$ to be the price of Netflix;

[3.] $x^0(t)$ is the price of Facebook, $x^1(t)$ to be the price of Twitter and $x^2(t)$ to be the price of Netflix;

Let $|M|$ be the number of data points in $\mathcal{T}$, and M_T be the number of data points in the interval $[t_0, t_0 + T] \subseteq \mathcal{T}$. We label historical data points by $M, M+1, ..., -1$, i.e. $M < 0$. Then we use the notation $\mathcal{T} = \{l\Delta \;:\; l = M, M+1, ..., -1\}$. As mentioned, overnight effects are neglected by setting $T = 1$ *day* and selecting t_0 such that it is the opening time of each trading day. Given that in our data there are $M_T = 130$ three minute samples in a day, we can then set $t_0 = k(M_T + 1)\Delta,\; k \in \{0, 1, ..., \lfloor \frac{|M|}{M_T} \rfloor\}$.

Although we have access to the entire timeframe $\mathcal{T}$, we note that the amount of data used in the calibration and estimation process affects the outcome of the model. Once δ and δ_0 are calibrated, an accumulation of data may widen the worst-case pruning constraints. This could indicate that some pruning constraints utilized in this thesis are not useful to use when constructing market models, since one would expect worst-case parameters to stabilize after an accumulation of data. We also mention that the shape of the convex hull generated from the set of possible asset changes should stabilize when accumulating data. That is, for the set $M = \{(\hat{\delta}^1 m^1, \hat{\delta}^2 m^2) : \exists (m^1, m^2, q, P) \in \mathcal{N}_E(x, \mathcal{T})\}$, $co(M)$ should not grow indefinitely with an accumulation of data. Figure 5.12 shows how some pruning constraints widen with an accumulation of data while Figure 5.11 (and numerous figures in the appendix) provide a visualization of the growth of $co(M)$.

When data is accumulated we note that the computation time to construct our market models increases significantly. For the purposes of this thesis, we limit the amount of data used in our estimation process. This limits characteristics of our models such as the number of children nodes produced at a given parent, overall size of our discrete grid, and the amount of pruning done at each iteration i, and consequently, the computation time for constructing a model will decrease.

Nonetheless we consider a wide range of investors who we anticipate estimate the future according to a worst case possible future (reflecting the investor's habits). In this way our models are constructed to be objective since our worst-case setting allows us to treat investor's habits, and their operational actions, in an unbiased manner. Although we note that this objectiveness is only satisfied when historical data is accumulated to exhibit stable model parameters, our reasoning to limiting the amount of historical data in this thesis should not confound our reader with the fact that the models constructed in this paper are considered to be objective.

Then, in order to distinguish between the amount of data used in a specific model estimation process we will denote the following time intervals:

- $\mathcal{T}^1$ is the entire dataset; from 2018-05-09 9:30 AM to 2018-10-15 3:57:00 PM,
- $\mathcal{T}^2$ is the most present 10 days of data; from 2018-10-02 9:30 AM to 2018-10-15 3:57:00 PM,

and then we will have $\mathcal{T}^2 \subseteq \mathcal{T}^1$. We could go about leaving $\mathcal{T}$ to indicate our entire historical dataset, however we use the notation $\mathcal{T}^1$ to indicate the difference in our future figures by indicating that we set $\mathcal{T} = \mathcal{T}^1$ or $\mathcal{T} = \mathcal{T}^2$.

Let us now show output for trajectory market models where we arbitrarily choose various parameters to show how they affect the construction of our models. Below we show output for δ-uncorrelated and δ-correlated models. Note that in each of the following sections we split the output into two subsections to distinguish between the two objectives observed in this thesis. The first subsection will deal with the construction of our trajectory sets, with the following subsection dealing with the superhedging methodology and trajectory sampling to determine the profit and losses associated with constructing a portfolio with initial value v.

6.2 δ-Uncorrelated Models - Currency as Numeraire

Let us now concern ourselves with the assets described in [1.] in the enumeration of the previous section and the interval $\mathcal{T}^2$ described in Section 6.1. We exhibit some properties of our trajectory market models constructed using the operational setting for δ-uncorrelated models.

6.2.1 Objective 1 - Constructing the Trajectory Set

We begin by selecting $\delta = 1.25$ and $\delta_0 = 0.5$. We set $\hat{\delta}^1 = \hat{\delta}^2 = \delta/2$ and $\hat{\nu}_0 = \delta$. First we perform the estimation process. The set of possible vector changes, given as $M =$

$\{(\hat{\delta}^1 m^1, \hat{\delta}^2 m^2) : \exists (m^1, m^2, q, P) \in \mathcal{N}_E(x, \mathcal{T}^2)\}$ is displayed in Figure 6.1. Notice that for this set, M, we have that $0 \in ri\big(co(M)\big)$. This was part of our criteria to satisfy when selecting calibrated values, which is discussed in Section 5.2. We also note that $|\mathcal{N}_E(x, \mathcal{T}^2)| = 58$.

Observed worst-case pruning constraints for the calibrated parameters are shown in Figure 6.2. Although not explicitly stated, it is shown that $i^*(x, \mathcal{T}^2) = 10$. In order to limit the number of rebalances in our trajectory set we set $N(\mathbf{X}) = 4$, which helps us regulate the computation time. This enables us to obtain data in reasonable time for the purpose of this thesis, since if we were to select $N(\mathbf{X}) = i^*(x, \mathcal{T}^2)$ then producing the trajectory set $\mathcal{X}$ would take days of computation time.

For each model type we began our construction with $X_0^1 = 183.82$, $X_0^2 = 331.62$, $t_0 = 0$, $T_0 = 0$, $W_0 = 0$ (note that $X_0^1 = 183.82$ and $X_0^2 = 331.62$ are the most present values we have in $\mathcal{T}^2$). The k^1 and k^2 values at each rebalancing i for nodes in our trajectory set are given in Figures 6.3, 6.4, and 6.5, which give the reader an idea of how the our asset prices change in $\mathcal{X}$ and the pruning occuring at each rebalancing i.

Table 6.1 shows the values obtained for each model type, and notice that Equation (4.7) given in Section 4.2.6 is satisfied. We also provide the average proportion of nodes pruned at each parent node for each rebalance, shown in Figure 6.8; this shows the amount of work the pruning constraints for each model type and our selected parameters.

6.2.2 Objective 2 - Superhedging Methodology

Finally, we produce histograms of the profit and losses of 1000 sampled trajectories for initial investments of size $v = X_0^2$ and $v = X_0^2 + 1.0$, which are seen in Figures 6.6 and 6.7, respectively. Notice that the proportion of trajectories which the investor profits nearly doubles when the investor adds \$1.0 of currency to their initial investment.

For clarity we indicate that the output from this section is given throughout Figures 6.1-6.8 and in Table 6.1.

6.3 δ-Correlated Models - Currency as Numeraire

We concern ourselves with the assets described in [1.] in the enumeration and the interval $\mathcal{T}^2$ described in Section 6.1 (once again $\mathcal{T}^2$ is chosen to limit computation time). In this section we exhibit some properties of our trajectory market models constructed using the operational setting for δ-correlated models.

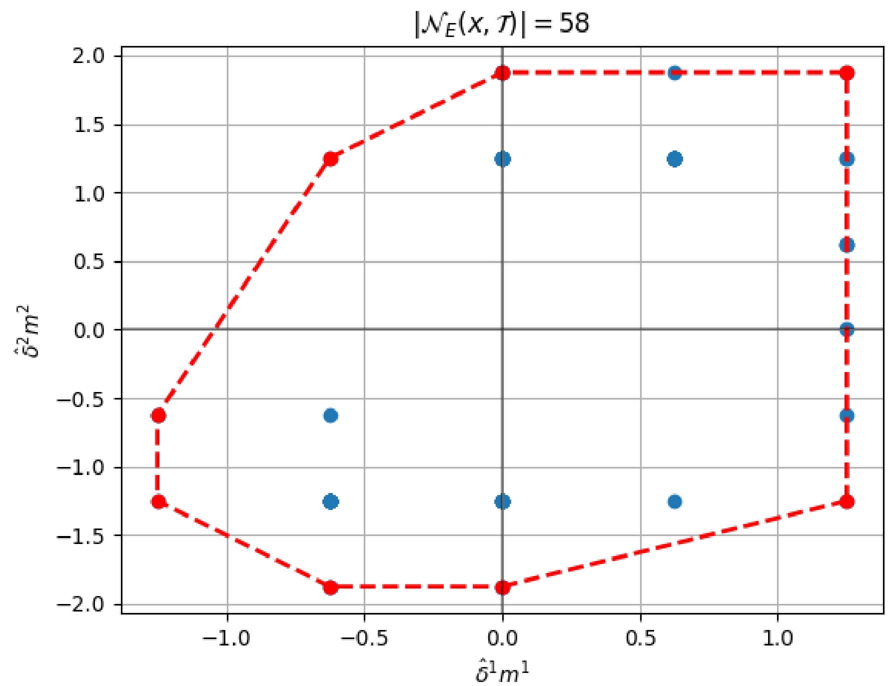


Figure 6.1: Using the assets described in [1.] (currency as numeraire, refer to Section 6.1) and the historical data subset $\mathcal{T}^2$ (refer to Section 6.1), the set $M = \{(\hat{\delta}^1 m^1, \hat{\delta}^2 m^2) : \exists (m^1, m^2, q, P) \in \mathcal{N}_E(x, \mathcal{T}^2)\}$ for the δ-uncorrelated model created with $\delta = 1.25$, $\delta_0 = 0.5$, $\hat{\delta}^1 = \hat{\delta}^2 = 0.625$, and $\hat{\nu}_0 = 1.25$ is shown. The convex hull of M is given as the dotted red line and red points as its vertices. Notice that $0 \in ri\big(co(M)\big)$.

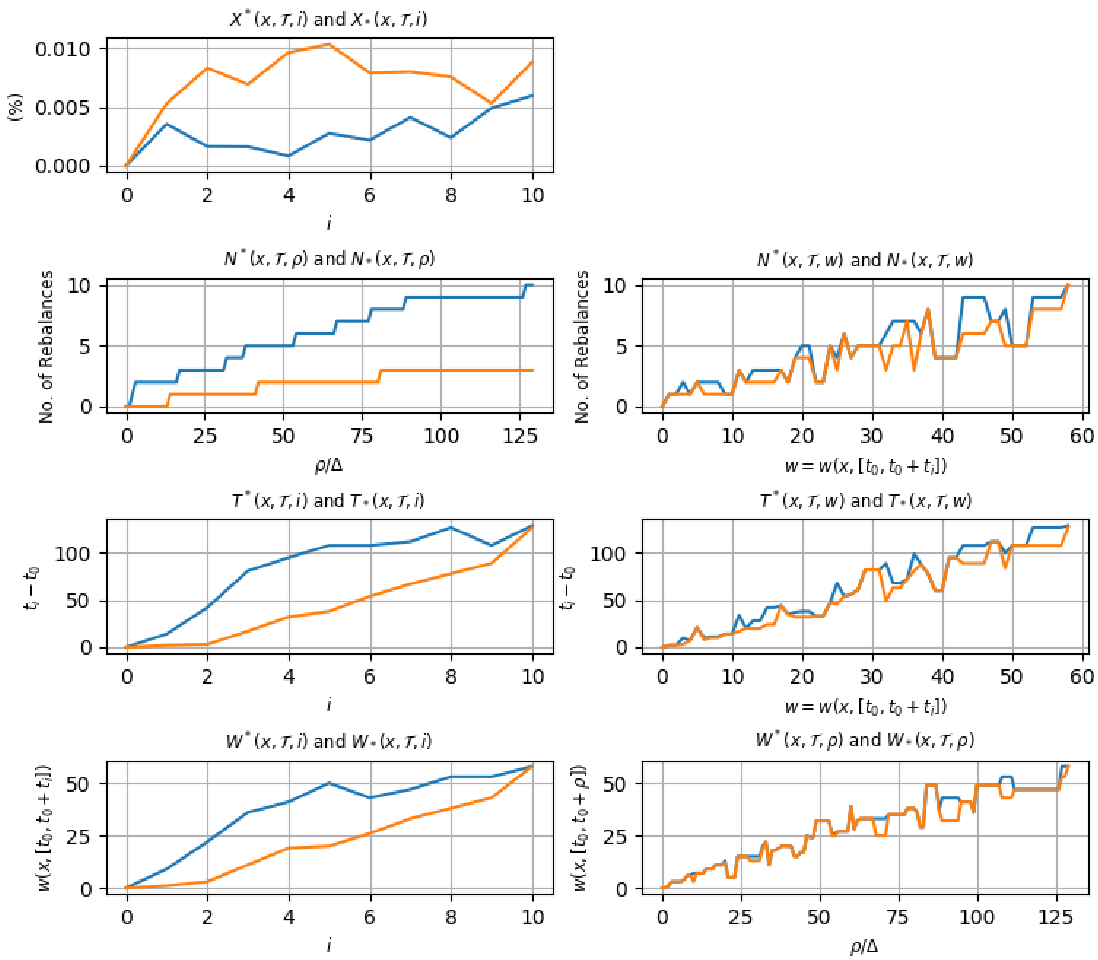


Figure 6.2: Using the assets described in [1.] (currency as numeraire, refer to Section 6.1) and the historical data subset $\mathcal{T}^2$ (refer to Section 6.1), the estimated pruning constraints with $\delta = 1.25$, $\delta_0 = 0.5$, $\hat{\delta}^1 = \delta/2$, $\hat{\delta}^2 = \delta/2$, and $\hat{\nu}_0 = \delta$ for a δ-uncorrelated model are shown. Here each axes is labeled on the subplots and the pruning constraint is given as each subplot's title. We have that $0 \leq i \leq i^*(x, \mathcal{T}^2) = 10$, $\rho \in [0, T]$, and w. w represents the values for the accumulated vector variation. Notice that in the plot for $N^*(x, \mathcal{T}, w)$ and $N_*(x, \mathcal{T}, w)$ there are instances when $N^*(x, \mathcal{T}, w) = N_*(x, \mathcal{T}, w)$. This indicates that we only observed one historical δ-movement resulting in a portfolio rebalancing at variation w. The same argument also applies for the plots $T^*(x, \mathcal{T}, w)$ and $T_*(x, \mathcal{T}, w)$, as well as $W^*(x, \mathcal{T}, \rho)$ and $W_*(x, \mathcal{T}, \rho)$.

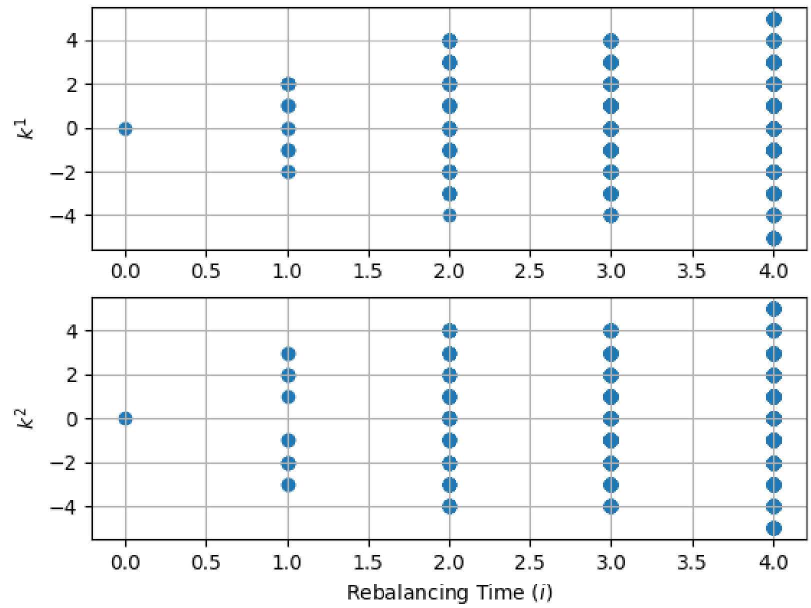


Figure 6.3: Using the assets described in [1.] (currency as numeraire, refer to Section 6.1) and the historical data subset $\mathcal{T}^2$ (refer to Section 6.1), the values of k^1 and k^2 for all nodes for the Type 0 δ-uncorrelated model created with $\delta = 1.25$, $\delta_0 = 0.5$, and $\hat{\delta}^1 = \hat{\delta}^2 = 0.625$, $\hat{\nu}_0 = 1.25$ are shown.

	$\underline{V}_0(\mathbf{X}_0, X^2, \mathcal{M})$	$\overline{V}_0(\mathbf{X}_0, X^2, \mathcal{M})$
	$X_0^2 = 331.62$	
$\mathcal{M} = \mathcal{M}^0$	327.37	335.37
$\mathcal{M} = \mathcal{M}^I$	328.495	334.745
$\mathcal{M} = \mathcal{M}^{II}$	329.016	333.912

Table 6.1: Although an abuse of notation, we use the notation $\mathcal{M} = \mathcal{M}^0$ to indicate that the corresponding row shows the values $\underline{V}_0(\mathbf{X}_0, X^2, \mathcal{M})$ and $\overline{V}_0(\mathbf{X}_0, X^2, \mathcal{M})$ for the Type 0 market model. The same applies for the next two rows but for Type I and II models, respectively. Using the assets described in [1.] (currency as numeraire, refer to Section 6.1) and the historical data subset $\mathcal{T}^2$ (refer to Section 6.1), the time 0 values for our asset X^2 for the δ-uncorrelated model created with $\delta = 1.25$, $\delta_0 = 0.5$, and $\hat{\delta}^1 = \hat{\delta}^2 = 0.625$, $\hat{\nu}_0 = 1.25$ and historical data subset $\mathcal{T}^2$ are shown. Notice that we always have $\underline{V}_0(\mathbf{X}_0, X^2, \mathcal{M}) \leq \overline{V}_0(\mathbf{X}_0, X^2, \mathcal{M})$ and that Equation (4.7) in Section 4.2.6 is satisfied.

6.3.1 Objective 1 - Constructing our Trajectory Set

We begin by selecting $\delta_{up} = \delta_{down} = 3.0$, $\delta_0 = 0.5$, $\hat{\delta}^1 = \hat{\delta}^2 = 0.5$ and $\hat{\nu}_0 = 1.0$. First we perform the estimation process. The set of possible vector changes, M (given in the previous

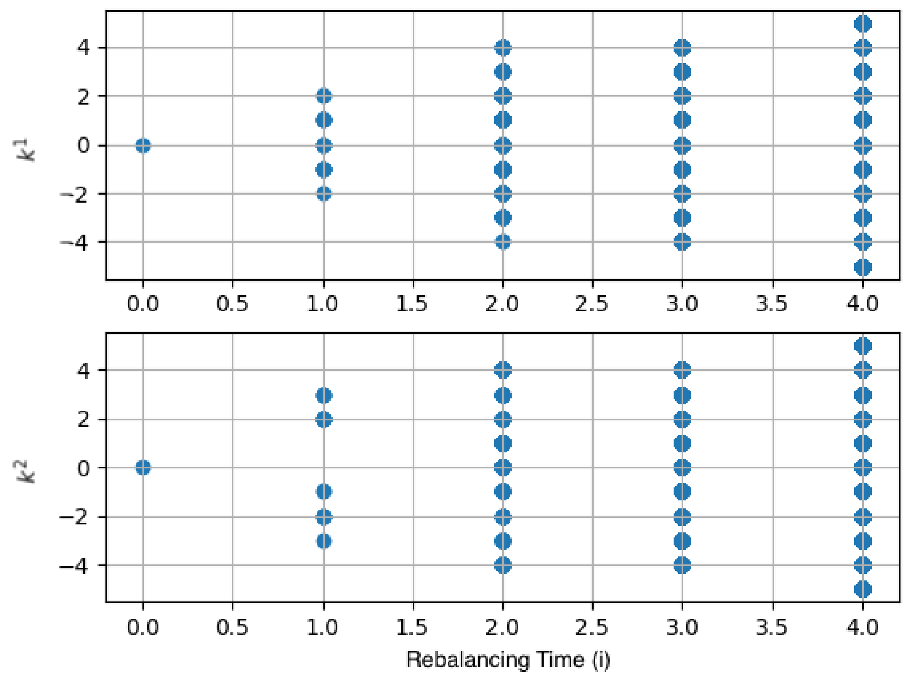


Figure 6.4: Using the assets described in [1.] (currency as numeraire, refer to Section 6.1) and the historical data subset $\mathcal{T}^2$ (refer to Section 6.1), the values of k^1 and k^2 for all nodes for the Type I δ-uncorrelated model created with $\delta = 1.25$, $\delta_0 = 0.5$, and $\hat{\delta}^1 = \hat{\delta}^2 = 0.625$, $\hat{\nu}_0 = 1.25$ are shown.

section), is displayed in Figure 6.9. Notice that for this set, M, we have that $0 \in ri\big(co(M)\big)$. Also notice that there are not any points in quadrants 2 and 4 on the 2D plot since we only allow our assets to move in the same direction. We also note that $|\mathcal{N}_E(x, \mathcal{T}^2)| = 41$.

Observed worst-case pruning constraints for the calibrated parameters are shown in Figure 6.10. Although not explicitly stated, it is shown that $i^*(x, \mathcal{T}^2) = 12$. Similar to the previous section, we limit the number of rebalances in our trajectory set by setting $N(\mathbf{X}) = 4$ since it helps us regulate the computation time. This enables us to obtain data in reasonable time for the purpose of this thesis, since if we were to select $N(\mathbf{X}) = i^*(x, \mathcal{T}^2)$ then producing the trajectory set $\mathcal{X}$ would take days of computation time.

For each model type we began our construction with $X_0^1 = 183.82$, $X_0^2 = 331.62$, $t_0 = 0$, $T_0 = 0$, $W_0 = 0$ (note that $X_0^1 = 183.82$ and $X_0^2 = 331.62$ are the most present values we have in $\mathcal{T}^2$). The k^1 and k^2 values at each rebalancing i for nodes in our trajectory set are given in Figures 6.11, 6.12, and 6.13, which give the reader an idea of how the our asset prices change in $\mathcal{X}$ and the pruning occuring at each rebalancing i. Notice that the k^1 and k^2 values in Figures 6.11 and 6.12 are exactly the same. Given that they utilize the same nodes this indicates that the pruning constraints used in the Type I model described here

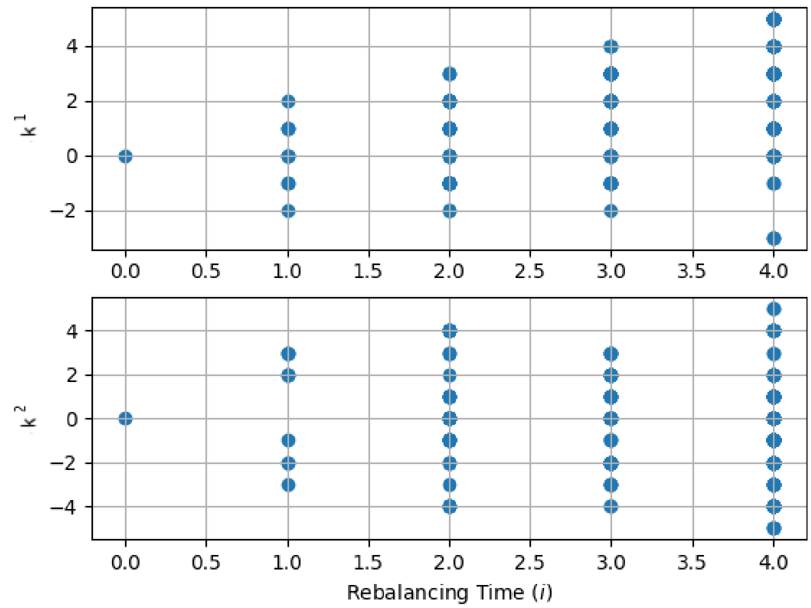


Figure 6.5: Using the assets described in [1.] (currency as numeraire, refer to Section 6.1) and the historical data subset $\mathcal{T}^2$ (refer to Section 6.1), the values of k^1 and k^2 for all nodes when creating a Type II δ-uncorrelated model created with $\delta = 1.25$, $\delta_0 = 0.5$, and $\hat{\delta}^1 = \hat{\delta}^2 = 0.625$, $\hat{\nu}_0 = 1.25$ are shown.

does not do any more pruning than the pruning constraints in the Type 0 models. This is then reflected in the time 0 values $\underline{V}_0$ and $\overline{V}_0$ for the models.

Table 6.2 shows the values obtained for each model type, and notice that Equation (4.7) given in Section 4.2.6 is satisfied. We also provide the average proportion of nodes pruned at each parent node for each rebalance, shown in Figure 6.17; this shows the amount of work the pruning constraints for each model type and our selected parameters.

6.3.2 Objective 2 - Superhedging Methodology

Finally, we produce histograms of the profit and losses of 1000 sampled trajectories for initial investments of size $v = X_0^2$ and $v = X_0^2 + 1.0$, which are seen in Figures 6.14 and 6.15, respectively. It is interesting to see that although the initial values for Type 0 and Type I models are equal, we observe a higher proportion of trajectories in Type I models which profit. We also include a similar figure showing that trajetories in $\mathcal{X}$ are superhedged when beginning with initial value $v = \overline{V}_0(\mathbf{X}_0, X^2, \mathcal{M})$, which is shown in Figure 6.16. Notice that the proportion of trajectories which the investor profits nearly doubles when the investor

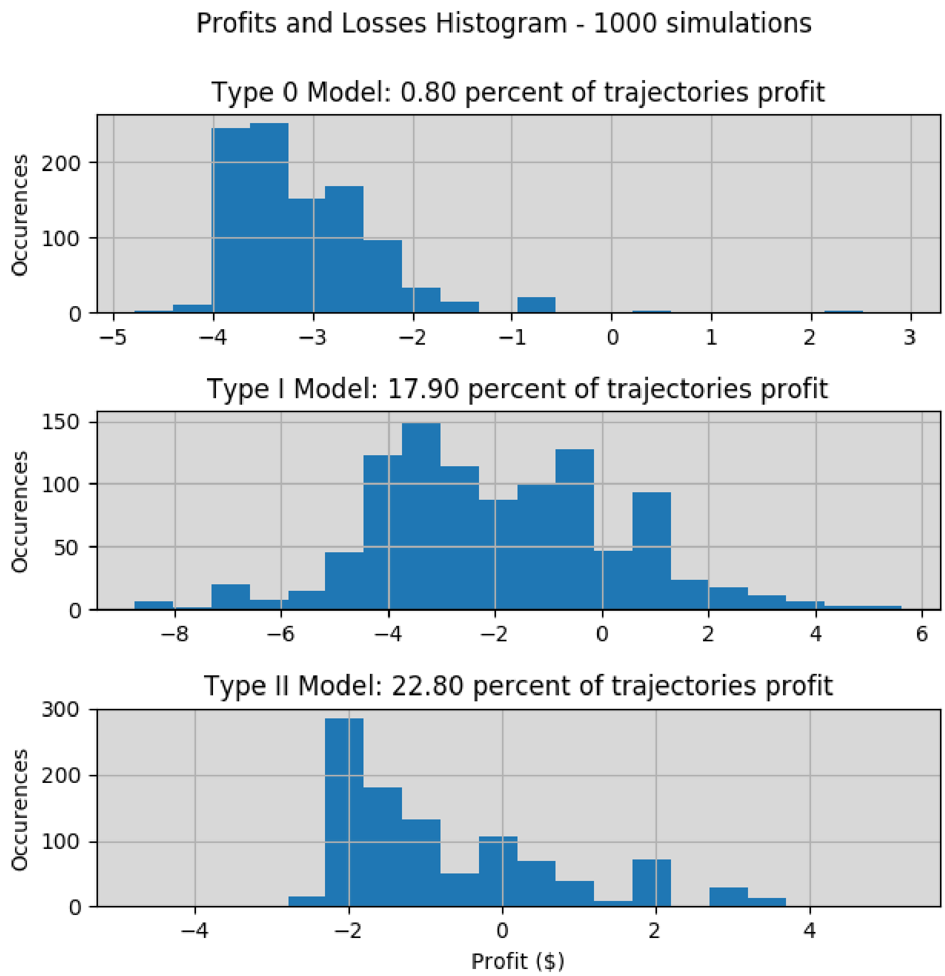


Figure 6.6: Using the assets described in [1.] (refer to Section 6.1) and the historical data subset $\mathcal{T}^2$ (refer to Section 6.1), the profits and losses for 1000 simulations in each δ-uncorrelated model type created with $\delta = 1.25$, $\delta_0 = 0.5$, and $\hat{\delta}^1 = \hat{\delta}^2 = 0.625$, and $\hat{\nu}_0 = 1.25$ are shown. Here we begin with initial investment $v = X_0^2$.

adds \$1.0 of currency to their initial investment.

For clarity we indicate that the output from this section is given throughout Figures 6.9-6.17 and in Table 6.2.

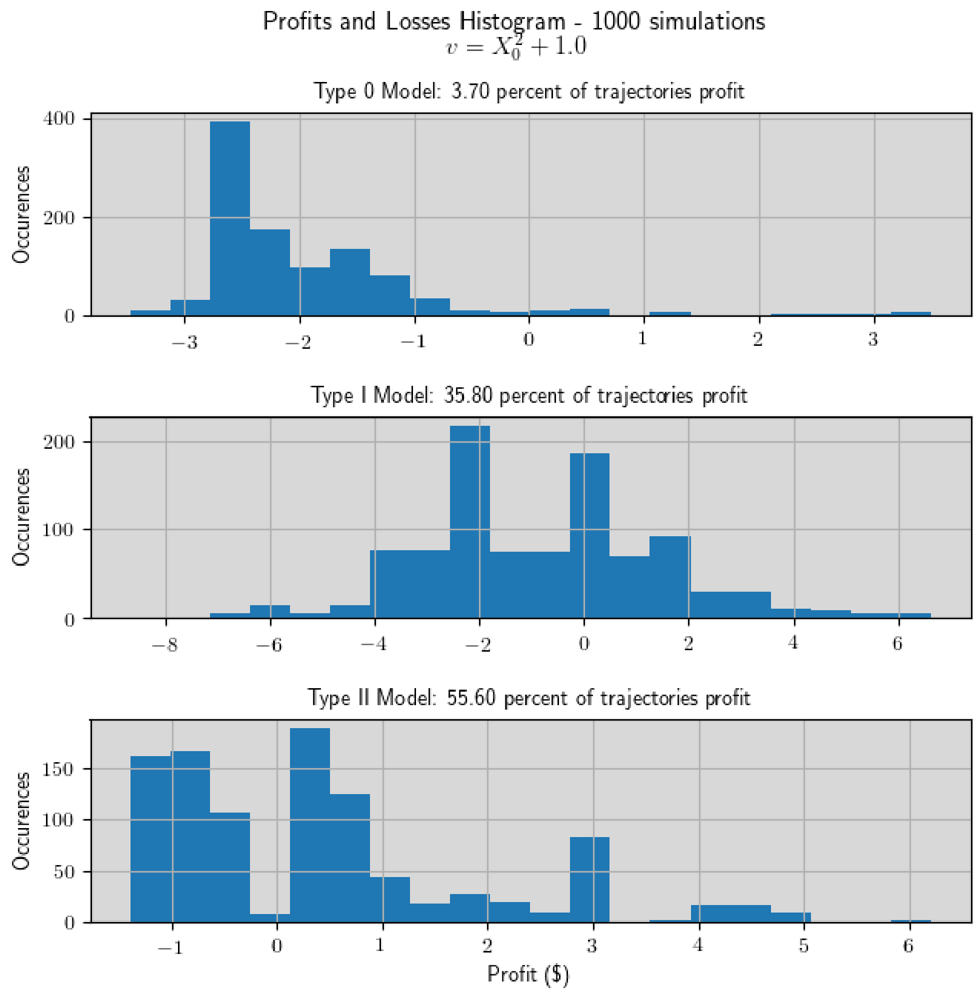


Figure 6.7: Using the assets described in [1.] (currency as numeraire) and the historical data subset $\mathcal{T}^2$ (refer to Section 6.1), the profits and losses for 1000 simulations in each δ-uncorrelated model type created with $\delta = 1.25$, $\delta_0 = 0.5$, and $\hat{\delta}^1 = \hat{\delta}^2 = 0.625$, and $\hat{\nu}_0 = 1.25$ are shown. Here we begin with initial investment $v = X_0^2 + 1.0$. Notice that with \$1.0 increase in portfolio value nearly doubles the amount of profitting trajectories.

6.4 δ-Uncorrelated Models: Twitter as Numeraire

We now go to show analogous results to the previous two sections, but with a change of numeraire. Concerning ourselves now with the assets described in [2.] in the enumeration and the interval $\mathcal{T}^2$ described in Section 6.1, we go about the same process as we described in the previous two sections. In Chapter 5, we provided parameter estimation results for

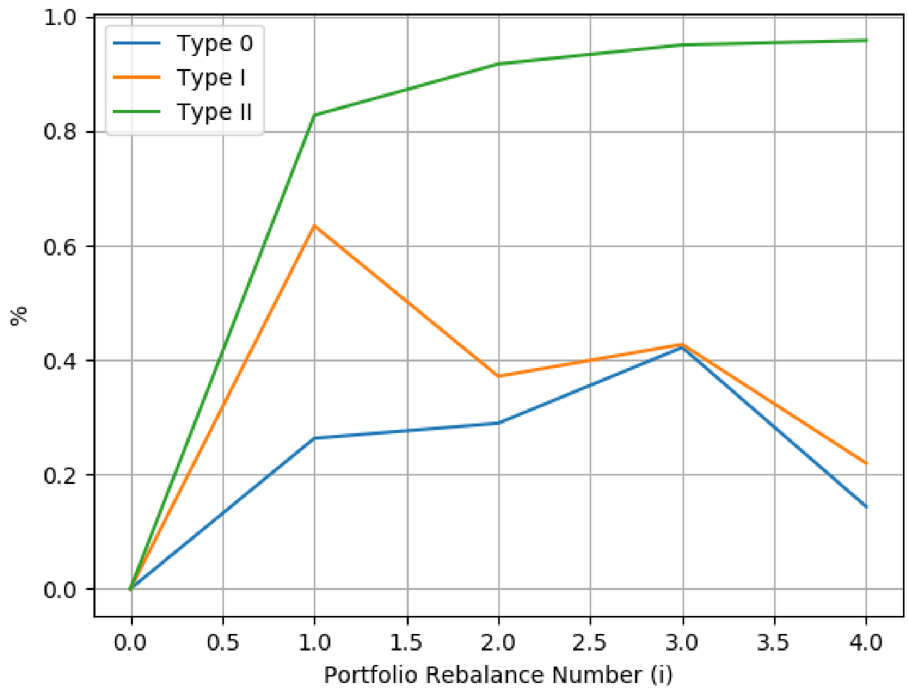


Figure 6.8: Using the assets described in [1.] (currency as numeraire, refer to Section 6.1) and the historical data subset $\mathcal{T}^2$ (refer to Section 6.1), the average percentage of children nodes pruned at each parent, averaged over each portfolio rebalancing i for each δ-uncorrelated model type created with $\delta = 1.25$, $\delta_0 = 0.5$, and $\hat{\delta}^1 = \hat{\delta}^2 = 0.625$, $\hat{\nu}_0 = 1.25$ are shown. Notice that, as a result of nesting the pruning constraints, each subsequent model does an increased amount of pruning at each portfolio rebalancing.

the models discussed in the previous two sections. We direct the reader to the Appendix to view parameter estimation results when using Twitter as a numeraire.

6.4.1 Objective 1 - Constructing the Trajectory Set

Once again, we note that the interval $\mathcal{T}^2$ is used to help lower computation times required to construct our trajectory market models. We begin by selecting $\delta = 0.05$ and $\delta_0 = 0.01$. We set $\hat{\delta}^1 = \hat{\delta}^2 = \delta/2$ and $\hat{\nu}_0 = \delta$. First we perform the estimation process. The set of possible vector changes, given as $M = \{(\hat{\delta}^1 m^1, \hat{\delta}^2 m^2) : \exists (m^1, m^2, q, P) \in \mathcal{N}_E(x, \mathcal{T}^2)\}$ is displayed in Figure 6.18. Notice that for this set, M, we have that $0 \in ri\big(co(M)\big)$. This was part of our criteria to satisfy when selecting calibrated values, which is discussed in Section 5.2. We also note that $|\mathcal{N}_E(x, \mathcal{T}^2)| = 68$.

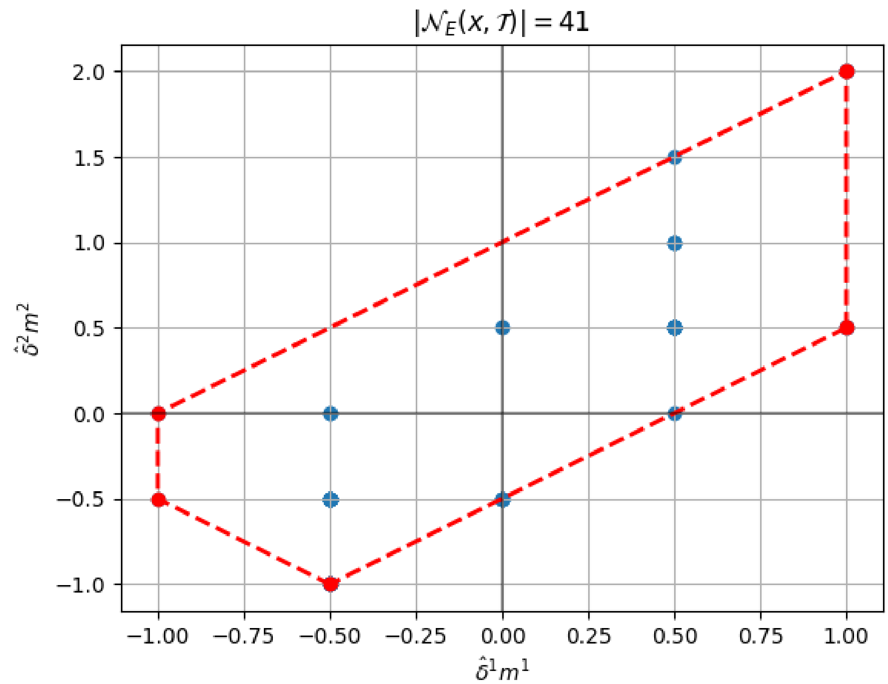


Figure 6.9: Using the assets described in [1.] (currency as numeraire, refer to Section 6.1) and the historical data subset $\mathcal{T}^2$ (refer to Section 6.1), the set $M = \{(\hat{\delta}^1 m^1, \hat{\delta}^2 m^2) : \exists (m^1, m^2, q, P) \in \mathcal{N}_E(x, \mathcal{T}^2)\}$ for the δ-correlated model created with $\delta = 3.0$, $\delta_0 = 0.5$, $\hat{\delta}^1 = \hat{\delta}^2 = 0.5$, and $\hat{\nu}_0 = 1.0$ is shown. The convex hull of M is given as the dotted red line and red points as its vertices. Notice that $0 \in ri\big(co(M)\big)$.

	$\underline{V}_0(\mathbf{X}_0, X^2, \mathcal{M})$	$\overline{V}_0(\mathbf{X}_0, X^2, \mathcal{M})$
	$X_0^2 = 331.62$	
$\mathcal{M} = \mathcal{M}^0$	329.953	333.508
$\mathcal{M} = \mathcal{M}^I$	329.953	333.508
$\mathcal{M} = \mathcal{M}^{II}$	330.151	333.453

Table 6.2: Although an abuse of notation, we use the notation $\mathcal{M} = \mathcal{M}^0$ to indicate that the corresponding row shows the values $\underline{V}_0(\mathbf{X}_0, X^2, \mathcal{M})$ and $\overline{V}_0(\mathbf{X}_0, X^2, \mathcal{M})$ for the Type 0 market model. The same applies for the next two rows but for Type I and II models, respectively. Using the assets described in [1.] (currency as numeraire, refer to Section 6.1) and the historical data subset $\mathcal{T}^2$ (refer to Section 6.1), time 0 values for our asset X^2 for the δ-correlated model created with $\delta_{up} = \delta_{down} = 3.0$, $\delta_0 = 0.5$, $\hat{\delta}^1 = \hat{\delta}^2 = 0.5$, and $\hat{\nu}_0 = 1.0$ are shown. Notice that we always have $\underline{V}_0(\mathbf{X}_0, X^2, \mathcal{M}) \leq \overline{V}_0(\mathbf{X}_0, X^2, \mathcal{M})$ and that Equation (4.7) in Section 4.2.6 is satisfied.

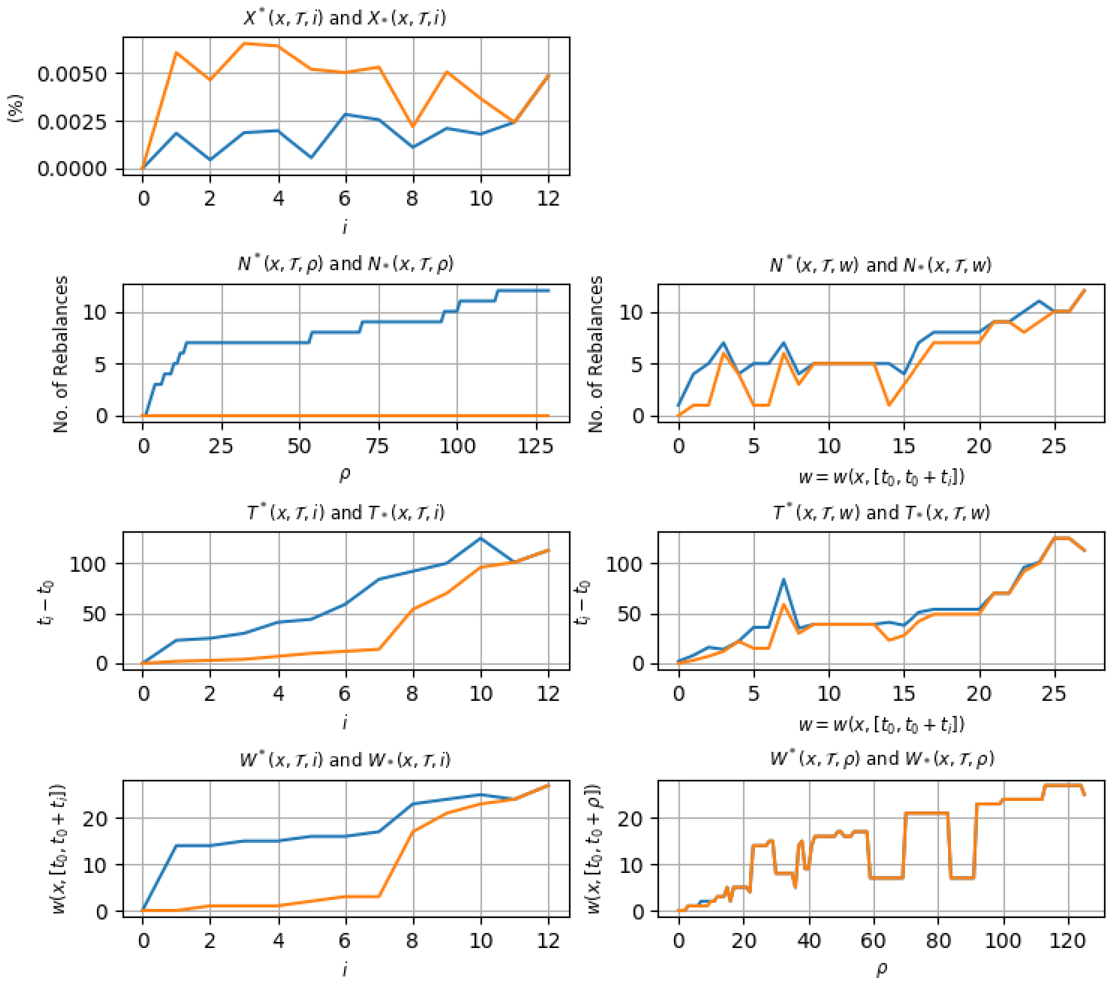


Figure 6.10: Using the assets described in [1.] (currency as numeraire, refer to Section 6.1) and the historical data subset $\mathcal{T}^2$ (refer to Section 6.1), the estimated pruning constraints for a δ-correlated model with the selection $\delta = 3.0$, $\delta_0 = 0.5$, and $\hat{\delta}^1 = \hat{\delta}^2 = 0.5$, and $\hat{\nu}_0 = 3.0$ are shwon. Although not explicitly stated we have that $i^* = 12$. Notice that in the plot for $N^*(x,\mathcal{T},w)$ and $N_*(x,\mathcal{T},w)$ there are instances when $N^*(x,\mathcal{T},w) = N_*(x,\mathcal{T},w)$. This indicates that we only observed one historical δ-movement resulting in the i'th rebalancing at variation w. The same argument also applies for the plots $T^*(x,\mathcal{T},w)$ and $T_*(x,\mathcal{T},w)$, as well as $W^*(x,\mathcal{T},\rho)$ and $W_*(x,\mathcal{T},\rho)$.

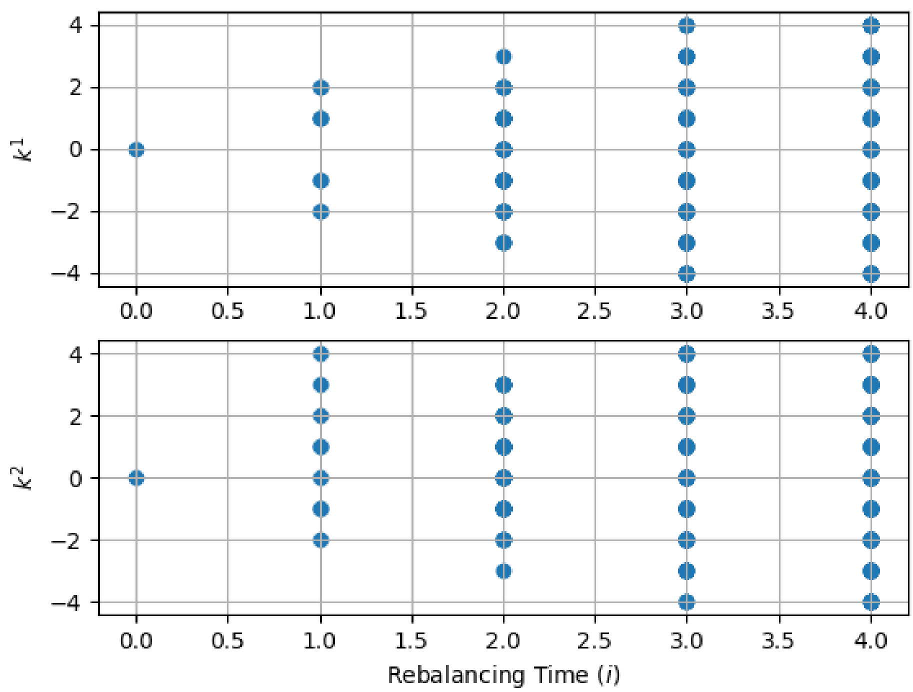


Figure 6.11: Using the assets described in [1.] (currency as numeraire, refer to Section 6.1) and the historical data subset $\mathcal{T}^2$ (refer to Section 6.1), the values of k^1 and k^2 for all nodes for the Type 0 δ-correlated model created with $\delta = 3.0$, $\delta_0 = 0.5$, $\hat{\delta}^1 = \hat{\delta}^2 = 0.5$, and $\hat{\nu}_0 = 3.0$ are shown.

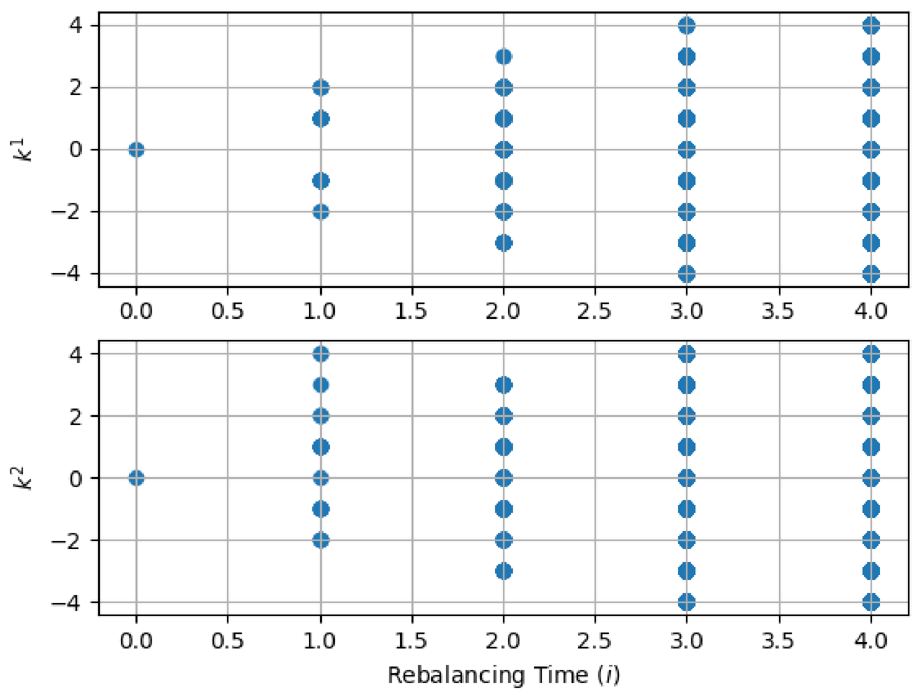


Figure 6.12: Using the assets described in [1.] (currency as numeraire, refer to Section 6.1) and the historical data subset $\mathcal{T}^2$ (refer to Section 6.1), the values of k^1 and k^2 for all nodes for the Type I δ-correlated model created with $\delta = 3.0$, $\delta_0 = 0.5$, $\hat{\delta}^1 = \hat{\delta}^2 = 0.5$, and $\hat{\nu}_0 = 3.0$ are shown.

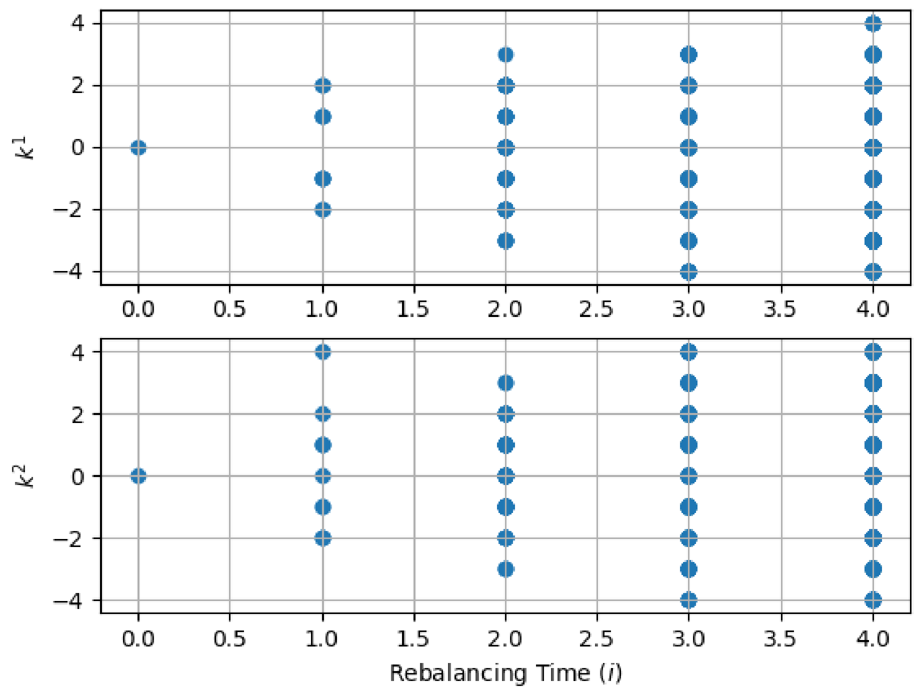


Figure 6.13: Using the assets described in [1.] (currency as numeraire, refer to Section 6.1) and the historical data subset $\mathcal{T}^2$ (refer to Section 6.1), the values of k^1 and k^2 for all nodes for the Type II δ-correlated model created with $\delta = 3.0$, $\delta_0 = 0.5$, $\hat{\delta}^1 = \hat{\delta}^2 = 0.5$, and $\hat{\nu}_0 = 3.0$ are shown.

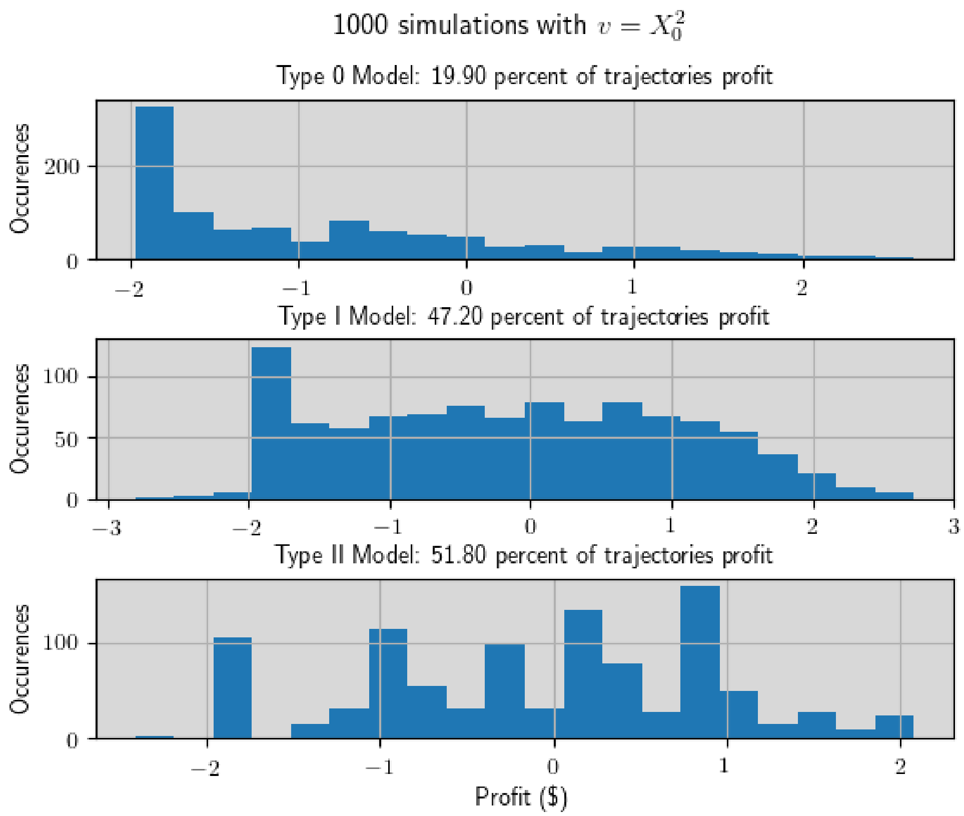


Figure 6.14: Using the assets described in [1.] (currency as numeraire) and the historical data subset $\mathcal{T}^2$ (refer to Section 6.1), the profits and losses for 1000 simulations in each δ-correlated model type are shown. Models are constructed by setting $\delta = 3.0$, $\delta_0 = 0.5$, $\hat{\delta}^1 = \hat{\delta}^2 = 0.5$, $\hat{\nu}_0 = 1.0$, $N(\mathbf{X}) = 4$. Here we begin with initial investment $v = X_0^2$.

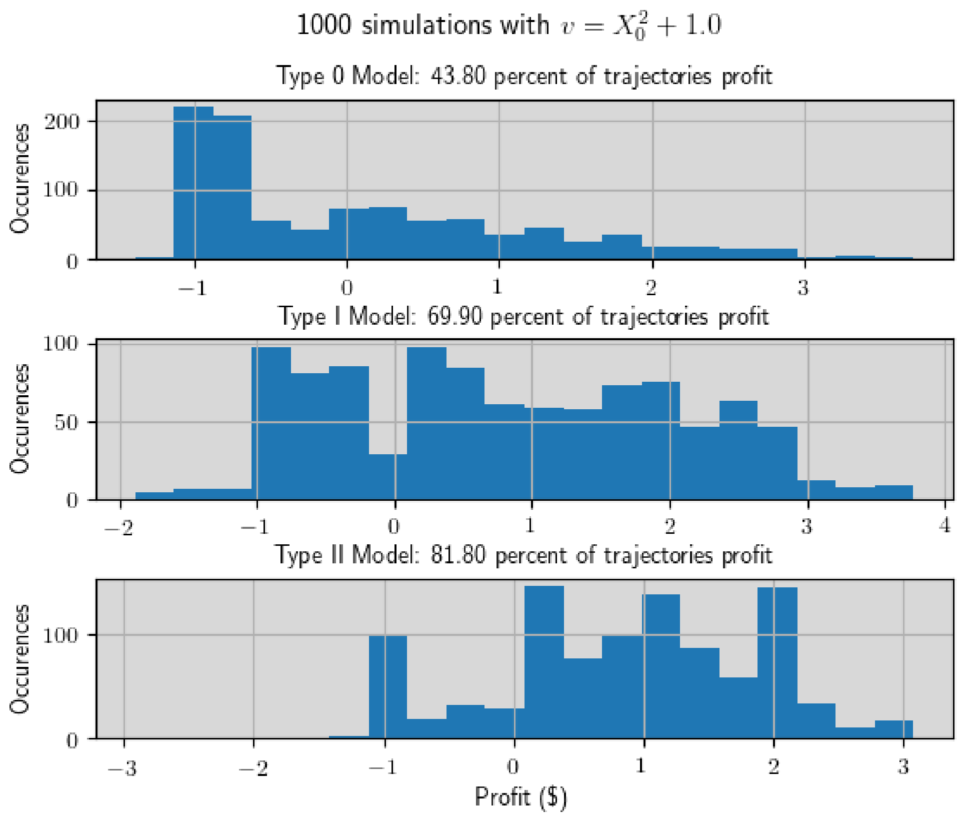


Figure 6.15: Using the assets described in [1.] (currency as numeraire, refer to Section 6.1) and the historical data subset $\mathcal{T}^2$ (refer to Section 6.1), the profits and losses for 1000 simulations in each δ-correlated model type are shown. Models are constructed by setting $\delta = 3.0$, $\delta_0 = 0.5$, $\hat{\delta}^1 = \hat{\delta}^2 = 0.5$, $\hat{\nu}_0 = 1.0$, and $N(\mathbf{X}) = 4$. Here we begin with initial investment $v = X_0^2 + 1.0$.

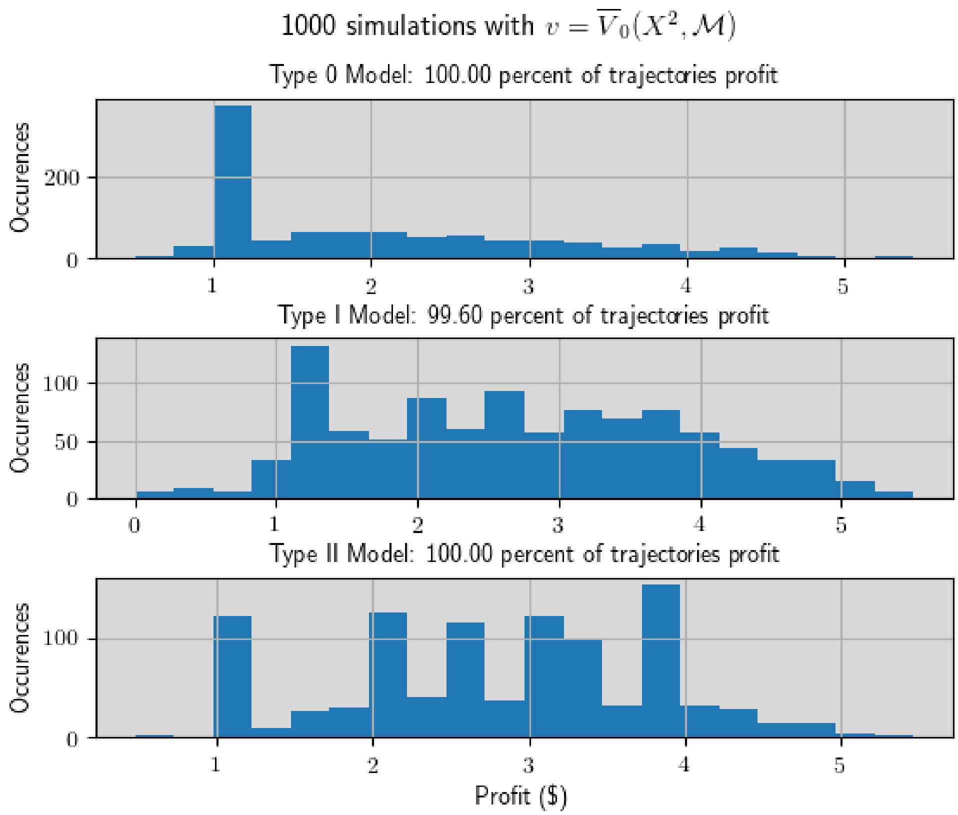


Figure 6.16: Using the assets described in [1.] (currency as numeraire, refer to Section 6.1) and the historical data subset $\mathcal{T}^2$ (refer to Section 6.1), the profits and losses for 1000 simulations in each δ-correlated model type. Models are constructed by setting $\delta = 3.0$, $\delta_0 = 0.5$, $\hat{\delta}^1 = \hat{\delta}^2 = 0.5$, $\hat{\nu}_0 = 1.0$, and $N(\mathbf{X}) = 4$. Notice that when we begin with initial investment $v = \overline{V}_0(\mathbf{X}_0, X^2, \mathcal{M})$, we superhedge X^2 using X^1, or rather, we always profit along any path $\mathbf{X} \in \mathcal{X}$. The 4 trajectories which do not profit in Type I models corresponds to having a profit of 0.

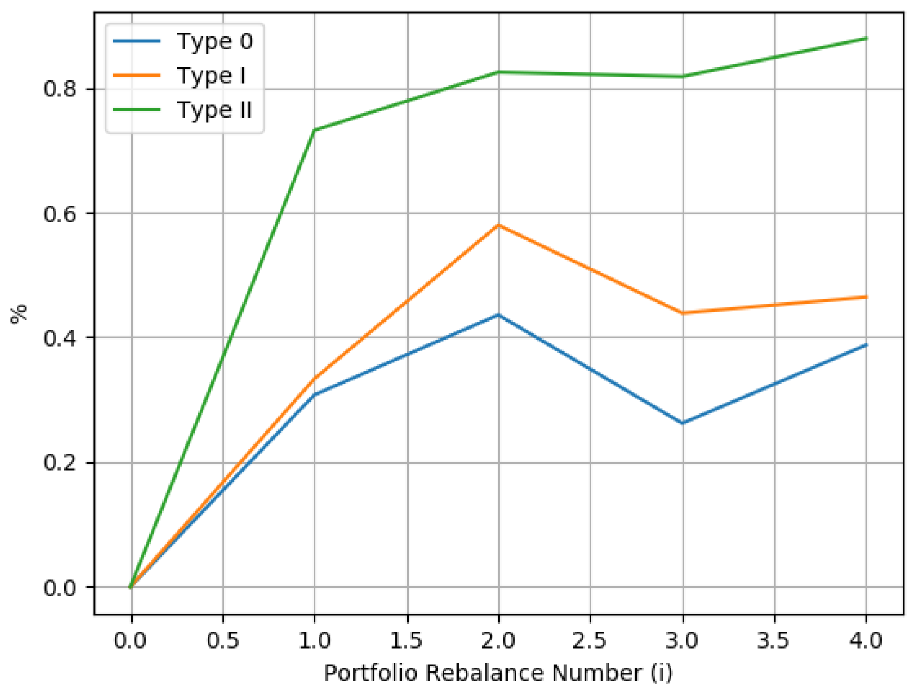


Figure 6.17: Using the assets described in [1.] (currency as numeraire, refer to Section 6.1) and the historical data subset $\mathcal{T}^2$ (refer to Section 6.1), we show the average percentage of children nodes pruned at each parent, averaged over each portfolio rebalancing i for each δ-correlated model type constructed by setting $\delta = 3.0$, $\delta_0 = 0.5$, $\hat{\delta}^1 = \hat{\delta}^2 = 0.5$, $\hat{\nu}_0 = 1.0$, and $N(\mathbf{X}) = 4$.

	$\underline{V}_0(\mathbf{X}_0, X^2, \mathcal{M})$	$\overline{V}_0(\mathbf{X}_0, X^2, \mathcal{M})$
	$X_0^2 = 331.62$	
$\mathcal{M} = \mathcal{M}^0$	326.827	336.412
$\mathcal{M} = \mathcal{M}^I$	326.827	336.412
$\mathcal{M} = \mathcal{M}^{II}$	328.562	334.632

Table 6.3: Although an abuse of notation, we use the notation $\mathcal{M} = \mathcal{M}^0$ to indicate that the corresponding row shows the values $\underline{V}_0(\mathbf{X}_0, X^2, \mathcal{M})$ and $\overline{V}_0(\mathbf{X}_0, X^2, \mathcal{M})$ for the Type 0 market model. The same applies for the next two rows but for Type I and II models, respectively. Using the assets described in [2.] (refer to Section 6.1) and the historical data subset $\mathcal{T}^2$ (refer to Section 6.1), the time 0 values for our asset X^2 for the δ-uncorrelated model created with $\delta = 0.05$, $\delta_0 = 0.01$, $\hat{\delta}^1 = \hat{\delta}^2 = 0.025$ and $\hat{\nu}_0 = 0.05$ are shown. Notice that we always have $\underline{V}_0(\mathbf{X}_0, X^2, \mathcal{M}) \leq \overline{V}_0(\mathbf{X}_0, X^2, \mathcal{M})$ and that Equation (4.7) in Section 4.2.6 is satisfied.

Observed worst-case pruning constraints for the chosen calibrated parameters are shown in Figure 6.19. It is not explicitly stated, but the figure indicates that we have $i^*(x.\mathcal{T}) = 13$. We once again set $N(\mathbf{X}) = 4$ to limit the computation time required to construct our market model.

For each model type we begin the construction with the same instantiations: $X_0^1 = 183.82$, $X_0^2 = 331.62$, $t_0 = 0$, $T_0 = 0$, $W_0 = 0$ (note that $X_0^1 = 183.82$ and $X_0^2 = 331.62$ are the most present values we have in $\mathcal{T}^2$). The k^1 and k^2 values at each rebalancing i for nodes in $\mathcal{X}$ are given in Figures 6.20, 6.21, and 6.22. Table 6.3 shows the values obtained for each model type, and notice that Equation (4.7) is satisfied. We also provide the average proportion of nodes pruned at each parent node for each rebalance, which is shown in Figure 6.26.

6.4.2 Objective 2 - Superhedging Methodology

Finally, we sampled 1000 trajectories for initial investments of size $v = X_0^2$, $v = \overline{V}_0(\mathbf{X}_0, X^2, \mathcal{M})$ and $v = \underline{V}_0(\mathbf{X}_0, X^2, \mathcal{M})$ which are seen in Figures 6.23, 6.24, and 6.25, respectively. Notice in Figure 6.23 that there is a greater proportion of profitting trajectories for Type 0 models than Type I models. This constrasts the analogous output in Sections 6.2 and 6.3 since we observed each subsequent model generating a larger proportion of profitting trajectories (when using $v = X_0^2$). Also note that the Type II models have nearly double the amount of profitting trajectories in our simulation.

For clarity we indicate that the output from this section is given throughout Figures 6.18-6.26 and in Table 6.3.

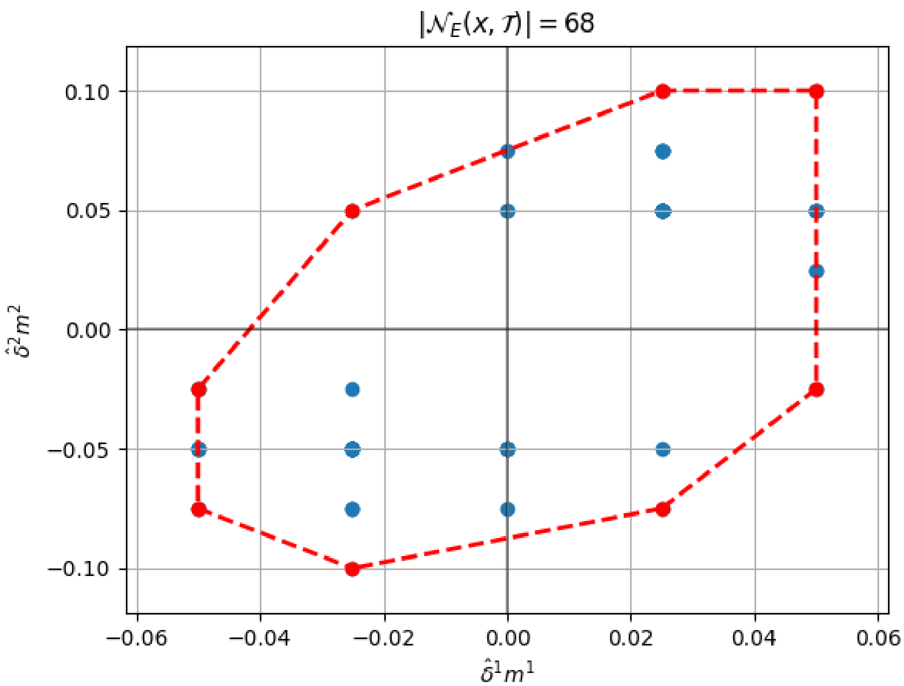


Figure 6.18: Using the assets described in [2.] in the enumeration of Section 6.1 (using Twitter as numeraire, refer to Section 6.1), we show the set $M = \{(\hat{\delta}^1 m^1, \hat{\delta}^2 m^2) : \exists (m^1, m^2, q, P) \in \mathcal{N}_E(x, \mathcal{T}^2)\}$ for the δ-uncorrelated model created with $\delta = 0.05$ and $\delta_0 = 0.01$. We set $\hat{\delta}^1 = \hat{\delta}^2 = 0.025$ and $\hat{\nu}_0 = 0.05$ and historical data subset $\mathcal{T}^2$ is shown. The convex hull of M is given as the dotted red line and red points as its vertices. Notice that $0 \in ri\big(co(M)\big)$.

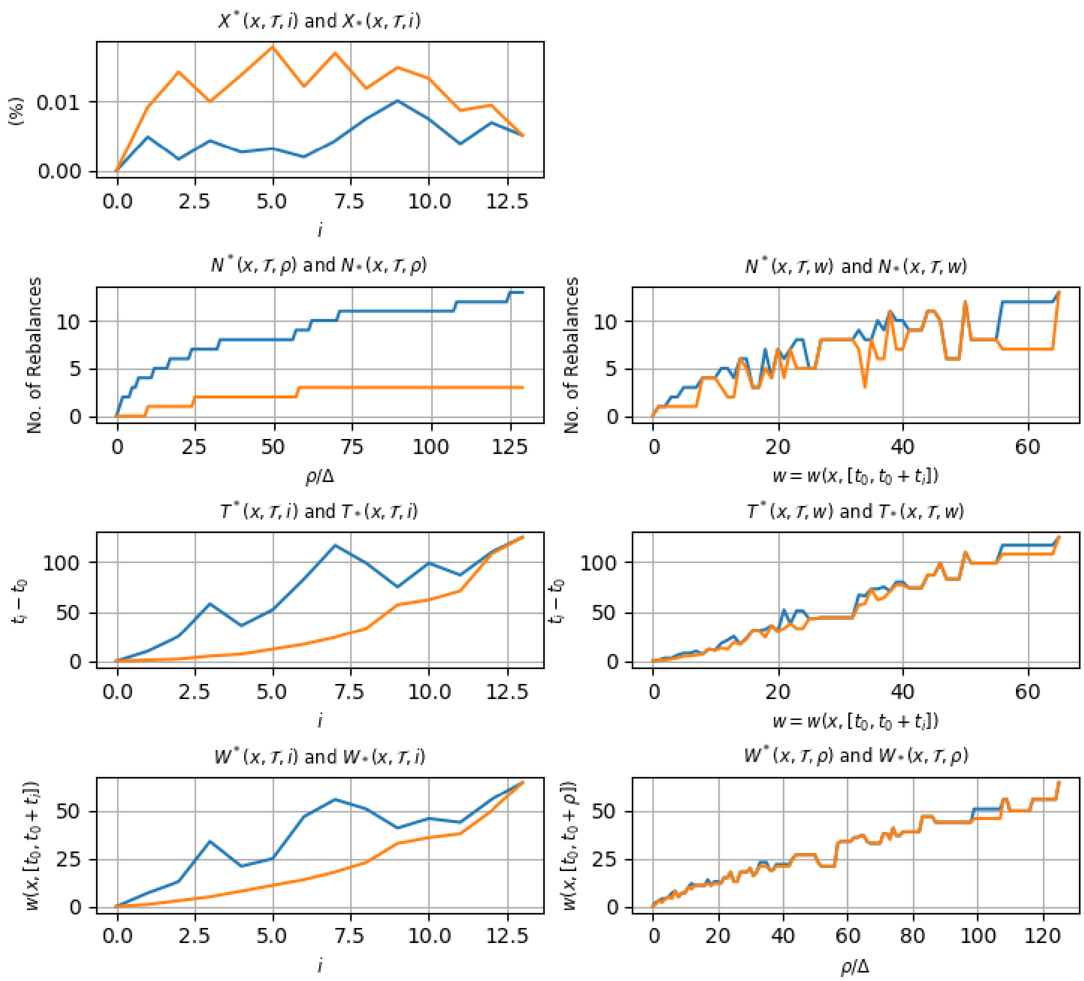


Figure 6.19: Using the assets described in [2.] (refer to Section 6.1) and the historical data subset $\mathcal{T}^2$ (refer to Section 6.1), we show the estimated pruning constraints. For this δ-uncorrelated model created with $\delta = 0.05$, $\delta_0 = 0.01$, $\hat{\delta}^1 = \hat{\delta}^2 = 0.025$ and $\hat{\nu}_0 = 0.05$, we have that $i^* = 12$. Notice that in the plot for $N^*(x, \mathcal{T}, w)$ and $N_*(x, \mathcal{T}, w)$ there are instances when $N^*(x, \mathcal{T}, w) = N_*(x, \mathcal{T}, w)$. This indicates that we only observed one historical δ-movement resulting in the i'th rebalancing at variation w. The same argument also applies for the plots $T^*(x, \mathcal{T}, w)$ and $T_*(x, \mathcal{T}, w)$, as well as $W^*(x, \mathcal{T}, \rho)$ and $W_*(x, \mathcal{T}, \rho)$.

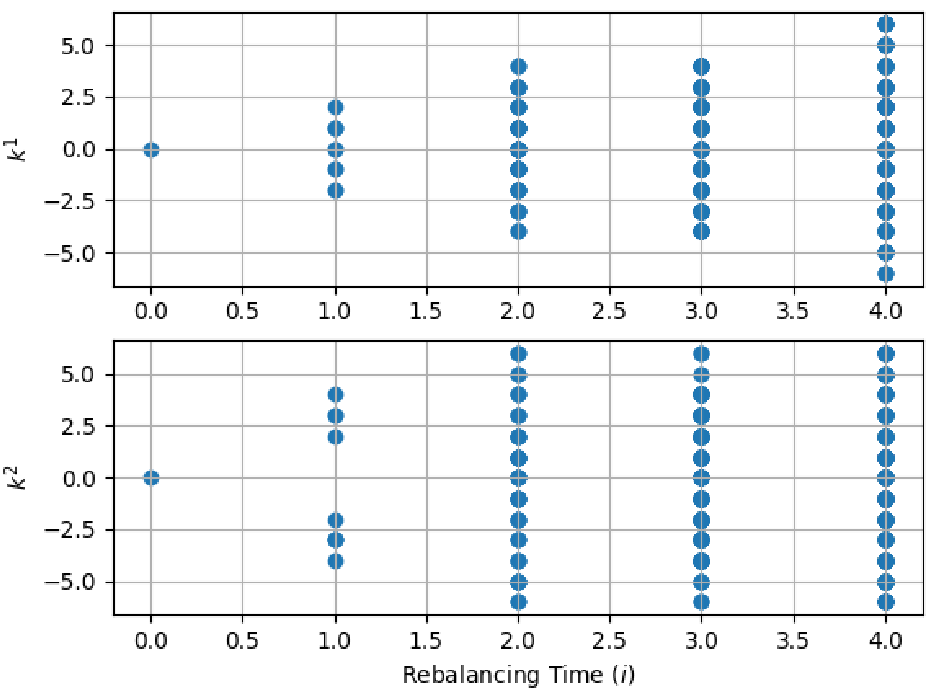


Figure 6.20: Using the assets described in [2.] (refer to Section 6.1) in the enumeration of Section 6.1 (using Twitter as numeraire) we show values of k^1 and k^2 for all nodes for the Type 0 δ-uncorrelated model created with $\delta = 0.05$, $\delta_0 = 0.01$, $\hat{\delta}^1 = \hat{\delta}^2 = 0.025$ and $\hat{\nu}_0 = 0.05$, while setting $N(\mathbf{X}) = 4$ and using historical data subset $\mathcal{T}^2$.

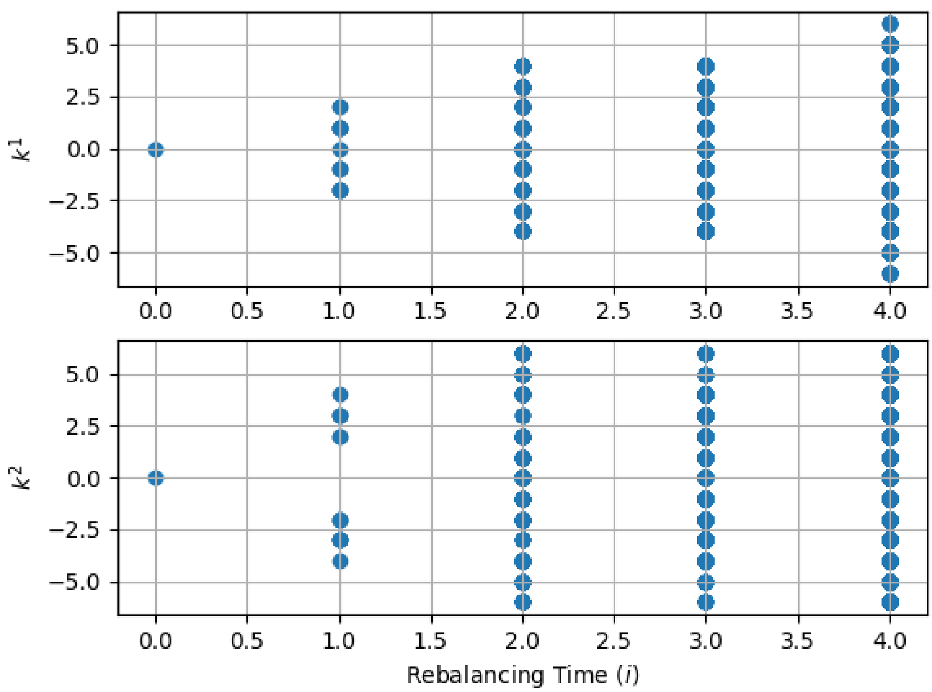


Figure 6.21: Using the assets described in [2.] (refer to Section 6.1) in the enumeration of Section 6.1 (using Twitter as numeraire) we show values of k^1 and k^2 for all nodes for the Type I δ-uncorrelated model created with $\delta = 0.05$, $\delta_0 = 0.01$, $\hat{\delta}^1 = \hat{\delta}^2 = 0.025$ and $\hat{\nu}_0 = 0.05$, while setting $N(\mathbf{X}) = 4$ and using historical data subset $\mathcal{T}^2$.

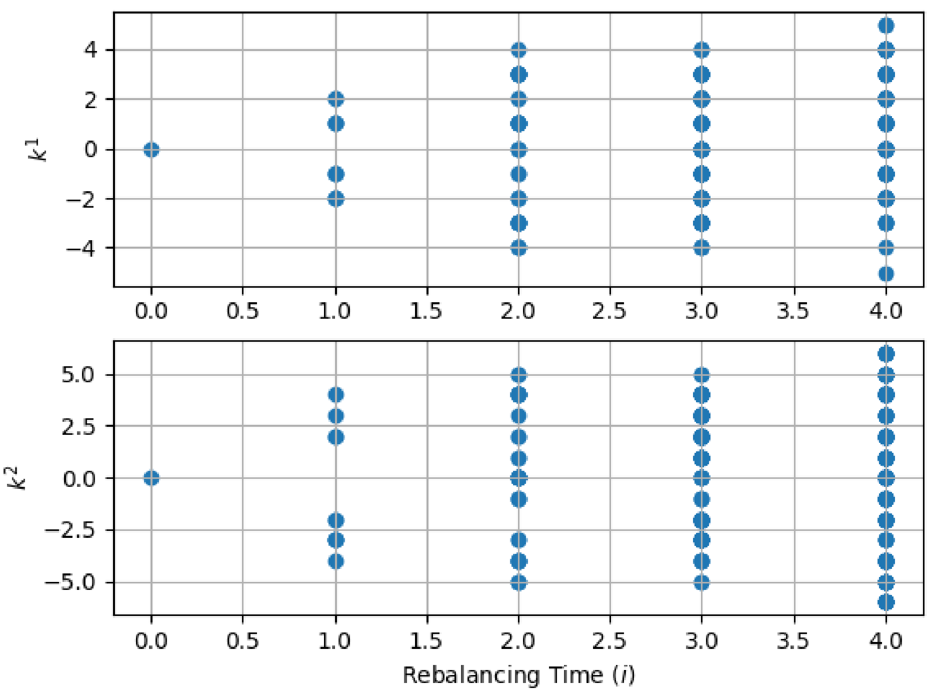


Figure 6.22: Using the assets described in [2.] (refer to Section 6.1) in the enumeration of Section 6.1 (using Twitter as numeraire) we show values of k^1 and k^2 for all nodes for the Type II δ-uncorrelated model created with $\delta = 0.05$, $\delta_0 = 0.01$, $\hat{\delta}^1 = \hat{\delta}^2 = 0.025$ and $\hat{\nu}_0 = 0.05$, while setting $N(\mathbf{X}) = 4$ and using historical data subset $\mathcal{T}^2$.

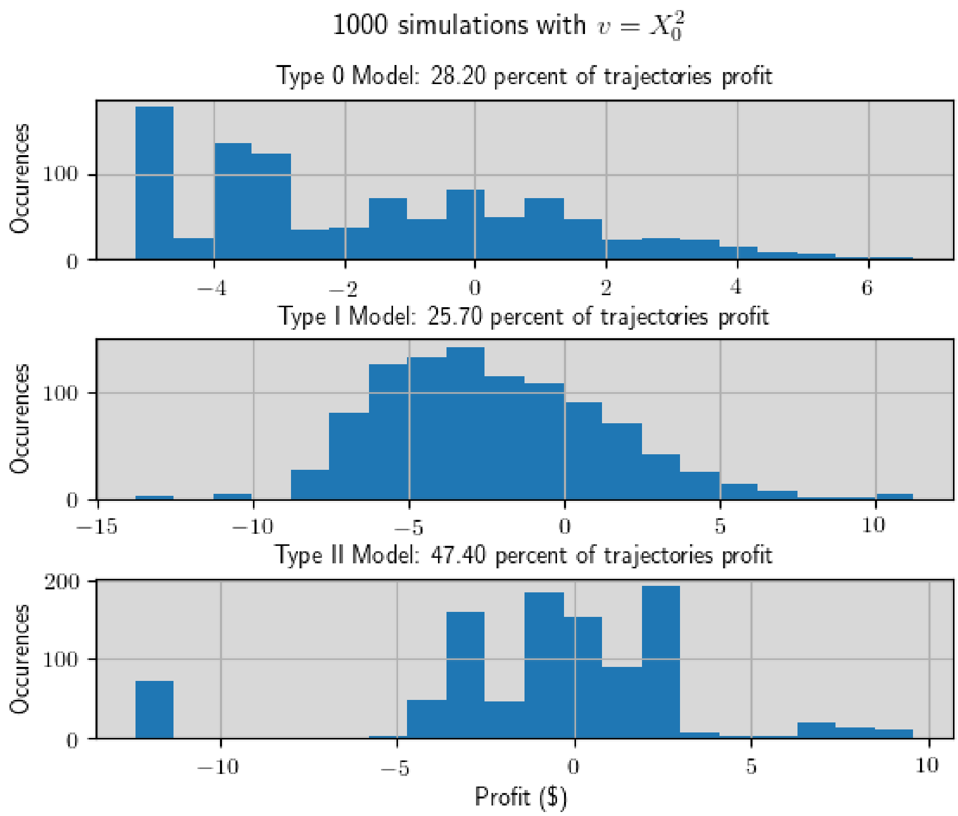


Figure 6.23: Profits and losses for 1000 simulations in each δ-uncorrelated model type when using the assets described in [2.] in the enumeration of Section 6.1 (using Twitter as numeraire). Models are constructed by setting $\delta = 0.05$, $\delta_0 = 0.01$, $\hat{\delta}^1 = \hat{\delta}^2 = 0.025$, $\hat{\nu}_0 = 0.05$, and $N(\mathbf{X}) = 4$. Here we begin with initial investment $v = X_0^2$.

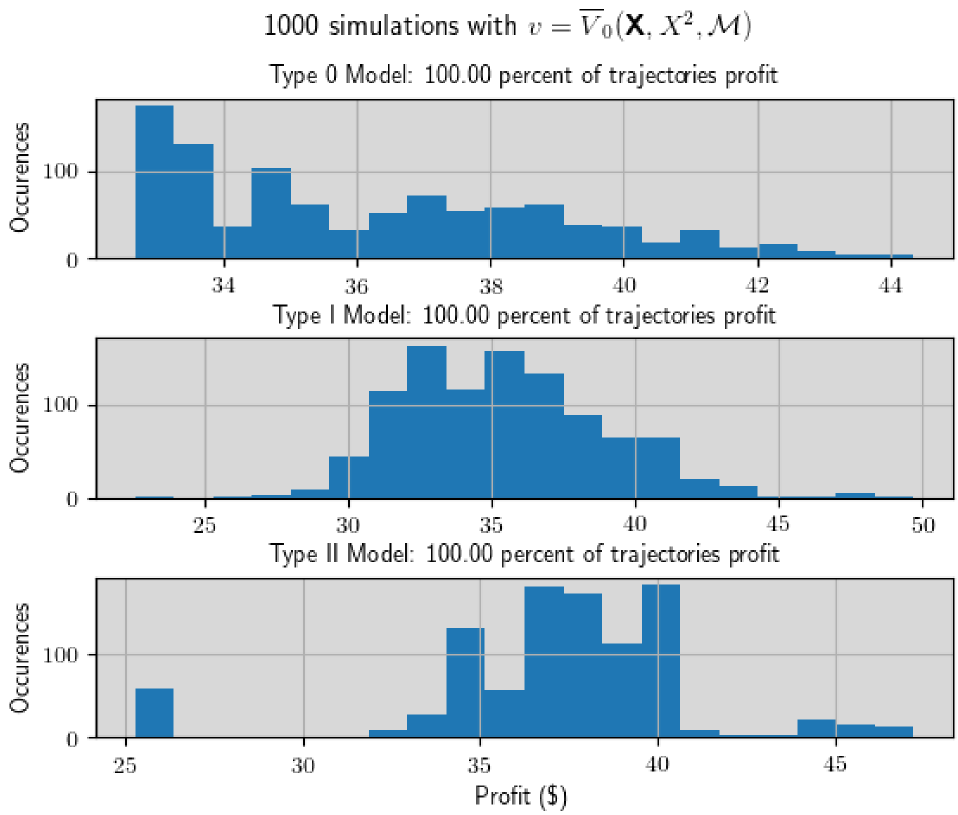


Figure 6.24: Profits and losses for 1000 simulations in each δ-uncorrelated model type when using the assets described in [2.] in the enumeration of Section 6.1 (using Twitter as numeraire). Models are constructed by setting $\delta = 0.05$, $\delta_0 = 0.01$, $\hat{\delta}^1 = \hat{\delta}^2 = 0.025$, $\hat{\nu}_0 = 0.05$, and $N(\mathbf{X}) = 4$. Here we begin with initial investment $v = \overline{V}_0(\mathbf{X}_0, X^2, \mathcal{M})$. Notice that with this initial investment, we always superhedge X^2 along the trajectories in our model when trading X^1.

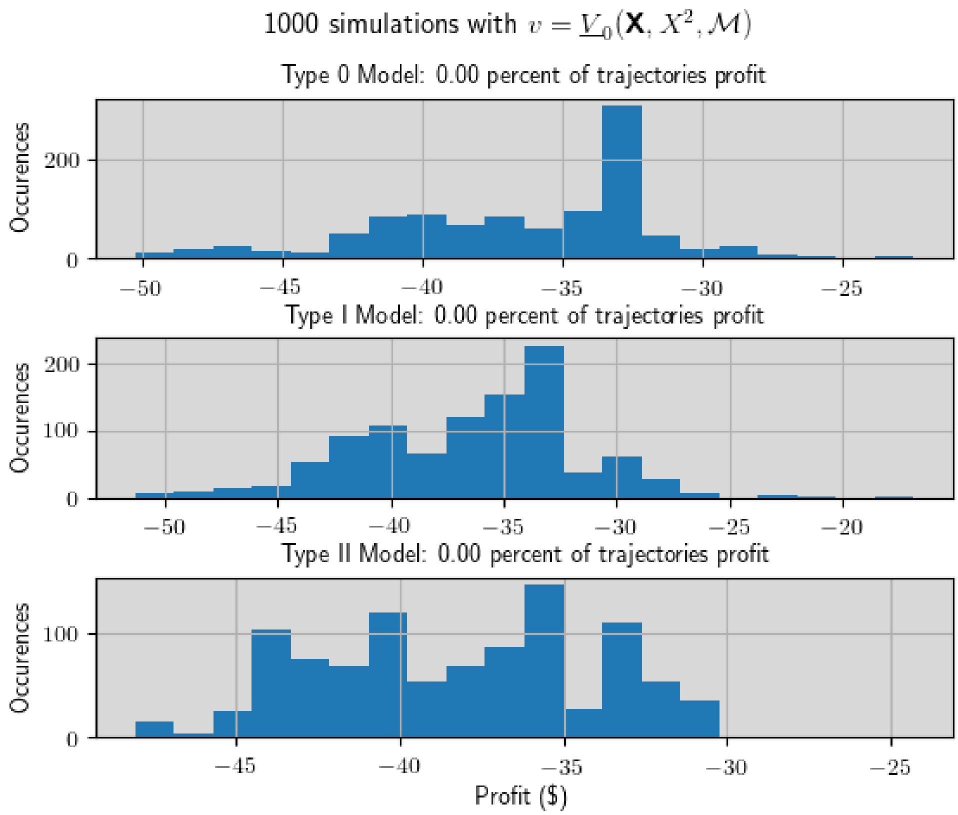


Figure 6.25: Profits and losses for 1000 simulations in each δ-uncorrelated model type when using the assets described in [2.] in the enumeration of Section 6.1 (using Twitter as numeraire). Models are constructed by setting $\delta = 0.05$ and $\delta_0 = 0.01$, $\hat{\delta}^1 = \hat{\delta}^2 = 0.025$, $\hat{\nu}_0 = 0.05$, and $N(\mathbf{X}) = 4$. Notice that when we begin with initial investment $v = \underline{V}_0(\mathbf{X}_0, X^2, \mathcal{M})$, we underhedge X^2 using X^1, or rather, we never profit along any path $\mathbf{X} \in \mathcal{X}$.

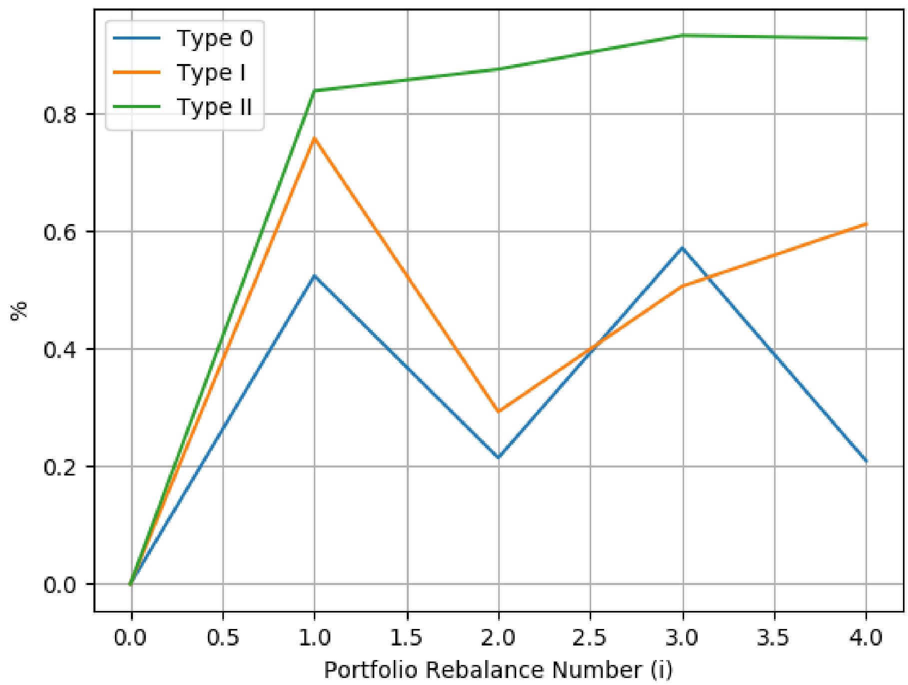


Figure 6.26: Average percentage of children nodes pruned at each parent, averaged over each portfolio rebalancing i for each $\delta = 0.05$ and $\delta_0 = 0.01$, $\hat{\delta}^1 = \hat{\delta}^2 = 0.025$, $\hat{\nu}_0 = 0.05$, $N(\mathbf{X}) = 4$ and historical data subset $\mathcal{T}^2$. Here we are using the assets described in [2.] in the enumeration of Section 6.1 (using Twitter as numeraire).

6.5 δ-Correlated Models: Twitter as Numeraire

Finally, we provide results for the construction of a δ-correlated model while using Twitter as a numeraire. The reader is directed to the Appendix to view parameter estimation results for this case. Concerning ourselves now with the assets described in [2.] in the enumeration and the interval $\mathcal{T}^2$ described in Section 6.1, we go about the same process as we described in the previous two sections.

6.5.1 Objective 1 - Constructing the Trajectory Set

Once again, we note that the interval $\mathcal{T}^2$ is used to help lower computation times required to construct our trajectory market models. We begin by selecting $\delta_{up} = \delta_{down} = 8.0$ and $\delta_0 = 0.075$. We set $\hat{\delta}^1 = \hat{\delta}^2 = 0.025$ and $\hat{\nu}_0 = 0.5$. First we perform the estimation process. The set of possible vector changes, given as $M = \{(\hat{\delta}^1 m^1, \hat{\delta}^2 m^2) : \exists (m^1, m^2, q, P) \in \mathcal{N}_E(x, \mathcal{T}^2)\}$ is displayed in Figure 6.27. Notice that for this set, M, we have that $0 \in ri\big(co(M)\big)$. This was part of our criteria to satisfy when selecting calibrated values, which is discussed in Section 5.2. We also note that $|\mathcal{N}_E(x, \mathcal{T}^2)| = 48$.

Observed worst-case pruning constraints for the chosen calibrated parameters are shown in Figure 6.28. It is not explicitly stated, but the figure indicates that we have $i^*(x.\mathcal{T}) = 91$. We once again set $N(\mathbf{X}) = 4$ to limit the computation time required to construct our market model.

For each model type we begin the construction with the same instantiations: $X_0^1 = 183.82$, $X_0^2 = 331.62$, $t_0 = 0$, $T_0 = 0$, $W_0 = 0$ (note that $X_0^1 = 183.82$ and $X_0^2 = 331.62$ are the most present values we have in $\mathcal{T}^2$). The k^1 and k^2 values at each rebalancing i for nodes in $\mathcal{X}$ are given in Figures 6.29, 6.30, and 6.31. Table 6.4 shows the values obtained for each model type, and notice that Equation (4.7) is satisfied. We also provide the average proportion of nodes pruned at each parent node for each rebalance, which is shown in Figure 6.35

6.5.2 Objective 2 - Superhedging Methodology

Finally, we sampled 1000 trajectories for initial investments of size $v = X_0^2$, $v = \overline{V}_0(\mathbf{X}_0, X^2, \mathcal{M})$ and $v = \underline{V}_0(\mathbf{X}_0, X^2, \mathcal{M})$ which are seen in Figures 6.32, 6.33, and 6.34, respectively. In these model types constructed in this section, we note that (similar to the previous section) there is a greater proportion of profitting trajectories sampled for Type I models than Type II models.

For clarity we indicate that the output from this section is given throughout Figures 6.27-6.35 and in Table 6.4.

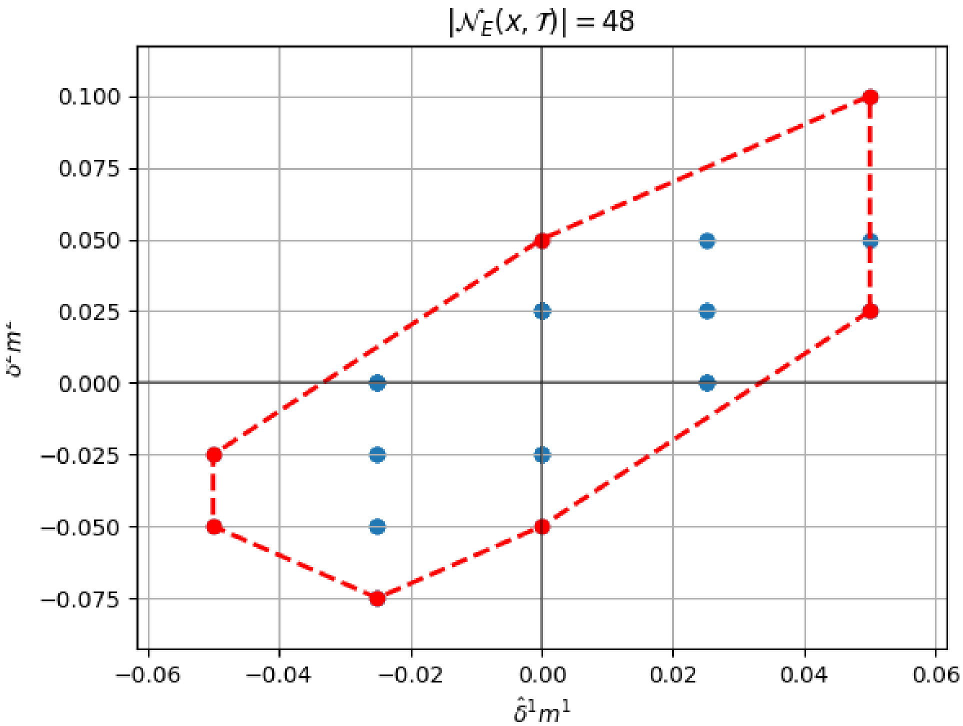


Figure 6.27: Using the assets described in [2.] in the enumeration of Section 6.1 (using Twitter as numeraire), we show the set $M = \{(\hat{\delta}^1 m^1, \hat{\delta}^2 m^2) : \exists (m^1, m^2, q, P) \in \mathcal{N}_E(x, \mathcal{T}^2)\}$ for the δ-correlated model created with $\delta_{up} = \delta_{down} = 8.0$, $\delta_0 = 0.075$, $\hat{\delta}^1 = \hat{\delta}^2 = 0.025$, $\hat{\nu}_0 = 0.5$ and historical data subset $\mathcal{T}^2$. The convex hull of M is given as the dotted red line and red points as its vertices. Notice that $0 \in ri\big(co(M)\big)$.

	$\underline{V}_0(\mathbf{X}_0, X^2, \mathcal{M})$	$\overline{V}_0(\mathbf{X}_0, X^2, \mathcal{M})$
	$X_0^2 = 331.62$	
$\mathcal{M} = \mathcal{M}^0$	327.741	334.963
$\mathcal{M} = \mathcal{M}^I$	327.741	333.564
$\mathcal{M} = \mathcal{M}^{II}$	328.470	333.393

Table 6.4: Although an abuse of notation, we use the notation $\mathcal{M} = \mathcal{M}^0$ to indicate that the corresponding row shows the values $\underline{V}_0(\mathbf{X}_0, X^2, \mathcal{M})$ and $\overline{V}_0(\mathbf{X}_0, X^2, \mathcal{M})$ for the Type 0 market model. The same applies for the next two rows but for Type I and II models, respectively. Time 0 values for our asset X^2 for the δ-correlated model created with $\delta_{up} = \delta_{down} = 8.0$, $\delta_0 = 0.075$, $\hat{\delta}^1 = \hat{\delta}^2 = 0.025$, $\hat{\nu}_0 = 0.5$ and historical data subset $\mathcal{T}^2$ is shown. Notice that we always have $\underline{V}_0(\mathbf{X}_0, X^2, \mathcal{M}) \leq \overline{V}_0(\mathbf{X}_0, X^2, \mathcal{M})$ and that Equation (4.7) in Section 4.2.6 is satisfied.

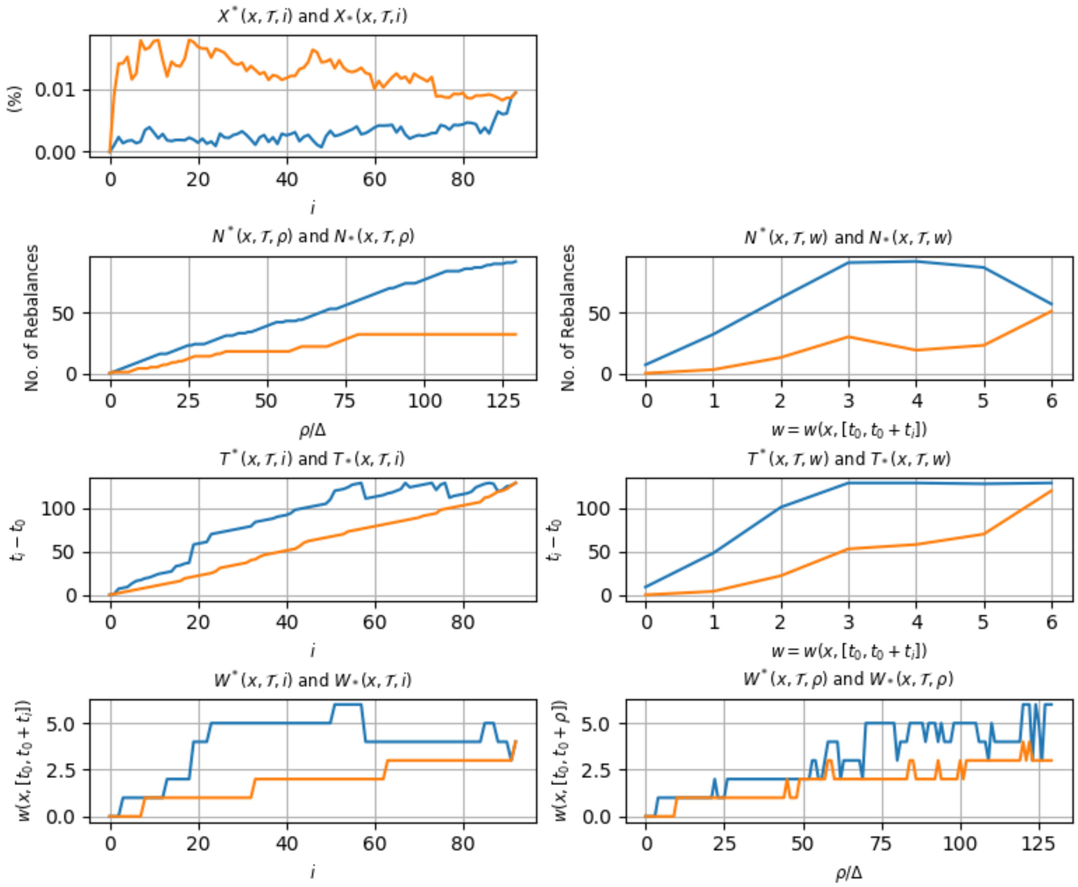


Figure 6.28: Estimated pruning constraints when using the assets described in [2.] in the enumeration of Section 6.1 (using Twitter as numeraire). For this δ-correlated model created with $\delta_{up} = \delta_{down} = 8.0$, $\delta_0 = 0.075$, $\hat{\delta}^1 = \hat{\delta}^2 = 0.025$, $\hat{\nu}_0 = 0.5$ and historical data subset $\mathcal{T}^2$, we have that $i^* = 91$. Notice that the plots for $N^*(x, \mathcal{T}, w)$ and $N_*(x, \mathcal{T}, w)$, $T^*(x, \mathcal{T}, w)$ and $T_*(x, \mathcal{T}, w)$, as well as $W^*(x, \mathcal{T}, \rho)$ and $W_*(x, \mathcal{T}, \rho)$ are much smoother than the analogous plots given in the previous sections. This could be an indication that constructing trajectory sets involving more portfolio rebalances might provide better pruning constraints.

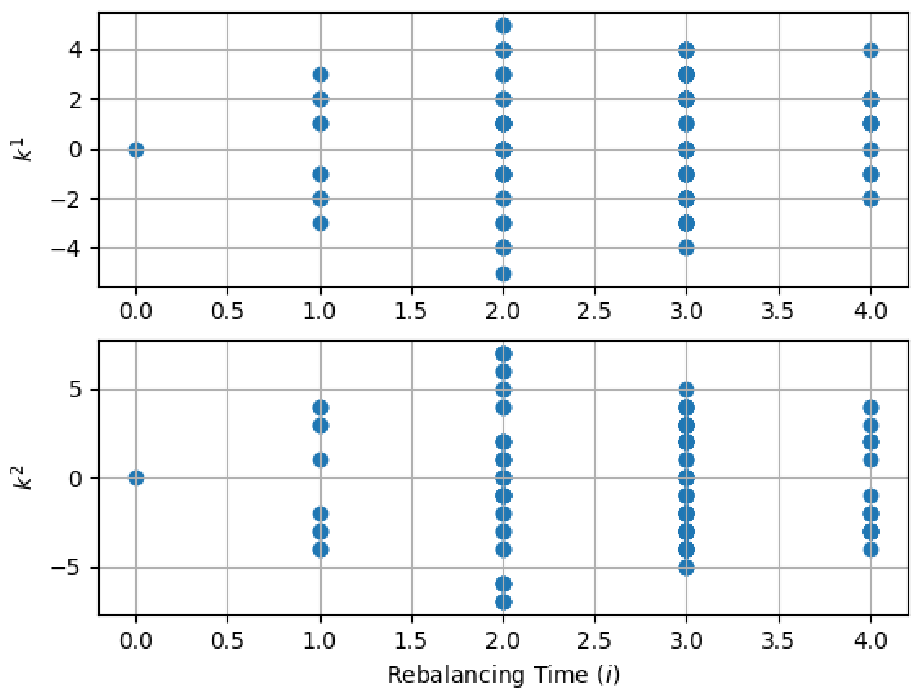


Figure 6.29: Using the assets described in [2.] in the enumeration of Section 6.1 (using Twitter as numeraire) we show values of k^1 and k^2 for all nodes for the Type 0 δ-correlated model created with $\delta_{up} = \delta_{down} = 8.0$, $\delta_0 = 0.075$, $\hat{\delta}^1 = \hat{\delta}^2 = 0.025$, $\hat{\nu}_0 = 0.5$, while setting $N(\mathbf{X}) = 4$ and using historical data subset $\mathcal{T}^2$.

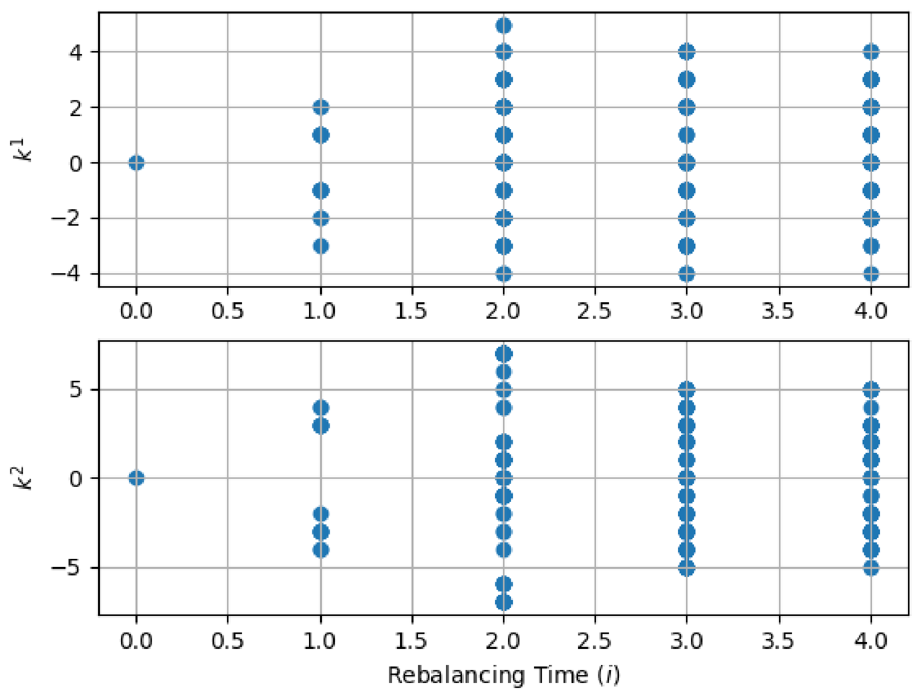


Figure 6.30: Using the assets described in [2.] in the enumeration of Section 6.1 (using Twitter as numeraire) we show values of k^1 and k^2 for all nodes for the Type I δ-correlated model created with $\delta_{up} = \delta_{down} = 8.0$, $\delta_0 = 0.075$, $\hat{\delta}^1 = \hat{\delta}^2 = 0.025$, $\hat{\nu}_0 = 0.5$, while setting $N(\mathbf{X}) = 4$ and using historical data subset $\mathcal{T}^2$.

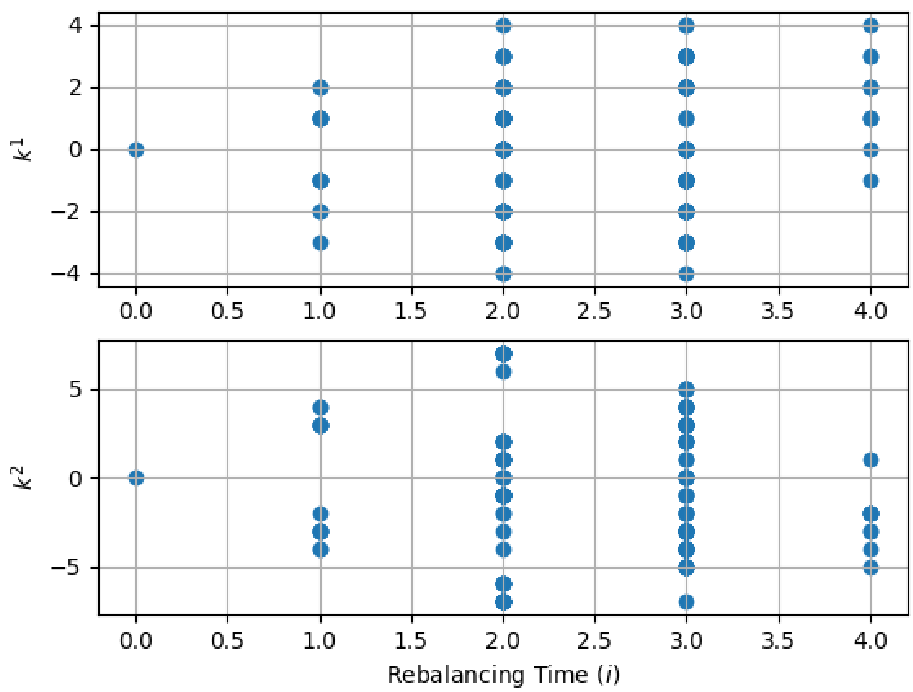


Figure 6.31: Using the assets described in [2.] in the enumeration of Section 6.1 (using Twitter as numeraire) we show values of k^1 and k^2 for all nodes for the Type II δ-correlated model created with $\delta_{up} = \delta_{down} = 8.0$, $\delta_0 = 0.075$, $\hat{\delta}^1 = \hat{\delta}^2 = 0.025$, $\hat{\nu}_0 = 0.5$, while setting $N(\mathbf{X}) = 4$ and using historical data subset $\mathcal{T}^2$.

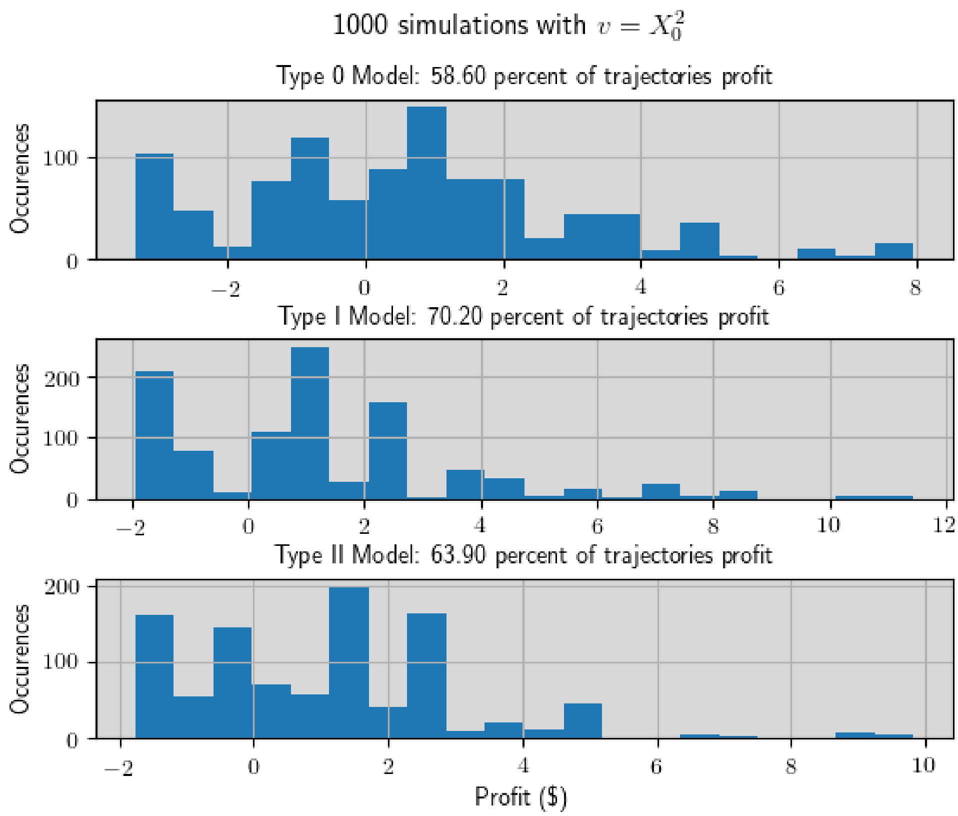


Figure 6.32: Profits and losses for 1000 simulations in each δ-correlated model type when using the assets described in [2.] in the enumeration of Section 6.1 (using Twitter as numeraire). Models are constructed by setting $\delta_{up} = \delta_{down} = 8.0$, $\delta_0 = 0.075$, $\hat{\delta}^1 = \hat{\delta}^2 = 0.025$, $\hat{\nu}_0 = 0.5$, and $N(\mathbf{X}) = 4$. Here we begin with initial investment $v = X_0^2$.

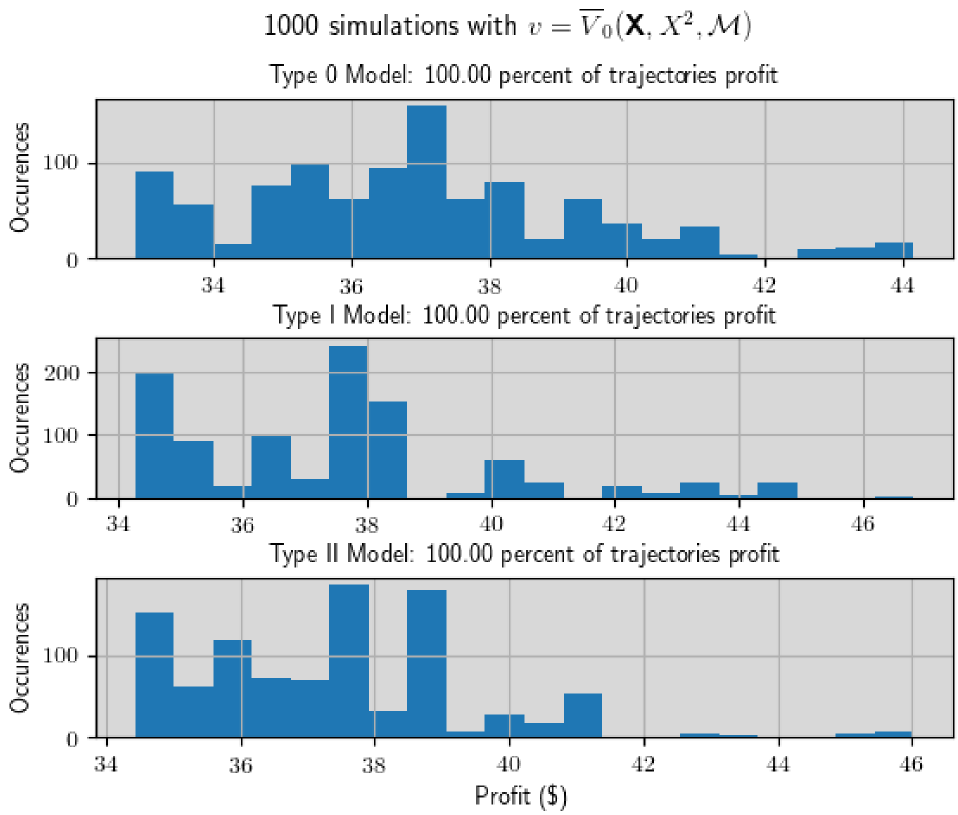


Figure 6.33: Profits and losses for 1000 simulations in each δ-correlated model type when using the assets described in [2.] in the enumeration of Section 6.1 (using Twitter as numeraire). Models are constructed by setting $\delta_{up} = \delta_{down} = 8.0$, $\delta_0 = 0.075$, $\hat{\delta}^1 = \hat{\delta}^2 = 0.025$, $\hat{\nu}_0 = 0.5$, and $N(\mathbf{X}) = 4$. Here we begin with initial investment $v = \overline{V}_0(\mathbf{X}_0, X^2, \mathcal{M})$. Notice that with this initial investment, we always superhedge X^2 along the trajectories in our model when trading X^1.

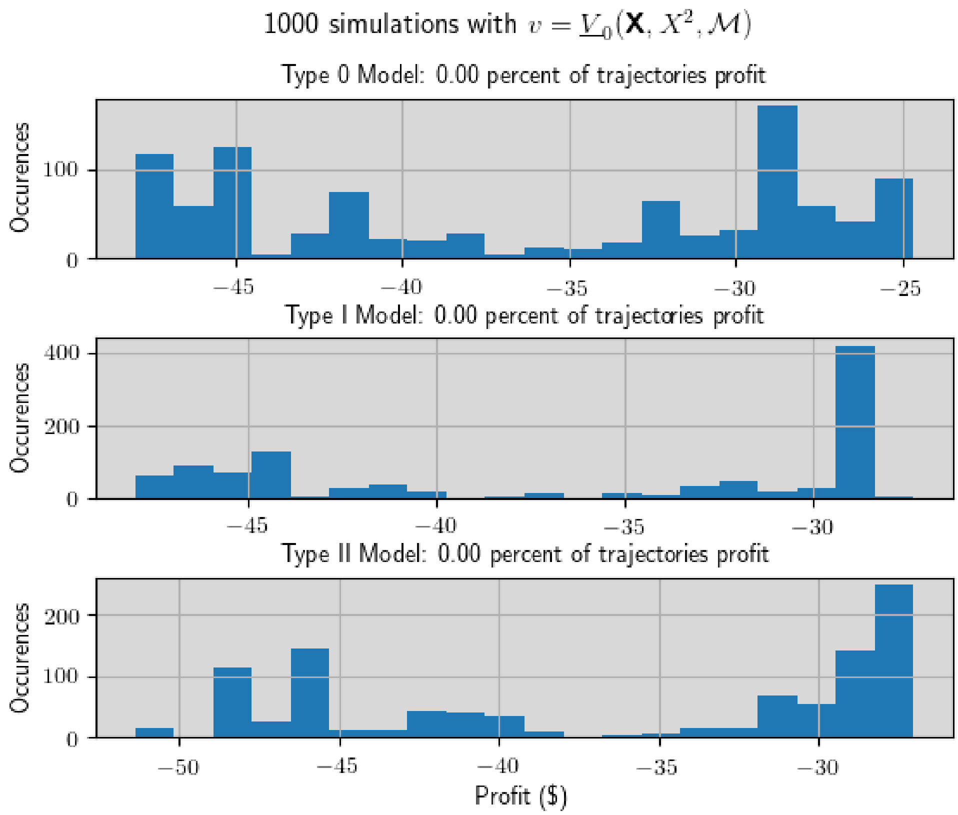


Figure 6.34: Profits and losses for 1000 simulations in each δ-correlated model type when using the assets described in [2.] in the enumeration of Section 6.1 (using Twitter as numeraire). Models are constructed by setting $\delta_{up} = \delta_{down} = 8.0$, $\delta_0 = 0.075$, $\hat{\delta}^1 = \hat{\delta}^2 = 0.025$, $\hat{\nu}_0 = 0.5$, and $N(\mathbf{X}) = 4$. Notice that when we begin with initial investment $v = \underline{V}_0(\mathbf{X}_0, X^2, \mathcal{M})$, we underhedge X^2 using X^1, or rather, we never profit along any path $\mathbf{X} \in \mathcal{X}$. Notice that with this initial investment, we always underhedge X^2 along the trajectories in our model when trading X^1.

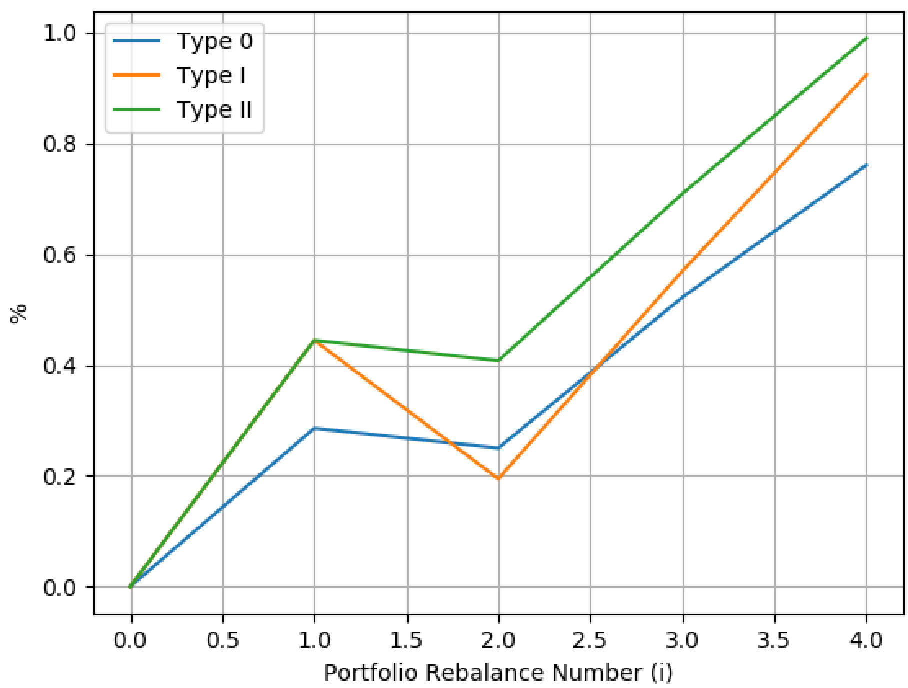


Figure 6.35: When using the assets described in [2.] in the enumeration of Section 6.1 (using Twitter as numeraire) we show the average percentage of children nodes pruned at each parent, averaged over each portfolio rebalancing i for each δ-correlated model type created with $\delta_{up} = \delta_{down} = 8.0$, $\delta_0 = 0.075$, $\hat{\delta}^1 = \hat{\delta}^2 = 0.025$, $\hat{\nu}_0 = 0.5$, $N(\mathbf{X}) = 4$ and historical data subset $\mathcal{T}^2$.

Chapter 7

Discussion and Conclusion

We now provide a discussion of the results above and refer to similar results shown in the Appendix. In this section we also discuss various items worthwhile to address in future work. As can be seen throughout the previous chapter, the output obtained from our trajectory market models is dependent on an investor's choice of inputs parameters. We provide model testing for each model type with δ-uncorrelated and δ-correlated operational assumptions for two different choices of numeraire. We also note that we provide additional output to supplement output shown throughout the main sections of the paper.

The first item to mention is that future models do not need to be created exactly as done in this thesis. Our models are general and may incorporate any type of assets, chart operations, model parameters and simulation techniques. An investor has the capability of manipulating trajectory set construction in simple ways such as incorporating a new financial observable, changing coordinate systems, or redefining the way a δ-movement (or δ_0-movement) is observed. Although we believe we construct $\mathcal{X}$ in a rather natural way, one might be interested in forcing each trajectory to reach terminal time T. This could be achieved by incorporating similar model construction assumptions as those given in Ferrando et al. [2019a] (the cases which arise in model construction that force trajectories to carry on until time T).

Given that we have constructed two different ways that an investor can define a δ-movement, we see from our provided figures that there are many contrasting characteristics between the two model types when observing charts. The simplest example would be to compare Figures 5.9 and 17, where $|\mathcal{N}_E(x, \mathcal{T})|$ decreases with an increase of δ in Figure 5.9, while $|\mathcal{N}_E(x, \mathcal{T})|$ increases with an increase of $\delta_{up} = \delta_{down}$ in Figure 17. It is also interesting to note that the values for $|\mathcal{N}_E(x, \mathcal{T})|$ stabilize after δ (or δ_{up} and δ_{down}) is increased to a certain point. For δ-uncorrelated models, the values for $|\mathcal{N}_E(x, \mathcal{T})|$ degenerate to 0, while the same values stabilize at about 52 elements (this stabilization is seen in Figure 17 in the Appendix).

This same behaviour is apparent in other parameters such as $N_*(x, \mathcal{T}, \rho)$ and $N^*(x, \mathcal{T}, \rho)$. When increasing δ in δ-correlated models, values for $N_*(x, \mathcal{T}, \rho)$ and $N^*(x, \mathcal{T}, \rho)$ (shown in 5.2) decrease until δ is large enough we observe $N^*(x, \mathcal{T}, \rho) = 0$. Conversely, when increasing δ_{up} and δ_{down} in δ-correlated models, values for $N_*(x, \mathcal{T}, \rho)$ and $N^*(x, \mathcal{T}, \rho)$ (shown in 5.6) increases until they stabilize at around $\delta_{up} = \delta_{down} \approx 8.0$. The parameters $T_*(x, \mathcal{T}, i)$ and $T^*(x, \mathcal{T}, i)$ also exhibit more values of δ_{up} and δ_{down} that correspond to trajectories ending at terminal time $t_0 + T$ (comparing Figures 5.3 and 5.7).

As expected, the constructed market models had values for $\underline{V}_0(\mathbf{X}_0, X^2, \mathcal{M}^0)$, $\overline{V}_0(\mathbf{X}_0, X^2, \mathcal{M}^0)$, $\underline{V}_0(\mathbf{X}_0, X^2, \mathcal{M}^I)$, $\overline{V}_0(\mathbf{X}_0, X^2, \mathcal{M}^I)$, $\underline{V}_0(\mathbf{X}_0, X^2, \mathcal{M}^{II})$, $\overline{V}_0(\mathbf{X}_0, X^2, \mathcal{M}^{II})$ which satisfied the inequality given in Equation (4.7). It is seen in some models created throughout Chapter 6 that $\underline{V}_0(\mathbf{X}_0, X^2, \mathcal{M}^0) = \underline{V}_0(\mathbf{X}_0, X^2, \mathcal{M}^I)$ or $\overline{V}_0(\mathbf{X}_0, X^2, \mathcal{M}^0) = \overline{V}_0(\mathbf{X}_0, X^2, \mathcal{M}^I)$. This indicates that the pruning constraints incorporated in Type I models did not prune enough nodes to cause the price bounds to satisfy $\underline{V}_0(\mathbf{X}_0, X^2, \mathcal{M}^0) < \underline{V}_0(\mathbf{X}_0, X^2, \mathcal{M}^I)$ and $\overline{V}_0(\mathbf{X}_0, X^2, \mathcal{M}^0) > \overline{V}_0(\mathbf{X}_0, X^2, \mathcal{M}^I)$. In contrast to the Type I pruning constraints not tightening price bounds (compared to Type 0 models), pruning constraints used in each of the Type II models constructed in this thesis pruned trajectory sets and yielded initial values satisfying $\underline{V}_0(\mathbf{X}_0, X^2, \mathcal{M}^I) < \underline{V}_0(\mathbf{X}_0, X^2, \mathcal{M}^{II})$ and $\overline{V}_0(\mathbf{X}_0, X^2, \mathcal{M}^I) > \overline{V}_0(\mathbf{X}_0, X^2, \mathcal{M}^{II})$.

We turn the reader's attention to Figure 3.3 and note that with the addition of historical data in the estimation process the worst-case pruning constraints widen. This, and the fact that we find $\underline{V}_0(\mathbf{X}_0, X^2, \mathcal{M}^0) = \underline{V}_0(\mathbf{X}_0, X^2, \mathcal{M}^I)$ and $\overline{V}_0(\mathbf{X}_0, X^2, \mathcal{M}^0) = \overline{V}_0(\mathbf{X}_0, X^2, \mathcal{M}^I)$, would indicate that the pruning constraints $N^*(x, \mathcal{T}, \rho)$, $N_*(x, \mathcal{T}, \rho)$, $T^*(x, \mathcal{T}, i)$, and $T_*(x, \mathcal{T}, i)$ are non-informative. Future research could involve finding combinations of assets which provide stable pruning constraints with the accumulation of historical data.

In this thesis we set $N(\mathbf{X}) = 4$ to limit the computation time required to construct and value trajectory sets. We expect that when setting $N(\mathbf{X}) = i^*$ (i.e. constructing markets without restricting the number of portfolio rebalances) our price bounds $[\underline{V}_0(\mathbf{X}_0, X^2, \mathcal{M}), \overline{V}_0(\mathbf{X}_0, X^2, \mathcal{M})]$ to widen. That is, we expect $\underline{V}_0(\mathbf{X}_0, X^2, \mathcal{M})$ to decrease and $\overline{V}_0(\mathbf{X}_0, X^2, \mathcal{M})$ to increase. Future work could entail using longer computation times to determine price bounds for models while not restricting the number of portfolio rebalances.

Output for profit and loss for trajectory path simulation is also worth mentioning. First, note that when we set $v = \overline{V}_0(\mathbf{X}_0, X^2, \mathcal{M})$ our portfolios superhedge the asset X^2 and when $v = \underline{V}_0(\mathbf{X}_0, X^2, \mathcal{M})$ our portfolios underhedge the asset X^2. This is seen in Figures 6.16, 6.16, 6.24, 6.25, 6.33, and 6.34. We also note that when setting $v = X_0^2$ the proportion of sampled trajectories which profit increases for each subsequent model (Type 0, I and II).

We also mention that there is a larger proportion of losing (not profitting) trajectories in the trajectory sets created in Section 6.2 (the reader may observe this in Figure 6.6). Given that, in our financial setup, we short X^2 while trading X^1, it is possible that when we observe cases such as in Figure 6.6 an investor could observe more profitting trajectories by longing X^2 and performing the inverse trades than what our algorithm provides (instead of buying we would sell, and vice-versa).

We constructed market models with two choices of numeraire to see the effect of selecting a risky numeraire. We were also interested to observe geometric characteristics of our use of the convex hull in the paper and the effect of changing a numeraire on the shape of the hulls. Although we show how the shape of our convex hulls stabilize after the addition of data in Figure 5.11 and throughout multiple figures in the Appendix, we do not provide any insight to selecting one numeraire over the other. Recall that the reference Filipovic [2007] shows that there is no optimal numeraire that provides an investor with a lower risk than any other numeraire.

Recall that the trajectory market models constructed in this thesis will never signal a market misprice (i.e. $X_0^2 \notin [\underline{V}_0(\mathbf{X}_0, X^2, \mathcal{M}), \overline{V}_0(\mathbf{X}_0, X^2, \mathcal{M})]$). An open problem to the reader would be going about constructing market models without instantiating models with knowledge of X_0^2. If models were to signal a market misprice, our models could help investors identify investment opportunities.

Our models might also provide an opportunity for investors to incorporate machine learning learning techniques in optimally calibrate model. We expect that machine learning could be used to optimize an investor's calibrated parameters, which then would yield models which minimize the risk associated with constructing a portfolio with initial value $v = X_0^2$. We also mentioned in Section 4.3 that trajectory paths were sampled from a population (i.e. $\mathcal{X}$) by imposing a uniform distribution on trajectories in our population. Future work may may also entail determining an optimal probability distribution to impose on trajectory paths in $\mathcal{X}$. Using a different probability distribution could then provide an investor a probability of gains.

Thus, we conclude that this thesis develops the basic framework required to construct a trajectory based market model with a purely observational approach to superhedge (and subhedge) an asset X^2 with a portfolio constructed of asset X^1. Our setting for constructing a trajectory set is left quite general and investors are free to set how they wish to construct models. One may select an arbitary number of stocks to construct a portfolio with and is not limited to a conventional riskless asset as the choice of numeraire. We adopt a worst-case point view which naturally restricts how our trajectories unfold. Framework for model construction allows for arbitrage opportunities to be included as long as we ensure nodes within our market are locally 0-neutral. This in turn provides an investor with meaningful prices which incorporates critical information such as future arbitrage opportunities. Thus the trajectory models proposed in this paper allow an investor to extract risk-taking information about how an asset's trajectory will unfold.

The construction of our market models, namely the Type II market models, is quite computationally intensive due to the size of discrete grid our $\mathbf{X}_i$ provide. Although it is not performed in this thesis, it would be of interest to construct models with the ability to set the maximum number of possible rebalances to our worst-case estimate. In order to do this, computation times could decrease if one could integrate parallel dynamic computing into the construction of $\mathcal{X}$. With faster computation an investor would be able to use a larger discretization of observable parameters (i.e. decrease values for $\hat{\delta}^1$, $\hat{\delta}^2$ and $\hat{\nu}_0$).

We also identify several areas which an investor might be able to incorporate machine learning techniques to aid with calibration and the possibility of signalling market misprices. There is also the opportunity to impose a preferred - or perhaps estimated - probability distribution on a trajectory set to increase probability of gains. An open problem that remains after this thesis is determining how to construct multidimensional trajectory market models without knowledge of the initial price X_0^2.

Appendices

Appendices

In this appendix we visualize the historical data for the assets we utilize in this thesis. There are a number of different representations shown here. First, we show the historical data for the assets $s(t) = \big(s^0(t), s^1(t), s^2(t)\big)$, where $s^0(t)$, $s^2(t)$, and $s^2(t)$ represent the historical stock prices for Twitter, Facebook and Netflix, respectively. We also show this same data on the different time intervals ($\mathcal{T}^1$ and $\mathcal{T}^2$) which are described in Section 6.1.

Assets Used

We begin by providing the reader with a visual representation of the data used throughout the thesis in our estimation processes.

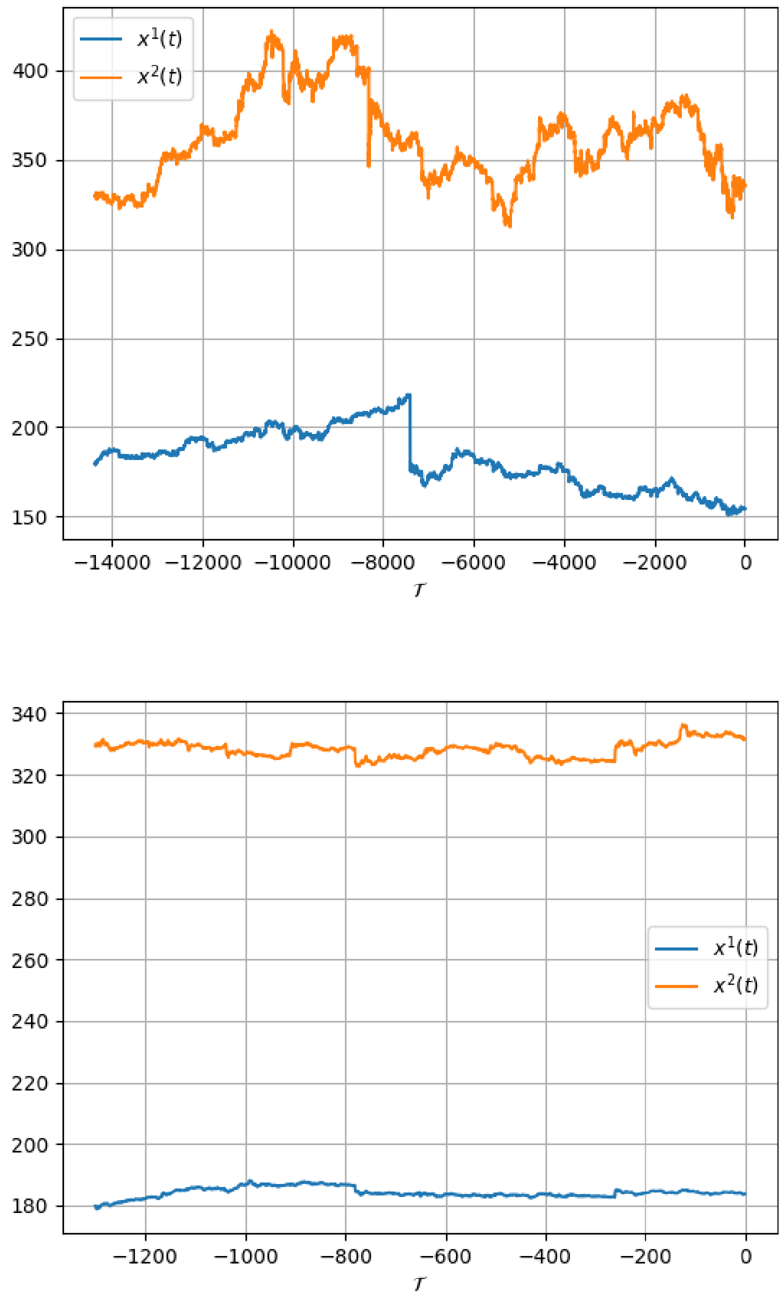
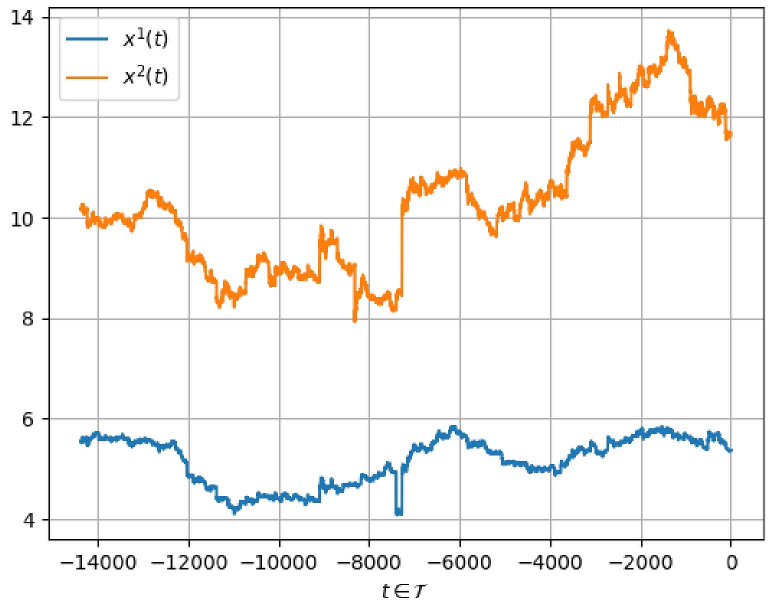


Figure 1: Here we have that $x^1(t)$ and $x^2(t)$ represent the stock values for Facebook and Netflix, respectively, while using currency as numeraire ([1.] in enumeration from Section 6.1). We use the dollar currency as numeraire in this case. The top subplot shows asset values throughout the interval $\mathcal{T} = \mathcal{T}^1$while the lower subplot uses $\mathcal{T} = \mathcal{T}^2$ as given in Section 6.1.



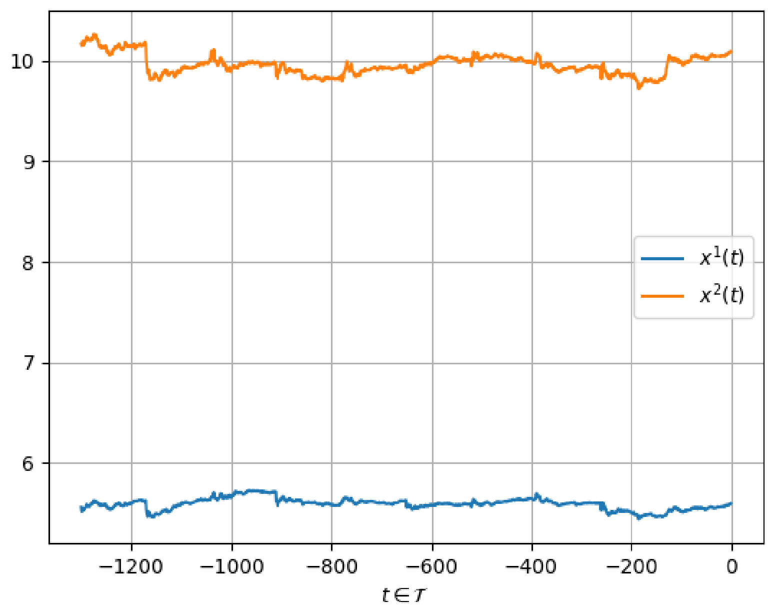


Figure 2: Here we have that $x^1(t)$ and $x^2(t)$ represent the stock values for the stock prices of Facebook and Netflix discounted by the stock price of Twitter, respectively ([2.] in enumeration from Section 6.1). The top subplot shows asset values throughout the interval $\mathcal{T} = \mathcal{T}^1$ while the lower subplot uses $\mathcal{T} = \mathcal{T}^2$ as given in Section 6.1.

Growth of the Convex Hull

In Figure 5.11 we provided a visualization of how the shape of $M = \{(m^1\hat{\delta}^1, m^2\hat{\delta}^2) : \exists(m^1, m^2, q, P) \in \mathcal{N}_E(x, \mathcal{T})\}$ changes as more data is accrued to the estimation process. Here we provide analogous output displaying how the convex hull of M grows when using more data, and different numeraires. It is seen in the following figures that there becomes a point where the shape of $co(M)$ stabilizes after a certain amount of data is added.

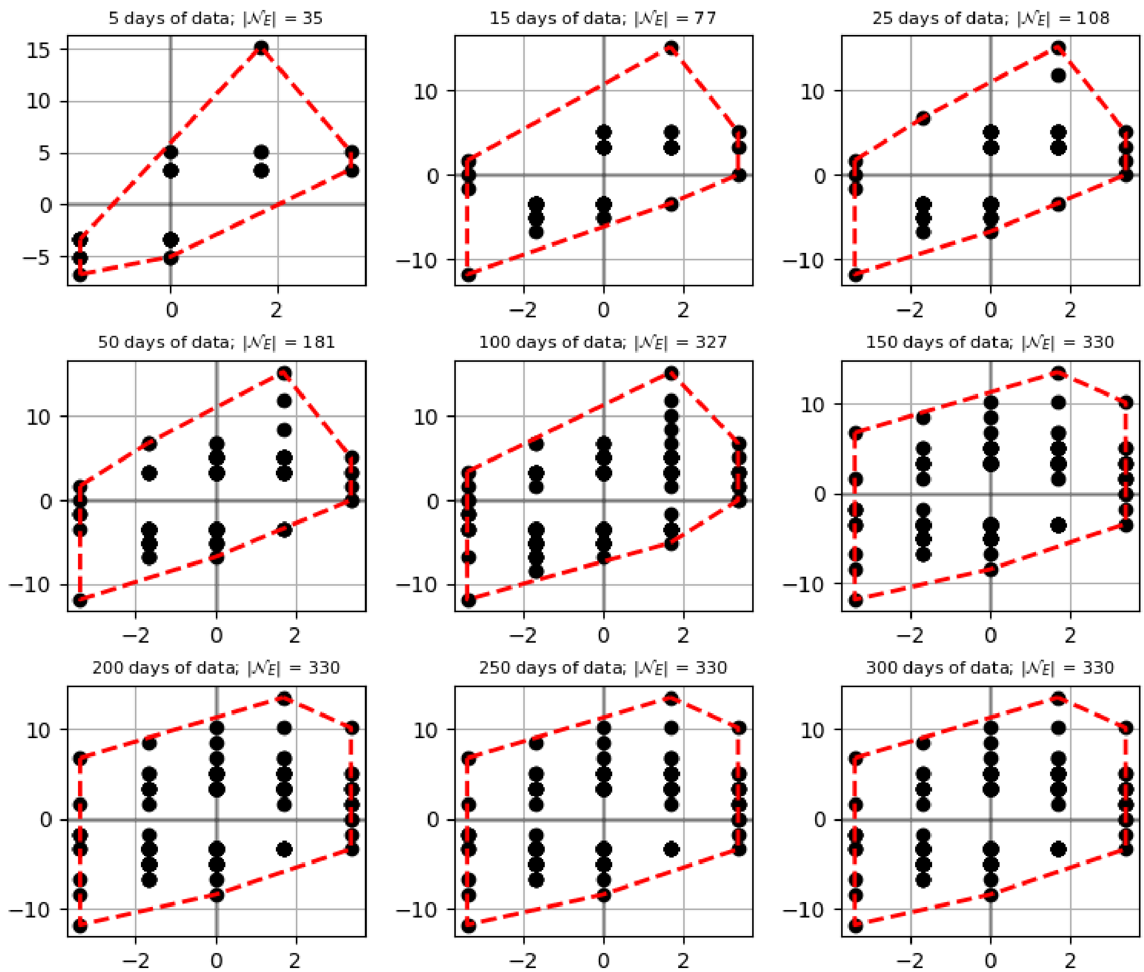


Figure 3: For $M = \{(m^1\hat{\delta}^1, m^2\hat{\delta}^2) : \exists(m^1, m^2, q, P) \in \mathcal{N}_E(x, \mathcal{T})\}$ we show how $co(M)$ grows with the increase of data. Here the assets historical time interval $\mathcal{T} = \mathcal{T}^1$ and data described in [1.] (currency as numeraire). We select the following values for the estimation process: $\delta = 3.378$, $\delta_0 = 0$, $\hat{\delta}^1 = \hat{\delta}^2 = \delta/2$ and $\hat{\nu}_0 = \delta$. Each subplot has values for $m^1\hat{\delta}^1$ along the x-axis and $m^2\hat{\delta}^2$ along the y-axis. Notice the convex hull's stability after incorporating about 150 days of data. Note that this figure is similar to Figure 5.11, but using more historical data. Each subplot has values for $m^1\hat{\delta}^1$ on the x-axis and $m^2\hat{\delta}^2$ along the y-axis.

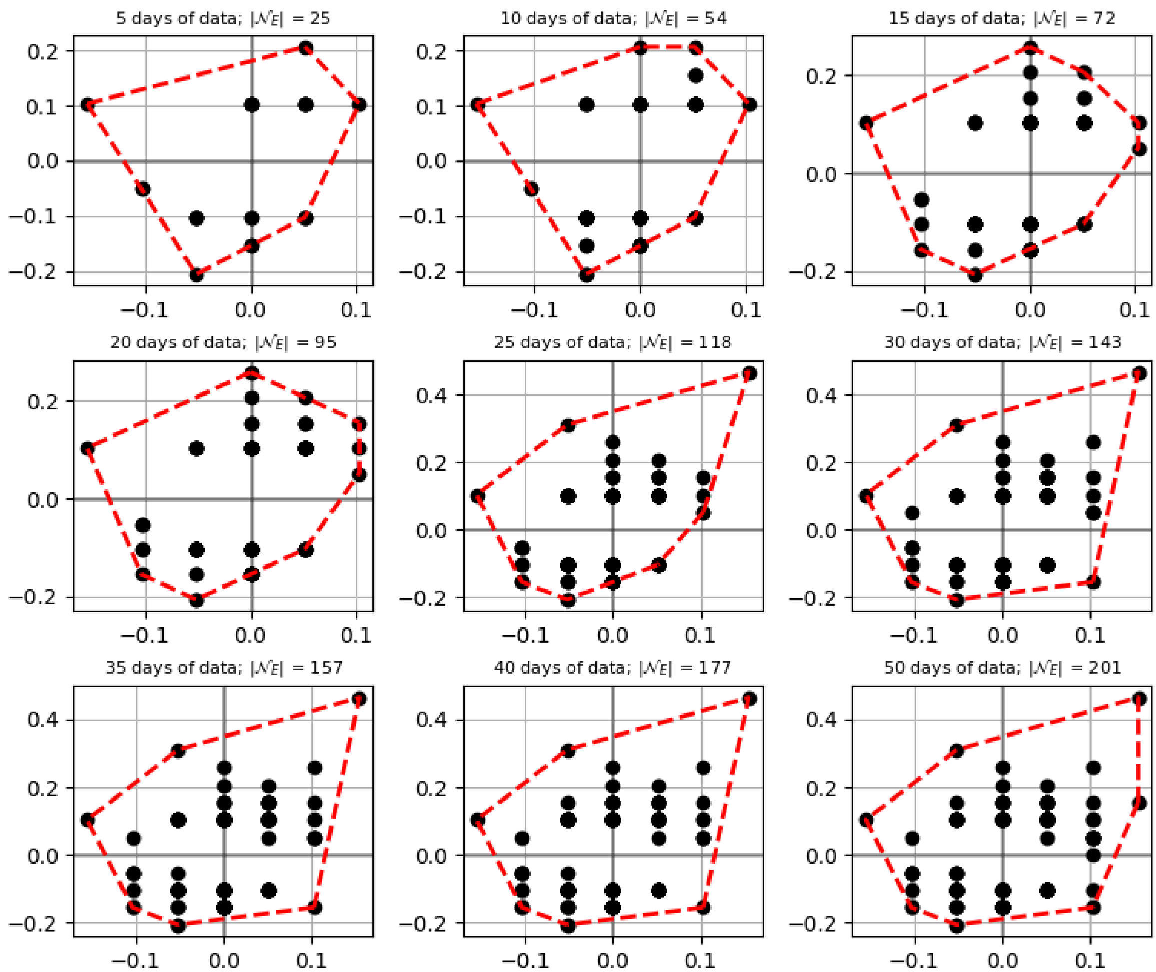


Figure 4: For $M = \{(m^1\hat{\delta}^1, m^2\hat{\delta}^2) : \exists(m^1, m^2, q, P) \in \mathcal{N}_E(x, \mathcal{T})\}$ we show how $co(M)$ grows with the increase of data. Here we use historical time interval $\mathcal{T} = \mathcal{T}^1$ and data described in [2.] (Twitter as numeraire). We select the following values for the estimation process: $\delta = 0.103$, $\delta_0 = \delta_{min} = 5.159 \times 10^{-7}$, $\hat{\delta}^1 = \hat{\delta}^2 = \delta/2$ and $\hat{\nu}_0 = \delta$. Each subplot has values for $m^1\hat{\delta}^1$ along the x-axis and $m^2\hat{\delta}^2$ along the y-axis. Notice the convex hull's stability after incorporating about 30 days of data. Each subplot has values for $m^1\hat{\delta}^1$ on the x-axis and $m^2\hat{\delta}^2$ along the y-axis.

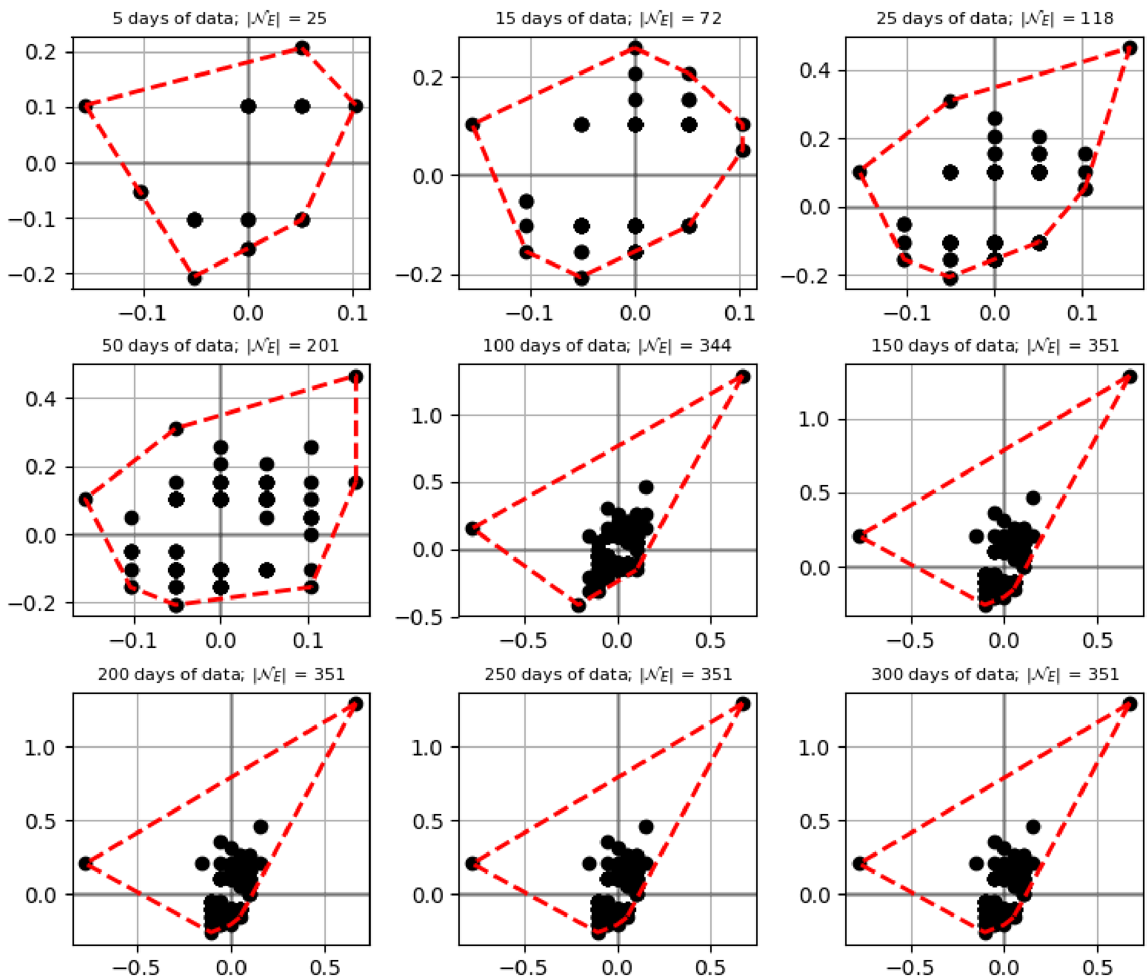


Figure 5: For $M = \{(m^1\hat{\delta}^1, m^2\hat{\delta}^2) : \exists(m^1, m^2, q, P) \in \mathcal{N}_E(x, \mathcal{T})\}$ we show how $co(M)$ grows with the increase of data. Here we use historical time interval $\mathcal{T} = \mathcal{T}^1$ and data described in [2.] (Twitter as numeraire). We select the following values for the estimation process: $\delta = 0.103$, $\delta_0 = \delta_{min} = 5.159 \times 10^{-7}$, $\hat{\delta}^1 = \hat{\delta}^2 = \delta/2$ and $\hat{\nu}_0 = \delta$. Each subplot has values for $m^1\hat{\delta}^1$ along the x-axis and $m^2\hat{\delta}^2$ along the y-axis. Notice the convex hull's stability after incorporating about 150 days of data. Note that this figure is similar to Figure 4, but using more historical data. Each subplot has values for $m^1\hat{\delta}^1$ on the x-axis and $m^2\hat{\delta}^2$ along the y-axis.

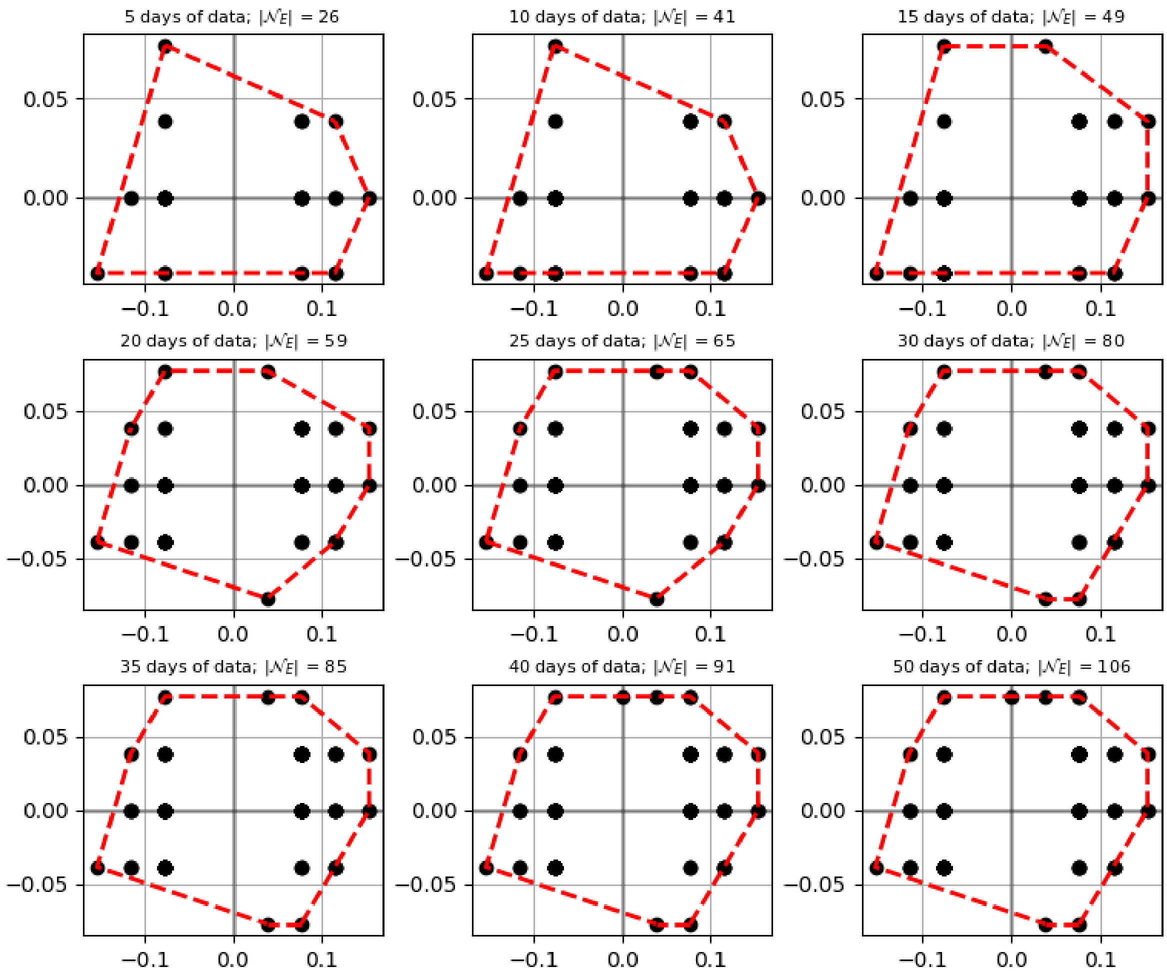


Figure 6: For $M = \{(m^1\hat{\delta}^1, m^2\hat{\delta}^2) : \exists (m^1, m^2, q, P) \in \mathcal{N}_E(x, \mathcal{T})\}$ we show how $co(M)$ grows with the increase of data. Here we use historical time interval $\mathcal{T} = \mathcal{T}^1$ and data described in [3.]. We select the following values for the estimation process: $\delta = 7.688 \times 10^2$, $\delta_0 = \delta_{min} = 2.212 \times 10^6$, $\hat{\delta}^1 = \hat{\delta}^2 = \delta/2$ and $\hat{\nu}_0 = \delta$. Each subplot has values for $m^1\hat{\delta}^1$ along the x-axis and $m^2\hat{\delta}^2$ along the y-axis. Notice the convex hull's stability after incorporating about 30 days of data. Each subplot has values for $m^1\hat{\delta}^1$ on the x-axis and $m^2\hat{\delta}^2$ along the y-axis.

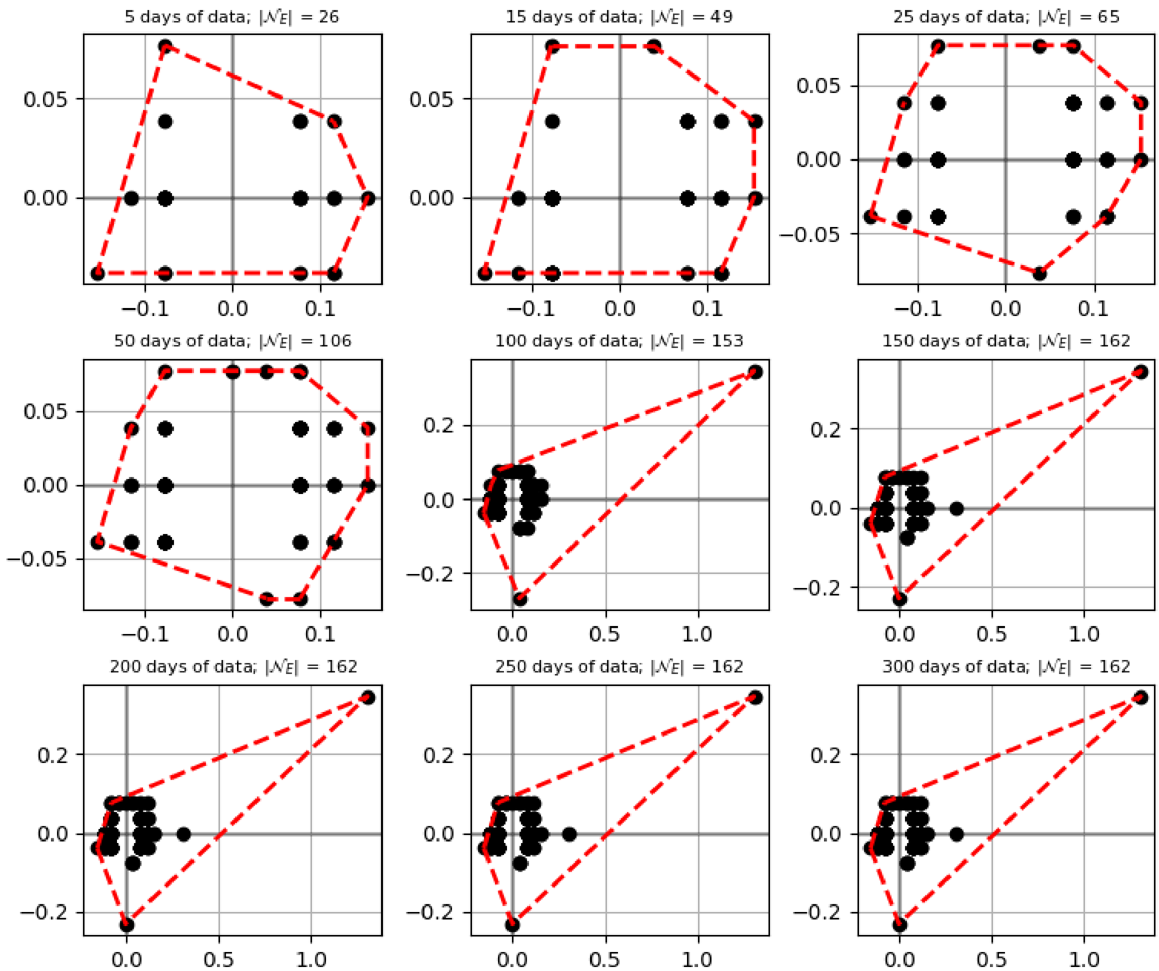


Figure 7: For $M = \{(m^1\hat{\delta}^1, m^2\hat{\delta}^2) : \exists(m^1, m^2, q, P) \in \mathcal{N}_E(x, \mathcal{T})\}$ we show how $co(M)$ grows with the increase of data. Here we use historical time interval $\mathcal{T} = \mathcal{T}^1$ and data described in [3.]. We select the following values for the estimation process: $\delta = 7.688 \times 10^2$, $\delta_0 = \delta_{min} = 2.212 \times 10^6$, $\hat{\delta}^1 = \hat{\delta}^2 = \delta/2$ and $\hat{\nu}_0 = \delta$. Each subplot has values for $m^1\hat{\delta}^1$ along the x-axis and $m^2\hat{\delta}^2$ along the y-axis. Notice the convex hull's stability after incorporating about 100 days of data.Note that this figure is similar to Figure 6, but using more historical data. Each subplot has values for $m^1\hat{\delta}^1$ on the x-axis and $m^2\hat{\delta}^2$ along the y-axis.

Parameter Estimation - Using Currency (US \$) as Numeraire

Output for $\mathcal{N}_E(x, \mathcal{T})$

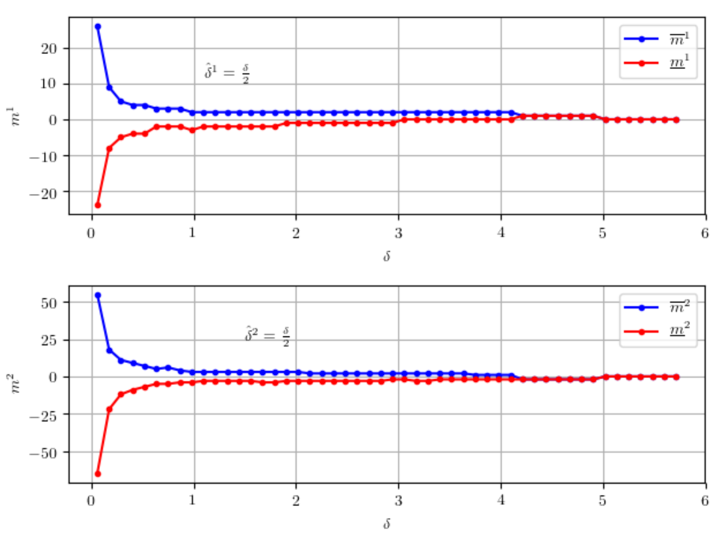


Figure 8: $\overline{m}^1$, $\underline{m}^1$, $\overline{m}^2$, and $\underline{m}^2$ vs. δ for δ-uncorrelated models.

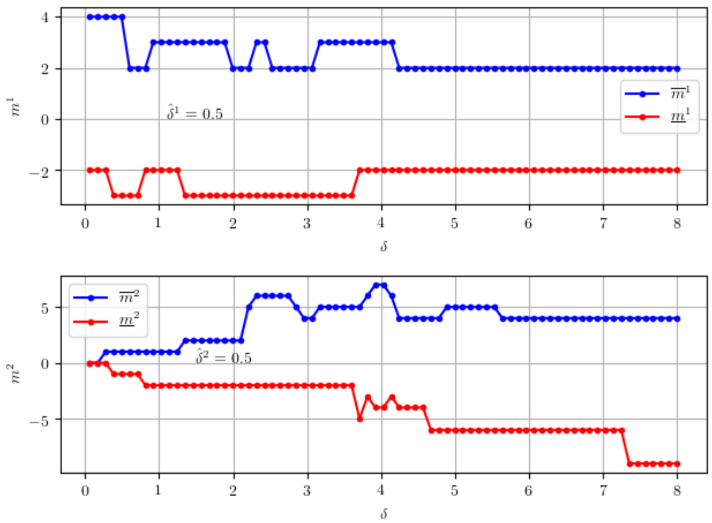


Figure 9: $\overline{m}^1$, $\underline{m}^1$, $\overline{m}^2$, and $\underline{m}^2$ vs. δ for δ-correlated models. Here we have that $\delta_{up} = \delta_{down} = \delta$ where δ represents the value along the x-axis in the figure.

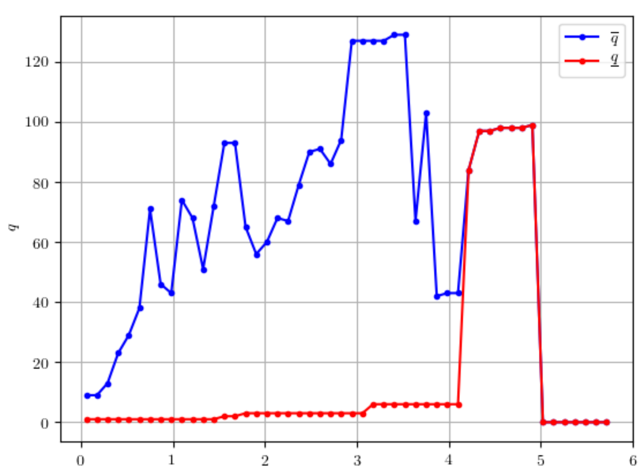


Figure 10: $\overline{q}$, and $\underline{q}$ vs. δ for δ-uncorrelated models.

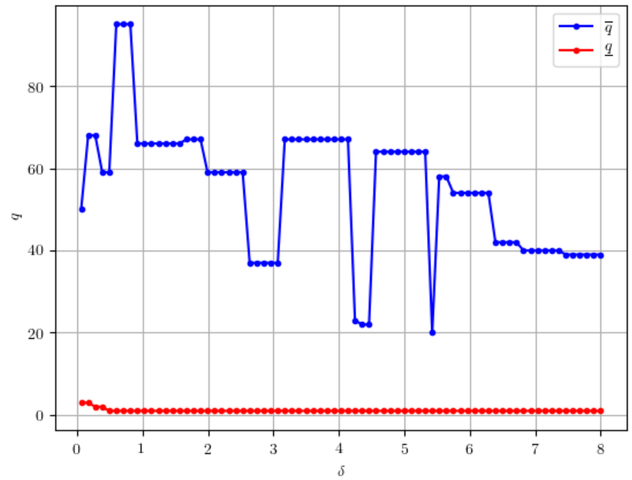


Figure 11: $\overline{q}$, and $\underline{q}$ vs. δ for δ-correlated models. Here we have that $\delta_{up} = \delta_{down} = \delta$ where δ represents the value along the x-axis in the figure.

Parameter Estimation - Using Twitter as Numeraire

In Chapter 5 we viewed various results for parameter estimation using the assets described in [1.] in the enumeration in Section 6.1; which uses a simple currency as numeraire. Since this paper is concerned with constructing trajectory market models with an arbitrary numeraire we show parameter estimation results for the assets described in [2.] in the enumeration in Section 6.1, which uses the stock price of Twitter as numeraire. We provide it in the appendix to avoid making the bulk of the main paper too long. Note that this output is analogous to output shown throughout Chapter 5 which uses currency as a numeraire.

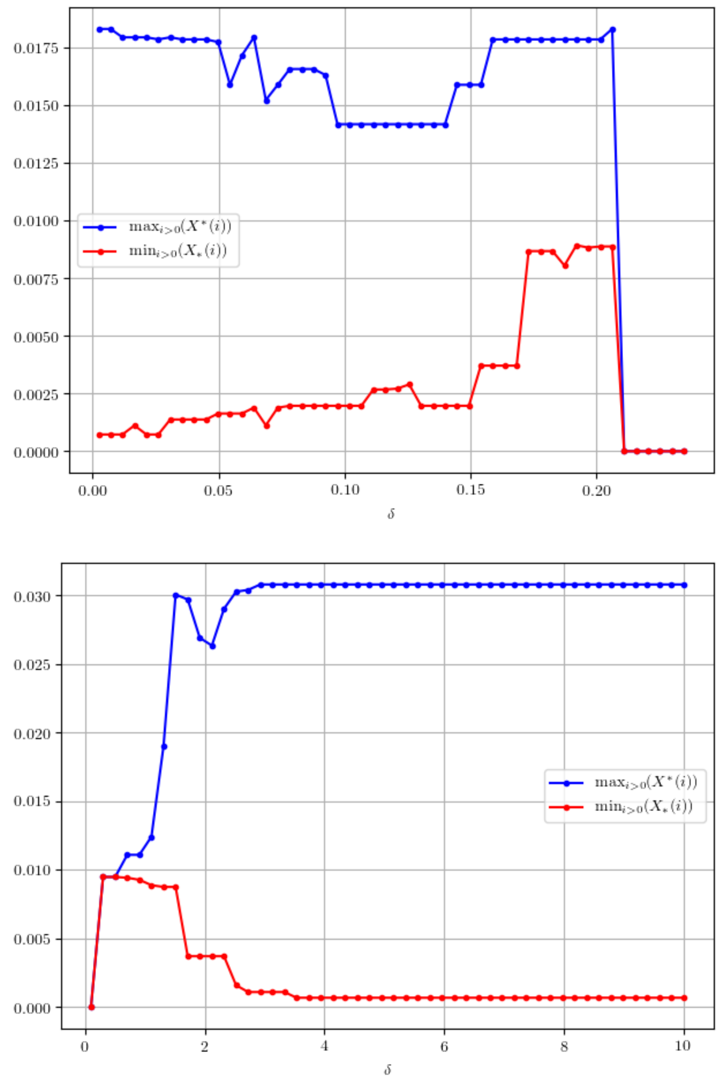


Figure 12: $\max_i\big(X^*(x,\mathcal{T},i)\big)$ and $\min_i\big(X_*(x,\mathcal{T},i)\big)$ vs. δ for δ-uncorrelated (top plot) and δ-correlated (bottom plot) models. Here we have that $\mathcal{T}=\mathcal{T}^2$ as given in Chapter 6. This represents the maximum and minimum amount X_i may vary from X_0. Notice that there are instances of δ where we have that $\max_i\big(X^*(x,\mathcal{T},i)\big)=\min_i\big(X_*(x,\mathcal{T},i)\big)$. This indicates that if δ is calibrated to any of these values, there is only one $[t_0,t_0+T]\in\mathcal{T}$ where we observe a δ-movement. Parameters are as given in Sections 6.3 and 6.4.

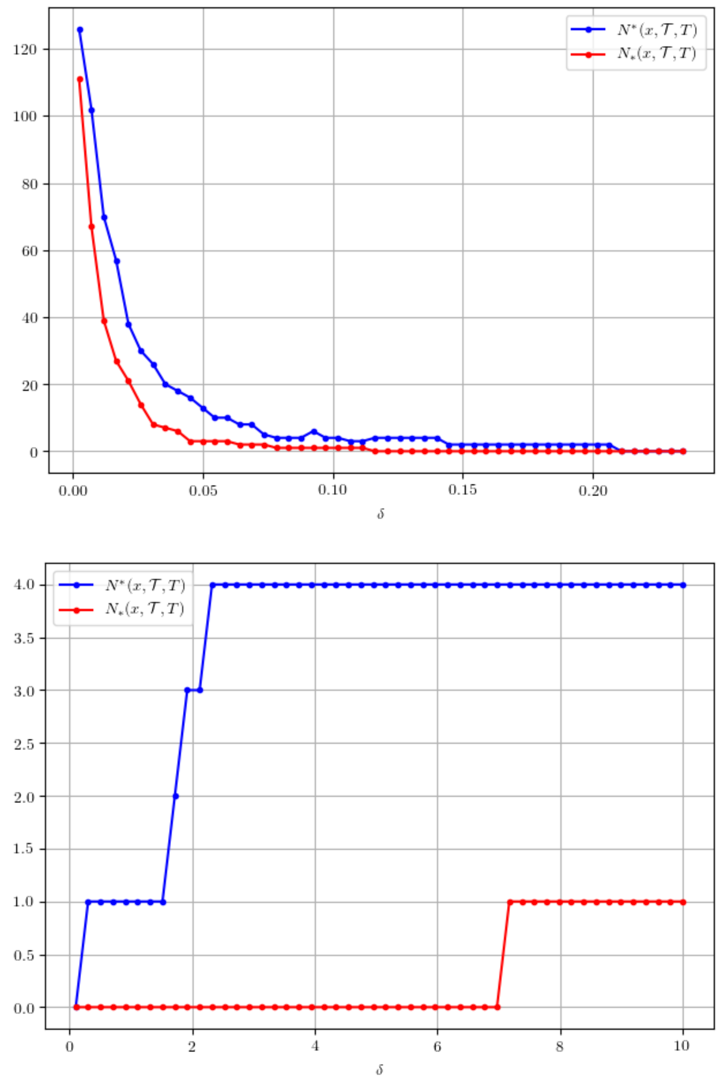


Figure 13: $N^*(x, \mathcal{T}, T)$ and $N_*(x, \mathcal{T}, T)$ vs. δ for δ-uncorrelated (top plot) and δ-correlated (bottom plot) models. Here we have that $\mathcal{T} = \mathcal{T}^2$. Greater the value of $N^*(x, \mathcal{T}, T) - N_*(x, \mathcal{T}, T)$ in this figure provide more stable output as it does not restrain the future entirely. i.e. choosing $\delta = 1.0$ allows trajectories to reach time T with at least 8 rebalances but less than 18 rebalances. Parameters are as given in Sections 6.3 and 6.4.

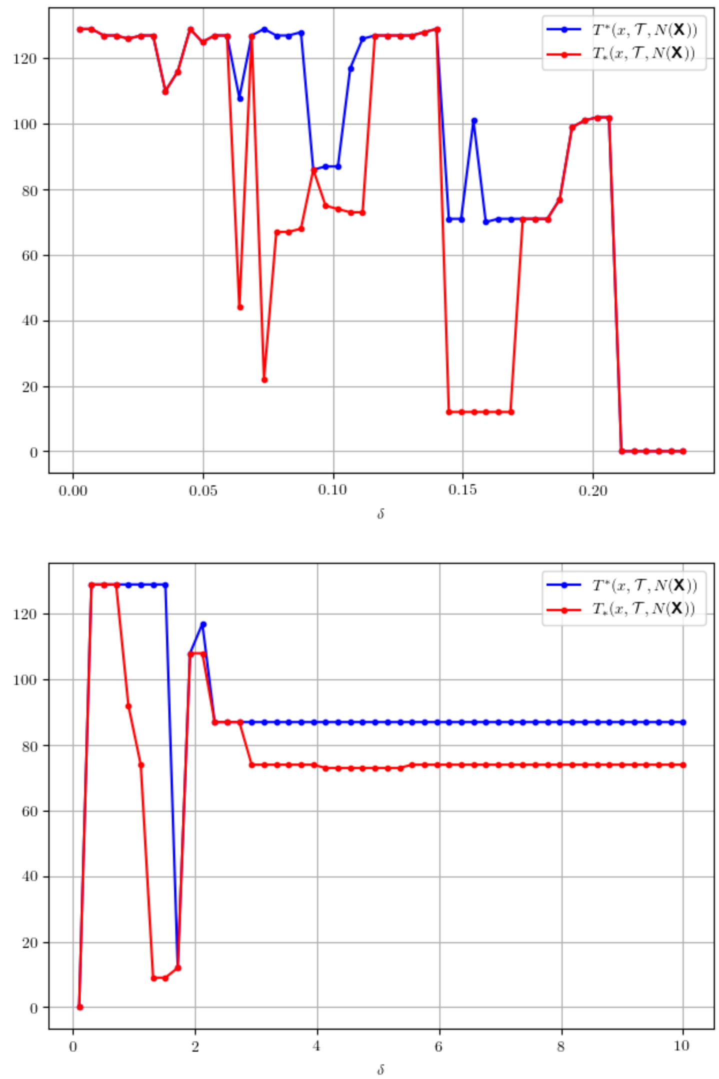


Figure 14: $T^*(x, \mathcal{T}, N(\mathbf{X}))$ and $T_*(x, \mathcal{T}, N(\mathbf{X}))$ vs. δ for δ-uncorrelated models. Here we have that $\mathcal{T} = \mathcal{T}^2$ as given in Chapter 6.

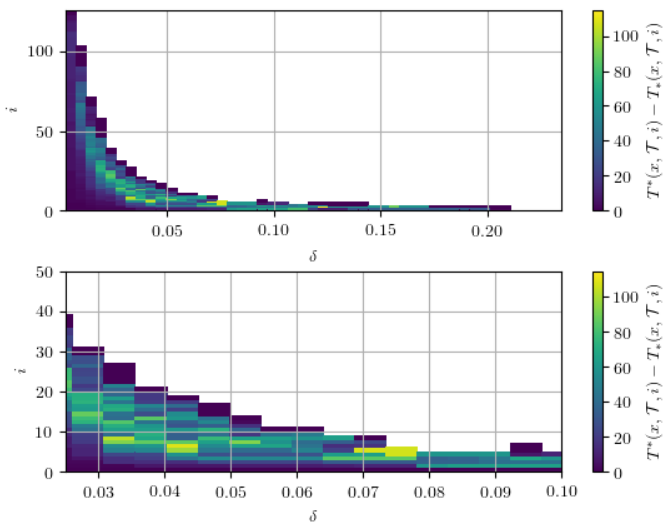


Figure 15: $\big(T^*(x,\mathcal{T},i) - T_*(x,\mathcal{T},i)\big)$ vs. δ for δ-uncorrelated models. Here we have that $\mathcal{T} = \mathcal{T}^2$ as given in Chapter 6. This plot shows how wide the pruning constraints $T^*(x,\mathcal{T},i)$ and $T_*(x,\mathcal{T},i)$ are at each δ-movement.

Output for $\mathcal{N}_E(x,\mathcal{T})$

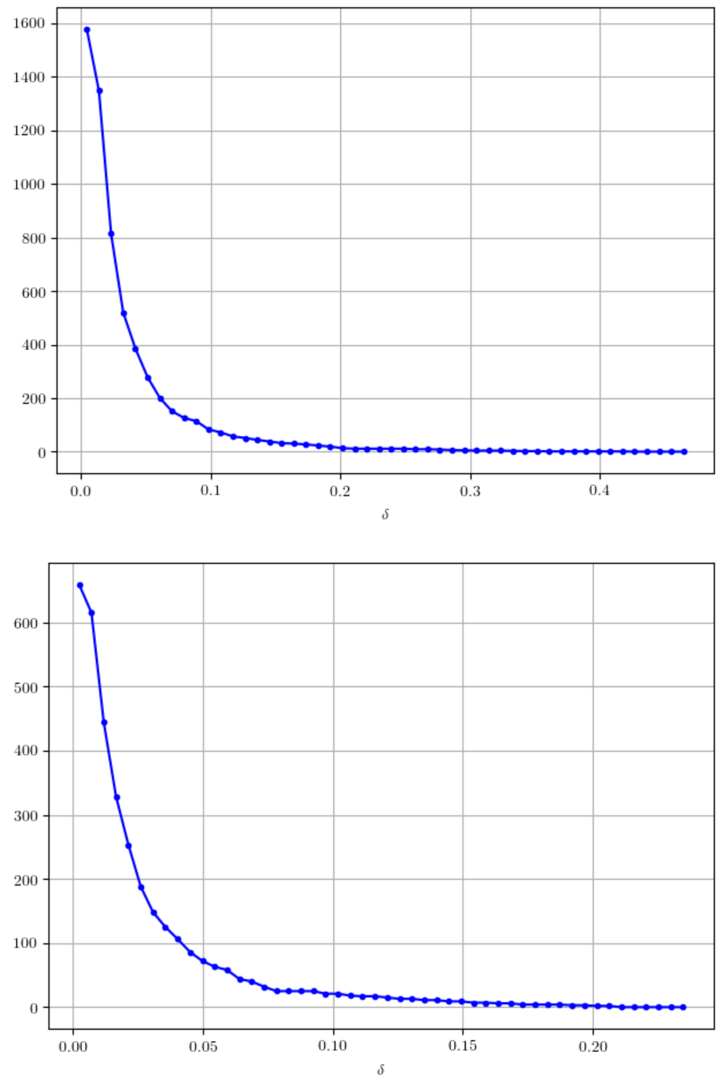
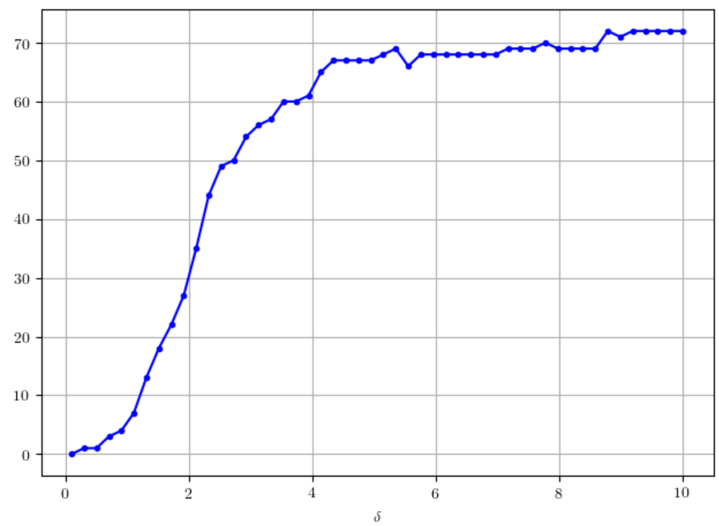


Figure 16: $|\mathcal{N}_E(x, \mathcal{T})|$ vs. δ for δ-uncorrelated models. We use two different time intervals of historical data in the plots provided. The first uses $\mathcal{T} = \mathcal{T}^1$ while the second plot uses $\mathcal{T} = \mathcal{T}^2$, where $\mathcal{T}^1$ and $\mathcal{T}^2$ are given in Chapter 6. Here we have $\delta \in [\delta_{min}, \delta_{max}]$ for each time interval used.



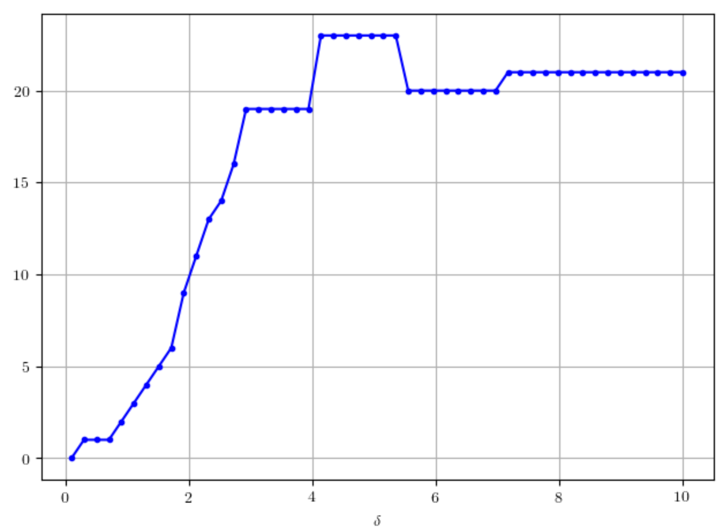


Figure 17: $|\mathcal{N}_E(x, \mathcal{T})|$ vs. δ for δ-correlated models. We use two different time intervals of historical data in the plots provided. The first uses $\mathcal{T} = \mathcal{T}^1$ while the second plot uses $\mathcal{T} = \mathcal{T}^2$, where $\mathcal{T}^1$ and $\mathcal{T}^2$ are given in Chapter 6. Here we have $\delta \in [0.01, 8.0]$ for each time interval used. It is shown in a figure in the appendix that the size $|\mathcal{N}_E(x, \mathcal{T})|$ stabilizes after a certain δ. Note that we have $\delta_{up} = \delta_{down} = \delta$ where δ represents the value along the x-axis in the figure.

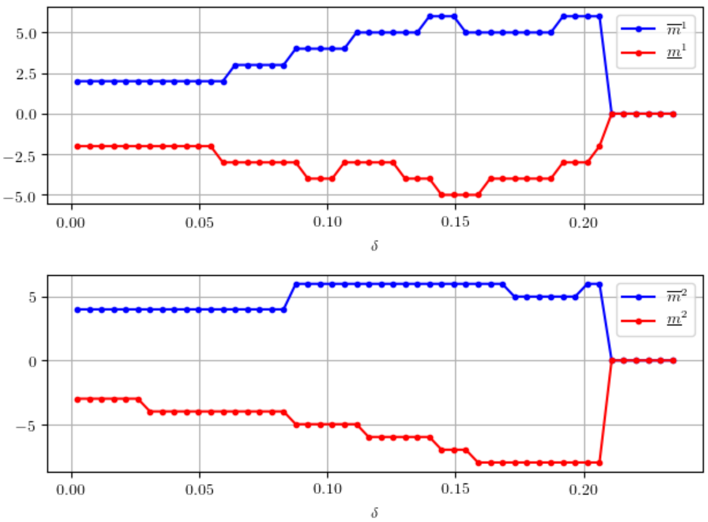


Figure 18: $\overline{m}^1$, $\underline{m}^1$, $\overline{m}^2$, and $\underline{m}^2$ vs. δ for δ-uncorrelated models.

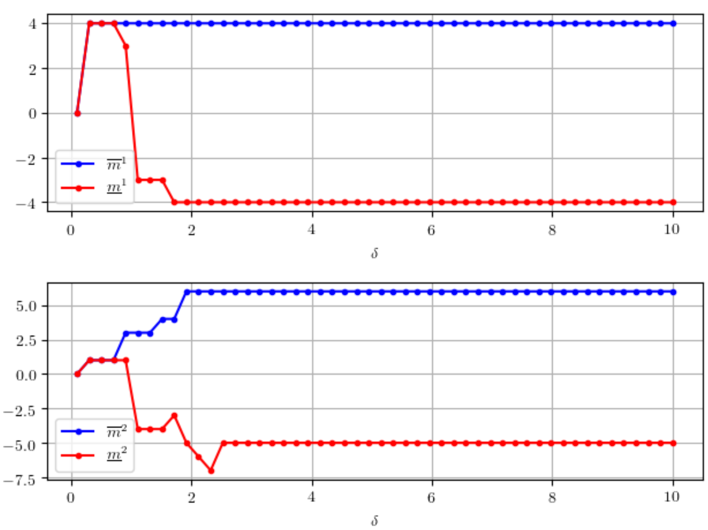


Figure 19: $\overline{m}^1$, $\underline{m}^1$, $\overline{m}^2$, and $\underline{m}^2$ vs. δ for δ-correlated models. Here we have that $\delta_{up} = \delta_{down} = \delta$ where δ represents the value along the x-axis in the figure.

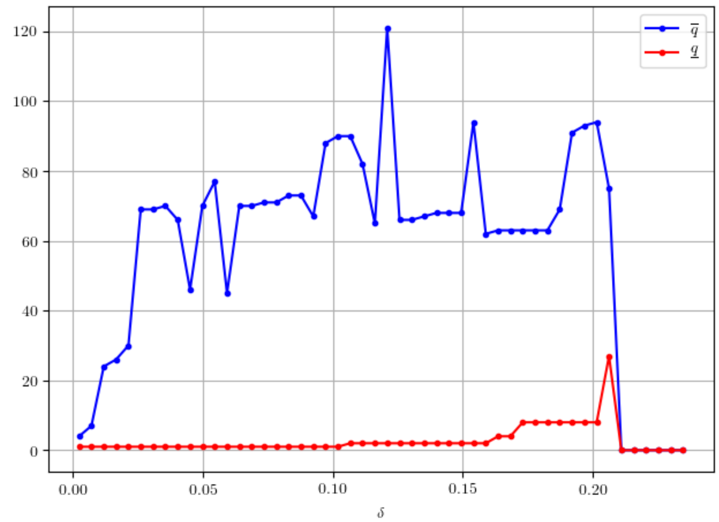


Figure 20: $\overline{q}$, and $\underline{q}$ vs. δ for δ-uncorrelated models.

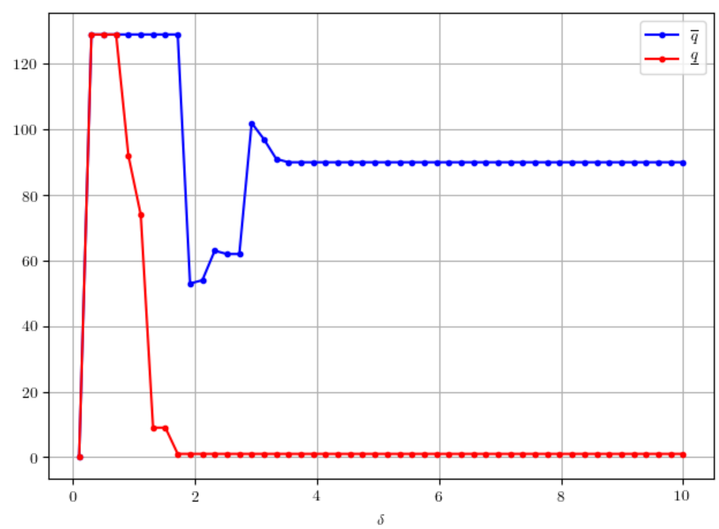


Figure 21: $\overline{q}$, and $\underline{q}$ vs. δ for δ-correlated models. Here we have that $\delta_{up} = \delta_{down} = \delta$ where δ represents the value along the x-axis in the figure.

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