LOCAL STABILITY ANALYSIS ON THE PREDATOR-PREY MODEL WITH INTRAGUILD PREDATION

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ABSTRACT

We consider the predator-prey system with a common consuming resource that was proposed by Holt and Polis in 1997 to introduce the the effects of intraguild predation in modelling community ecology. Some of the results suggest that strong intraguild predation can even foster the coexistence of species. In 2018, the spatiotemporal dynamics of the model proposed was further analyzed to illustrate the theoretical findings previously mentioned in 1997. In this thesis, we perform transformations to the system, in order to study a simplified equivalent system. The number of parameters is reduced without altering the biological meaning of the system or the dynamic behaviour. The local stability of the model is studied at each of the two positive boundary equilibria and at the positive interior equilibrium by finding the intervals of the parameters involved. The behaviour of the system will depend on which intervals the parameters fall. The emphasis is put on the ranges of the predation rate assuming, there is less that can be done to influence the parameters representing the natural birth and death rates of the

prey and predator. By using the qualitative theory for autonomous planar systems, we show under which conditions each positive boundary equilibria can be a saddle, saddle node, or stable, and the interior positive equilibrium is locally asymptotically stable. Under certain conditions the positive interior equilibrium is a stable node. It is interesting to note that when the consumption of the common resources are equal for the predator and prey species then we would be dealing with a symmetric system.

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Chapter 1

Introduction

1.1 Definitions and Notation

Consider the following autonomous dynamical system

$$\begin{cases} \dot{x}(t) &= f(x(t), y(t)), \\ \dot{y}(t) &= g(x(t), y(t)) \end{cases} \tag{1.1.1}$$

subject to the initial condition

$$(x(0), y(0)) = (x_0, y_0),$$

where $t \geq 0$ and $f, g : \mathbb{R}^2 \to \mathbb{R}$ are functions.

We denote by $\dot{x}$ the derivative of the function x related to time t, and by $C^1(\mathbb{R}^2)$ the Banach space of functions defined on $\mathbb{R}^2$ whose first-order partial derivatives are continuous on $\mathbb{R}^2$. We always assume that $f, g \in C^1(\mathbb{R}^2)$.

Definition 1.1.1. $(x(t), y(t))$ is said to be a *solution* of (1.1.1) if $x, y \in$

$C^1(\mathbb{R}^2)$ and satisfies both equations of (1.1.1). A solution $(x(t), y(t))$ is said to be positive if for all $t \geq 0$, $x(t), y(t) \geq 0$ [3].

Definition 1.1.2. $(x^*, y^*) \in \mathbb{R}^2$ is said to be an *equilibrium* of (1.1.1) if it satisfies $f(x^*, y^*) = 0$ and $g(x^*, y^*) = 0$. An equilibrium point (x^*, y^*) is said to be positive if both $x^*, y^* \geq 0$; a boundary, if $x^* = 0$ or $y^* = 0$; positive interior, if both $x^*, y^* > 0$ [3]. In other words, the (x^*, y^*) equilibrium solution is a boundary when the point lies on the x or y axis. In predator prey models, bounary equilibria refer to a situation where at least one of the species is extinct.

1.2 Local Stability Analysis

In order to analyze the local asymptotic stability near an equilibrium of a system of first order autonomous non-linear scalar differential equations, we can use the method of linearization [3], which is described below.

Consider the autonomous dynamical system of non-linear equations with two variables, x and y:

$$\begin{cases} \dot{x}(t) = f(x, y), \\ \dot{y}(t) = g(x, y). \end{cases} \tag{1.2.1}$$

Assume the system (1.2.1) has an equilibrium (x^*, y^*).

We may expand the functions f and g using the Taylor series centered at (x^*, y^*) where $u = x - x^*$ and $v = y - y^*$ [3]:

$$\frac{du}{dt} = f(x^*, y^*) + f_x(x^*, y^*)u + f_y(x^*, y^*)v + f_{xx}(x^*, y^*)\frac{u^2}{2} + f_{xy}(x^*, y^*)uv + \dots$$
$$\frac{dv}{dt} = g(x^*, y^*) + g_x(x^*, y^*)u + g_y(x^*, y^*)v + g_{xx}(x^*, y^*)\frac{u^2}{2} + g_{xy}(x^*, y^*)uv + \dots$$

Since $f(x^*, y^*) = 0$ and $g(x^*, y^*) = 0$ at equilibrium we have

$$\frac{du}{dt} = f_x(x^*, y^*)u + f_y(x^*, y^*)v + f_{xx}(x^*, y^*)\frac{u^2}{2} + f_{xy}(x^*, y^*)uv + \dots$$
$$\frac{dv}{dt} = g_x(x^*, y^*)u + g_y(x^*, y^*)v + g_{xx}(x^*, y^*)\frac{u^2}{2} + g_{xy}(x^*, y^*)uv + \dots$$

Assuming that $f_{xx}(x^*, y^*)\frac{u^2}{2} + f_{xy}(x^*, y^*)uv + ..$ and $g_{xx}(x^*, y^*)\frac{u^2}{2} + g_{xy}(x^*, y^*)uv$ is negligible for an approximation of the function f and g close to the equilibrium, we have

$$\frac{du}{dt} = f_x(x^*, y^*)u + f_y(x^*, y^*)v$$
$$\frac{dv}{dt} = g_x(x^*, y^*)u + g_y(x^*, y^*)v$$

Definition 1.2.1. The following system is said to be *linearized about the equilibrium* (x^*, y^*):

$$\frac{d\vec{X}}{dt} = A\vec{X}$$

where $\vec{X} = (u, v)^T$ and A is the Jacobian matrix evaluated at the equilibrium (x^*, y^*) [3].

We denote by $A(x, y)$ the Jacobian matrix of f and g at (x, y), that is,

$$A(x,y) = \begin{pmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \\ \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} \end{pmatrix} \tag{1.2.2}$$

and by $|A(x, y)|$ and $\mathrm{tr}(A(x, y))$ its determinant and trace, respectively [3].

The eigenvalues of $A(x, y)$ are determined by the roots of the characteristic polynomial:

$$P(\lambda) = \lambda^2 - \mathrm{tr}\,(x, y)\lambda + |A(x, y)|$$

The behaviour of the solutions of a linear system can be studied near the equilibrium (x^*, y^*) by the eigenvalues, λ_1, λ_2, of $A(x^*, y^*)$. Eigenvalues are not limited to real numbers.

Definition 1.2.2. An equilibrium is called a *node* when the eigenvalues are real numbers with the same sign and may be distinct or equal, $\lambda_1 \leq \lambda_2 < 0$ or $0 < \lambda_1 \leq \lambda_2$ [3]. A node is called *stable* when solutions approach the equilibrium as $t \to \infty$: in this case, $\lambda_1 \leq \lambda_2 < 0$. It is called an *unstable node* when the solution does not converge as $t \to \infty$; in this case, $0 < \lambda_1 \leq \lambda_2$.

Definition 1.2.3. An equilibrium is called a *saddle* when the eigenvalues are real numbers with opposite signs [3]. Such that $\lambda_1\lambda_2 < 0$, e.g. $\lambda_1 < 0 < \lambda_2$. Solutions will approach a saddle point initially, but in general solutions will not stay near a saddle point over time.

Definition 1.2.4. An equilibrium is called *locally asymptotically stable* when eigenvalues are negative or have negative real parts [3]. In this case, solutions approach the equilibrium as $t \to \infty$.

The results stated in local stability analysis section apply to non-linear planar systems of first order differential equations. The results require that the system may be linearized about an equilibrium, in this case the non-linear system behaves similiarly to a linear system, with some exceptions. The results have been commonly used to study the local stability of biological models based on non-linear systems, for example, in [38, 40, 8, 24, 7, 19, 39, 26]. We later will use these qualitative theories when discussing local stabilities. [42]

1.3 The Proposed Model

In this thesis we consider the following predator-prey model:

$$\begin{cases} \dot{N} = N\big(\frac{bs}{cP+sN} - dP - h\big), \\ \dot{P} = P\big(\frac{bc}{cP+sN} + dN - g\big). \end{cases} \tag{1.3.1}$$

$N(t)$ and $P(t)$ represent the densities of the prey and predator respectively. The parameters b, c and s represent the consumption of the predator and the prey species for common resources. Coefficient d measures the predation rate, g and h are the natural death rates of the predator and the prey respectively. All the parameters are positive constants.

Chapter 2

Positive Equilibria

We first use the following transformation:

$$\alpha = \frac{c}{s}$$

to change system (1.3.1) into the following equivalent system:

$$\begin{cases} \dot{x} = x\left(\frac{\beta}{\alpha y + x} - \gamma y - \delta\right) := f(x,y), \\ \dot{y} = y\left(\frac{\alpha\beta}{\alpha y + x} + \gamma x - \sigma\right) := g(x,y), \end{cases} \tag{2.0.1}$$

where $\beta = b$, $\gamma = d$, $\delta = h$, $\sigma = g$, $x(t) = N(t)$ and $y(t) = P(t)$ have the same biological meanings as b, d, h, g, $N(t)$ and $P(t)$. From (1.3.1), we see that under the above transformation, the constant s is normalized to one, in system (2.0.1). Note that (1.3.1) is reduced from 6 parameters to 5 which helps symplify the analysis of the model.

Proof. Multiplying the numerator and denominator by $1/s$ results in the following terms:

$$\begin{cases} \frac{bs}{cP+sN} = \frac{b(s/s)}{(c/s)P+(s/s)N} = \frac{b}{(c/s)P+N}, \\ \frac{bc}{cP+sN} = \frac{b(c/s)}{(c/s)P+(s/s)N} = \frac{b(c/s)}{(c/s)P+N}. \end{cases} \tag{2.0.2}$$

By (2.0.2) and with the new variables $\alpha = \frac{c}{s}$, $\beta = b$, $\gamma = d$, $\delta = h$, $\sigma = g$, $x(t) = N(t)$ and $y(t) = P(t)$, it follows that the model represented by (1.3.1) is reperesented by the equivalent model (2.0.1). $\square$

Using suitable transformations to reduce the number of parameters in predator-prey models and SIR models (Susceptible, Infectious, or Recovered in epidemiology models) has been widely used in [38, 24, 39, 22, 19, 7, 8, 40, 26].

Recall that $(x, y) \in \mathbb{R}^2$ is an equilibrium point of (2.0.1) if it satisfies $f(x, y) = 0$ and $g(x, y) = 0$. An equilibrium point (x, y) is said to be positive if $x, y \geq 0$ and to be a positive interior equilibrium point if $x, y > 0$. It is easy to verify that (x, y) is an equilibrium point of (2.0.1) if and only if (x, y) satisfies

$$\begin{cases} x\big(\frac{\beta}{\alpha y+x} - \gamma y - \delta\big) = 0, \\ y\big(\frac{\alpha\beta}{\alpha y+x} + \gamma x - \sigma\big) = 0. \end{cases} \tag{2.0.3}$$

The following notation for the reocurring combination of parameters will be used throughout the thesis and in the Theorems:

$$\sigma_0 = \frac{\alpha\delta}{1-\alpha}, \qquad \gamma_0 = \frac{\delta}{\beta}(\sigma - \alpha\delta),$$
$$\gamma_1 = \frac{\delta}{\beta}\big(\sigma - \alpha\delta + \delta\big), \quad \gamma_2 = \frac{\sigma}{\alpha\beta}(\sigma - \alpha\delta - \alpha\sigma), \quad \gamma_3 = \frac{\sigma}{\alpha\beta}(\sigma - \alpha\delta).$$

Theorem 2.0.1. Suppose $\alpha > 0$, $\beta > 0$, $\delta > 0$.

(1) If one of the following conditions holds,

(*i*) $\sigma > \alpha\delta$ and $0 < \gamma \leq \gamma_0$,

(*ii*) $0 < \sigma \leq \alpha\delta$ and $\gamma > 0$,

(*iii*) $\sigma > \alpha\delta$ and $\gamma \geq \gamma_3$,

then (2.0.1) has two positive equilibria $(x_1, y_1) = (\beta/\delta, 0)$ and $(x_2, y_2) = (0, \beta/\sigma)$.

(2) If $\sigma > \alpha\delta$ and $\gamma_0 < \gamma < \gamma_3$,

then (2.0.1) has three positive equilibria (x_1, y_1), (x_2, y_2) and (x^*, y^*), where

$$x^* = \frac{\alpha\beta}{\gamma(\sigma - \alpha\delta)}\big(\gamma_3 - \gamma\big) \quad \textit{and} \quad y^* = \frac{\beta}{\gamma(\sigma - \alpha\delta)}\big(\gamma - \gamma_0\big).$$

Proof. Note that the trivial solution, $x = 0$ and $y = 0$, is not taken into consideration because the first term in the equation $\frac{\beta}{\alpha y + x}$ would be undefined. It is easy to verify that if either $x = 0$ or $y = 0$, then $(0, \beta/\sigma)$ or $(\beta/\delta, 0)$ respectively, satisfy (2.0.3) and thus are equilibria of (2.0.1).

Suppose that $y = 0$ and $x \neq 0$, from the first equation of (2.0.3) we obtain,

$$\frac{\beta}{x} - \delta = 0 \qquad\qquad x = \frac{\beta}{\delta}$$

It follows that $(x_1, y_1) = (\beta/\delta, 0)$ is a solution of (2.0.3).

Suppose that $x = 0$ and $y \neq 0$, from the second equation of (2.0.3) we obtain,

$$\frac{\alpha\beta}{\alpha y} - \sigma = 0 \qquad\qquad \frac{\beta}{y} = \sigma \qquad\qquad y = \frac{\beta}{\sigma}$$

It follows that $(x_2, y_2) = (0, \beta/\sigma)$ is a solution of (2.0.3).

Suppose $x > 0$, $y > 0$ and $\sigma - \alpha\delta \neq 0$. We show that x^* and y^* satisfy (2.0.3).

Since, $\alpha \neq 0$, multiplying the second equation of (2.0.3) by $1/\alpha$ yields,

$$\begin{cases} \frac{\beta}{\alpha y + x} - \gamma y - \delta = 0, \\ \frac{\beta}{\alpha y + x} + \frac{\gamma}{\alpha} x - \frac{\sigma}{\alpha} = 0. \end{cases} \tag{2.0.4}$$

Subtracting, the first equation from the second equation of (2.0.4) we obtain

$$\frac{\beta}{\alpha y + x} + \frac{\gamma}{\alpha} x - \frac{\sigma}{\alpha} - \left(\frac{\beta}{\alpha y + x} - \gamma y - \delta \right) = 0.$$

Simplifying results in the following expression for y

$$\frac{\gamma}{\alpha} x - \frac{\sigma}{\alpha} + \gamma y + \delta = 0, \quad \gamma y = \frac{\sigma}{\alpha} - \delta - \frac{\gamma}{\alpha} x, \quad y = y^* = \frac{\sigma}{\alpha\gamma} - \frac{\delta}{\gamma} - \frac{1}{\alpha} x. \tag{2.0.5}$$

From the first equation of (2.0.4), multiplying both sides by $(\alpha y + x)$ we have,

$$\beta - (\gamma y + \delta)(\alpha y + x) = 0. \tag{2.0.6}$$

Substituting (2.0.5) into (2.0.6),

$$\begin{aligned} \beta - \left[\gamma \left(\frac{\sigma}{\alpha\gamma} - \frac{\delta}{\gamma} - \frac{1}{\alpha} x \right) + \delta \right] \left[\alpha \left(\frac{\sigma}{\alpha\gamma} - \frac{\delta}{\gamma} - \frac{1}{\alpha} x \right) + x \right] &= 0 \\ \beta - \gamma \left(\frac{\sigma}{\alpha\gamma} - \frac{1}{\alpha} x \right) \alpha \left(\frac{\sigma}{\alpha\gamma} - \frac{\delta}{\gamma} \right) &= 0 \\ \beta - \frac{\gamma}{\alpha} \left(\frac{\sigma}{\gamma} - x \right) \frac{\alpha}{\gamma} \left(\frac{\sigma}{\alpha} - \delta \right) &= 0 \\ \beta - \left(\frac{\sigma}{\gamma} - x \right) \left(\frac{\sigma - \alpha\delta}{\alpha} \right) &= 0. \end{aligned} \tag{2.0.7}$$

Solving for x from (2.0.7), together with $\sigma - \alpha\delta \neq 0$ yields

$$\beta = \left(\frac{\sigma}{\gamma} - x\right)\left(\frac{\sigma - \alpha\delta}{\alpha}\right), \qquad \frac{\alpha\beta}{\sigma - \alpha\delta} = \frac{\sigma}{\gamma} - x, \qquad x = \frac{\sigma}{\gamma} - \frac{\alpha\beta}{\sigma - \alpha\delta}.$$

Thus the expression we find for x is the following

$$x = x^* = \frac{1}{\gamma}\left(\sigma - \frac{\alpha\beta\gamma}{\sigma - \alpha\delta}\right). \tag{2.0.8}$$

We will use the expression for x^* in the following forms,

$$\begin{aligned} x^* &= \frac{\alpha\beta}{\gamma(\sigma - \alpha\delta)}\left(\frac{\sigma}{\alpha\beta}(\sigma - \alpha\delta) - \gamma\right), && (2.0.9) \\ &= \frac{\alpha\beta}{\gamma(\sigma - \alpha\delta)}\big(\gamma_3 - \gamma\big). && (2.0.10) \end{aligned}$$

Substituting (2.0.8) into (2.0.5), we have,

$$\begin{aligned} y^* &= \frac{\sigma}{\alpha\gamma} - \frac{\delta}{\gamma} - \frac{1}{\alpha}x^* \\ &= \frac{\sigma - \alpha\delta}{\alpha\gamma} - \frac{1}{\alpha\gamma}\left(\sigma - \frac{\alpha\beta\gamma}{\sigma - \alpha\delta}\right) \\ &= \frac{1}{\alpha\gamma}\left(\sigma - \alpha\delta - \sigma + \frac{\alpha\beta\gamma}{\sigma - \alpha\delta}\right) \\ &= \frac{1}{\gamma}\left(\frac{\beta\gamma}{\sigma - \alpha\delta} - \delta\right). \end{aligned} \tag{2.0.11}$$

From (2.0.11),

$$\begin{aligned} y^* &= \frac{\beta}{\gamma(\sigma - \alpha\delta)}\left(\gamma - \frac{\delta}{\beta}(\sigma - \alpha\delta)\right) && (2.0.12) \\ &= \frac{\beta}{\gamma(\sigma - \alpha\delta)}\big(\gamma - \gamma_0\big). && (2.0.13) \end{aligned}$$

Therefore,

$$(x^*, y^*) = \left(\frac{\alpha\beta}{\gamma(\sigma - \alpha\delta)}(\gamma_3 - \gamma), \frac{\beta}{\gamma(\sigma - \alpha\delta)}(\gamma - \gamma_0)\right),$$

satisfies (2.0.3) and thus is an equilibrium point of (2.0.1).

(1) Under conditions (i), we have $\sigma > \alpha\delta$ and $0 < \gamma \leq \gamma_0$ and under conditions (ii) we have $0 < \sigma \leq \alpha\delta$ and $\gamma > 0$. By (2.0.13), and (2.0.12) respectively, it follows that $y^* \leq 0$. Similarly, if (iii) holds, then $\sigma > \alpha\delta$ and $\gamma \geq \gamma_3$. By (2.0.10) we have $x^* \leq 0$. So, (x^*, y^*) is not a positive equilibria under (1). Then, system (2.0.1) has only two positive equilibria points (x_1, y_1) and (x_2, y_2).

(2) Under conditions (2) we have, $\sigma - \alpha\delta > 0$, and $\gamma_0 < \gamma < \gamma_3$. By (2.0.10) we have $x^* > 0$ and by (2.0.13), we have $y^* > 0$. Hence, (x^*, y^*) is a positive equilibrium point of (2.0.1). Thus, (2.0.1) has three positive equilibria, (x_1, y_1), (x_2, y_2) and (x^*, y^*). $\square$

Chapter 3

Local Stability Analysis

In this section, we analyze the local stability of each positive equilibrium of (2.0.1) by using the well-known qualitative theory for autonomous planar systems [41, 42].

We recall some well-known results on local stability of the following system:

$$\begin{cases} \dot{x}(t) = f(x(t), y(t)), \\ \dot{y}(t) = g(x(t), y(t)), \end{cases} \tag{3.0.1}$$

where $f, g \in C^1(\mathbb{R}^2)$. We denote by $A(x, y)$ the Jacobian matrix of f and g at (x, y), that is,

$$A(x, y) = \begin{pmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \\ \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} \end{pmatrix} \tag{3.0.2}$$

and by $|A(x, y)|$ and $\mathrm{tr}(A(x, y))$ its determinant and trace, respectively.

The following results have been widely employed to study the local stability and phase portraits for predator-prey models, for example, in [7, 8, 19, 24, 26] and susceptible-infective-removed epidemic models in [38, 39, 40].

Lemma 3.0.1. Let (x^*, y^*) be an equilibrium of (3.0.1). Then the following assertions hold.

(i) If $|A(x^*, y^*)| < 0$, then (x^*, y^*) is a saddle.

(ii) If $|A(x^*, y^*)| > 0$ and $(\mathrm{tr}(A(x^*, y^*)))^2 - 4|A(x^*, y^*)| \geq 0$, then (x^*, y^*) is a node. It is stable if $\mathrm{tr}(A(x^*, y^*)) < 0$ and unstable if $\mathrm{tr}(A(x^*, y^*)) > 0$.

(iii) Assume that $|A(x^*, y^*)| > 0$. If $\mathrm{tr}(A(x^*, y^*)) < 0$, then (x^*, y^*) is locally asymptotically stable; if $\mathrm{tr}(A(x^*, y^*)) = 0$, then it is stable and if $\mathrm{tr}(A(x^*, y^*)) > 0$, then it is unstable.

A map $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (f(x, y), g(x, y))$ is said to be regular if T is one to one and onto, T and T^{-1} are continuous and $|A(x, y)| \neq 0$ on $\mathbb{R}^2$. If T is regular, then the following transformation

$$\begin{cases} x_1 = f(x, y), \\ y_1 = g(x, y) \end{cases} \tag{3.0.3}$$

is said to be a regular transformation. If (3.0.1) is changed into another system under suitable regular transformations, then the two systems are said to be equivalent. It is known (for example see [41]) that under regular transformations, the topological structures of solutions of a planar system near equilibria including a variety of dynamics like saddles, topological saddles, nodes, saddle nodes, foci, centers, or cusps remain unchanged.

We need the following result which was proved in [19, Proposition 3.2].

Lemma 3.0.2. Let (x^*, y^*) be an equilibrium of (3.0.1). Assume that $|A(x^*, y^*)| = 0$, $\mathrm{tr}(A(x^*, y^*)) \neq 0$ and (3.0.1) is equivalent to the following system

$$\begin{cases} \dot{x}_1 = p(x_1, y_1), \\ \dot{y}_1 = \varrho y_1 + q(y_1, x_1) \end{cases} \tag{3.0.4}$$

with an isolated equilibrium $(0,0)$, where $p(x_1, y_1) = \sum_{i+j=2, i,j\geq 0}^{\infty} a_{ij} x_1^i y_1^j$ and $q(x_1, y_1) = \sum_{i+j=2, i,j\geq 0}^{\infty} b_{ij} x_1^i y_1^j$ are convergent power series. If $\varrho \neq 0$ and $a_{20} \neq 0$, then (x^*, y^*) is a saddle node.

A function $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ is said to be *analytic* in an open set Ω if it has a convergent Taylor series in some neighborhood of each point in Ω (see [42, p.69]).

When $|A(x^*, y^*)| = \mathrm{tr}(A(x^*, y^*)) = 0$ and $A(x^*, y^*) \neq 0$, under suitable regular transformations, (3.0.1) is equivalent to the following form

$$\begin{cases} \dot{x} = y, \\ \dot{y} = a_k x^k [1 + h(x)] + b_n x^n y [1 + g(x)] + y^2 R(x, y) \end{cases} \tag{3.0.5}$$

with equilibrium $(0,0)$, where h, g and R are analytic in a neighborhood of $(0,0)$, $h(0) = g(0) = 0$, $k \geq 2$, $a_k \neq 0$ and $n \in \mathbb{N}$.

Lemma 3.0.3 ([41, 42])**.** Let (x^*, y^*) be an equilibrium of (3.0.1) and $|A(x^*, y^*)| = \mathrm{tr}(A(x^*, y^*)) = 0$ and $A(x^*, y^*) \neq 0$. If (3.0.1) is equivalent to (3.0.5), $k = 2m+1 \in \mathbb{N}$ and $a_k > 0$, then (x^*, y^*) is a topological saddle.

Now, we use the above theoretical results to study phase portraits near each of the positive equilibria of (2.0.1) obtained in section 2.

By (2.0.1) and (3.0.2), we have

$$A(x,y) = \begin{pmatrix} \frac{\alpha\beta y}{(x+\alpha y)^2} - \gamma y - \delta & -\frac{\alpha\beta x}{(x+\alpha y)^2} - \gamma x \\ \gamma y - \frac{\alpha\beta y}{(x+\alpha y)^2} & \frac{\alpha\beta x}{(x+\alpha y)^2} + \gamma x - \sigma \end{pmatrix}, \tag{3.0.6}$$

$$|A(x,y)| = -\frac{\alpha\beta(\delta x + \sigma y)}{(x+\alpha y)^2} + \gamma(\sigma y - \delta x) + \delta\sigma, \tag{3.0.7}$$

and

$$\mathrm{tr}(A(x,y)) = \frac{\alpha\beta(x+y)}{(x+\alpha y)^2} + \gamma(x-y) - (\delta+\sigma). \tag{3.0.8}$$

Proof. Taking the partial derivatives of the functions $f(x,y)$ and $g(x,y)$, defined in (2.0.1) we obtain,

$$\frac{\partial f}{\partial x} = \frac{\beta}{\alpha y + x} - \gamma y - \delta - x\left(\frac{\beta}{(\alpha y + x)^2}\right) \quad = \frac{\beta(\alpha y + x) - x\beta}{\alpha y + x} - \gamma y - \delta$$

$$= \frac{\alpha\beta y}{(x+\alpha y)^2} - \gamma y - \delta \tag{3.0.9}$$

$$\frac{\partial f}{\partial y} = x\left(-\frac{\alpha\beta}{(\alpha y + x)^2} - \gamma\right) \qquad = -\frac{\alpha\beta x}{(x+\alpha y)^2} - \gamma x \tag{3.0.10}$$

$$\frac{\partial g}{\partial x} = y\left(-\frac{\alpha\beta}{(\alpha y + x)^2} + \gamma\right) \qquad = \gamma y - \frac{\alpha\beta y}{(x+\alpha y)^2} \tag{3.0.11}$$

$$\frac{\partial g}{\partial y} = \frac{\alpha\beta}{\alpha y + x} + \gamma x - \sigma + y\left(-\frac{\alpha^2\beta}{(\alpha y + x)^2}\right) \quad = -\frac{\alpha\beta(x+\alpha y) - \alpha^2\beta y}{(\alpha y + x)^2} + \gamma x - \sigma$$

$$= \frac{\alpha\beta x}{(x+\alpha y)^2} + \gamma x - \sigma. \tag{3.0.12}$$

By (3.0.9), (3.0.10), (3.0.11) and (3.0.12), together with (3.0.2), it follows that the jacobian matrix of f and g at (x, y) is defined by (3.0.6).

By (3.0.6), we compute the determinant of $A(x, y)$,

$$\begin{aligned}
|A(x,y)| &= \left(\frac{\alpha\beta y}{(x+\alpha y)^2} - \gamma y - \delta\right)\left(\frac{\alpha\beta x}{(x+\alpha y)^2} + \gamma x - \sigma\right) \\
&- \left(\gamma y - \frac{\alpha\beta y}{(x+\alpha y)^2}\right)\left(-\frac{\alpha\beta x}{(x+\alpha y)^2} - \gamma x\right) \\
&= xy\Bigg[\left(\frac{\alpha\beta}{(x+\alpha y)^2} - \gamma - \delta/y\right)\left(\frac{\alpha\beta}{(x+\alpha y)^2} + \gamma - \sigma/x\right) \\
&- \left(\frac{\alpha\beta}{(x+\alpha y)^2} - \gamma\right)\left(\frac{\alpha\beta}{(x+\alpha y)^2} + \gamma\right)\Bigg] \\
&= xy\Bigg[\left(\frac{\alpha\beta}{(x+\alpha y)^2} - \gamma\right)\left(\frac{\alpha\beta}{(x+\alpha y)^2} + \gamma\right) - \sigma/x\left(\frac{\alpha\beta}{(x+\alpha y)^2} - \gamma\right) \\
&- \delta/y\left(\frac{\alpha\beta}{(x+\alpha y)^2} + \gamma\right) + \frac{\sigma\delta}{xy} - \left(\frac{\alpha\beta}{(x+\alpha y)^2} - \gamma\right)\left(\frac{\alpha\beta}{(x+\alpha y)^2} + \gamma\right)\Bigg] \\
&= xy\left[-\sigma/x\left(\frac{\alpha\beta}{(x+\alpha y)^2} - \gamma\right) - \delta/y\left(\frac{\alpha\beta}{(x+\alpha y)^2} + \gamma\right) + \frac{\sigma\delta}{xy}\right] \\
&= -\sigma y\left(\frac{\alpha\beta}{(x+\alpha y)^2} - \gamma\right) - \delta x\left(\frac{\alpha\beta}{(x+\alpha y)^2} + \gamma\right) + \sigma\delta \\
&= -\frac{\alpha\beta(\delta x + \sigma y)}{(x+\alpha y)^2} + \gamma(\sigma y - \delta x) + \delta\sigma,
\end{aligned}$$

and the trace of $A(x, y)$,

$$\begin{aligned}
\operatorname{tr}\big(A(x,y)\big) &= \frac{\alpha\beta y}{(x+\alpha y)^2} - \gamma y - \delta + \frac{\alpha\beta x}{(x+\alpha y)^2} + \gamma x - \sigma \\
&= \frac{\alpha\beta(x+y)}{(x+\alpha y)^2} + \gamma(x-y) - (\delta+\sigma).
\end{aligned}$$

$\square$

3.1 Analysis of the Positive Boundary Equilibria

Consider the equilibrium $(\beta/\delta, 0)$

Recall the notation $\gamma_1 = \frac{\delta}{\beta}(\sigma - \alpha\delta + \delta)$

Lemma 3.1.1. Suppose $\alpha > 0$, $\beta > 0$, $\delta > 0$, then the following assertions hold,

(1) If one of the following conditions holds,

(*i*) If $\alpha > 1$ and $\sigma > \delta(\alpha - 1)$ and $\gamma > \gamma_1$,

(*ii*) If $0 < \alpha \leq 1$ and $\sigma > 0$ and $\gamma > \gamma_1$,

(*iii*) If $\alpha > 1$ and $0 < \sigma \leq \delta(\alpha - 1)$ and $\gamma > 0$,

then $\mathrm{tr}((\beta/\delta, 0)) > 0$.

(2) If one of the following conditions holds,

(*i*) If $\alpha > 1$ and $\sigma > \delta(\alpha - 1)$ and $0 < \gamma < \gamma_1$,

(*ii*) If $0 < \alpha \leq 1$ and $\sigma > 0$ and $0 < \gamma < \gamma_1$,

then $\mathrm{tr}((\beta/\delta, 0)) < 0$.

(3) If one of the following conditions holds,

(*i*) If $\alpha > 1$ and $\sigma > \delta(\alpha - 1)$ and $\gamma = \gamma_1$,

(*ii*) If $0 < \alpha \leq 1$ and $\sigma > 0$ and $\gamma = \gamma_1$,

then, $\mathrm{tr}((\beta/\delta, 0)) = 0$.

Proof. By (3.0.8) with $(x, y) = (\beta/\delta, 0)$, we have

$$
\begin{aligned}
\mathrm{tr}(A(\beta/\delta,0)) &= \alpha\delta + \frac{\beta\gamma}{\delta} - (\sigma+\delta) \\
&= \frac{\beta}{\delta}\left[\gamma - \frac{\delta}{\beta}(\sigma-\alpha\delta) - \frac{\delta}{\beta}\alpha\delta\right] \\
&= \frac{\beta}{\delta}\left[\gamma - \frac{\delta}{\beta}(\sigma-\alpha\delta+\delta)\right] \qquad (3.1.1) \\
&= \frac{\beta}{\delta}(\gamma-\gamma_1). \qquad (3.1.2)
\end{aligned}
$$

(1) If (i) or (ii) holds, we have $\gamma > \gamma_1$. The result follows by (3.1.8). If (iii) holds, we have $\alpha > 1$ and $0 < \sigma \le \delta(\alpha-1)$ and $\gamma > 0$.

$$\gamma_1 = \frac{\delta}{\beta}(\sigma-\alpha\delta+\delta) = \frac{\delta}{\beta}(\sigma-\delta(\alpha-1))$$

Hence, if $\frac{\delta}{\beta}(\sigma-\delta(\alpha-1)) \le 0$, the result follows by (3.1.1).

(2) If (i) or (ii) holds, we have $0 < \gamma < \gamma_1$. The result folows by (3.1.8).

(3) If (i) or (ii) holds, we have $\gamma = \gamma_1$. The result follows by (3.1.8). □

Lemma 3.1.2. Suppose $\alpha > 0$, $\beta > 0$, $\delta > 0$, then the following assertions hold,

(1) If one of the following conditions holds,

(i) If $\sigma > \alpha\delta$ and $\gamma > \gamma_0$,

(ii) If $0 < \sigma \le \alpha\delta$ and $\gamma > 0$,

then $|A(\beta/\delta,0)| < 0$.

(2) If $\sigma > \alpha\delta$ and $0 < \gamma < \gamma_0$, then $|A(\beta/\delta,0)| > 0$.

(3) If $\sigma > \alpha\delta$ and $\gamma = \gamma_0$, then $|A(\beta/\delta,0)| = 0$.

Proof. By (3.0.7) with $(x,y) = (\beta/\delta,0)$ we have,

$$
\begin{aligned}
|A(\beta/\delta, 0)| &= -\alpha\delta^2 - \gamma\beta + \sigma\delta \\
&= \beta\left[\frac{\delta}{\beta}(\sigma - \alpha\delta) - \gamma\right] \qquad (3.1.3) \\
&= \beta(\gamma_0 - \gamma). \qquad (3.1.4)
\end{aligned}
$$

(1) If (i) or (ii) holds, then $\frac{\delta}{\beta}(\sigma - \alpha\delta) - \gamma < 0$. Hence, by (3.1.3), we have $|A(\beta/\delta, 0)| < 0$.

(2) Under conditions (2), we have $0 < \gamma < \gamma_0$. The result follows by (3.1.4).

(3) Since $\gamma = \gamma_0$, the result follows by (3.1.4).

□

Theorem 3.1.3. Suppose that $\alpha > 0$, $\beta > 0$ and $\delta > 0$.

(1) If one of the following conditions hold,

(i) If $\sigma > \alpha\delta$ and $\gamma > \gamma_0$,

(ii) If $0 < \sigma \leq \alpha\delta$ and $\gamma > 0$,

then the equilibrium point $(\beta/\delta, 0)$ is a saddle.

(2) If $\sigma > \alpha\delta$ and $0 < \gamma < \gamma_0$, then $(\beta/\delta, 0)$ is a stable node.

(3) If $\sigma > \alpha\delta$, and $\gamma = \gamma_0$, then the equilibrium $(\beta/\delta, 0)$ is a saddle-node.

Proof. By (3.0.7) with $(x, y) = (\beta/\delta, 0)$ we have,

$$
\begin{aligned}
|A(\beta/\delta, 0)| &= -\alpha\delta^2 - \gamma\beta + \sigma\delta \\
&= \beta\left[\frac{\delta}{\beta}(\sigma - \alpha\delta) - \gamma\right] \qquad (3.1.5) \\
&= \beta(\gamma_0 - \gamma). \qquad (3.1.6)
\end{aligned}
$$

(1) If (i) or (ii) holds, then $\frac{\delta}{\beta}(\sigma - \alpha\delta) - \gamma < 0$. By (3.1.5), we have $|A(\beta/\delta, 0)| < 0$. The result follows from Lemma (3.0.1) (i).

By (3.0.8) with $(x, y) = (\beta/\delta, 0)$, we have

$$
\begin{aligned}
\mathrm{tr}(A(\beta/\delta, 0)) &= \alpha\delta + \frac{\beta\gamma}{\delta} - (\sigma + \delta) \\
&= \frac{\beta}{\delta}\left[\gamma - \frac{\delta}{\beta}(\sigma - \alpha\delta) - \frac{\delta}{\beta}\alpha\delta\right] \\
&= \frac{\beta}{\delta}\left[\gamma - \frac{\delta}{\beta}(\sigma - \alpha\delta + \delta)\right] \qquad (3.1.7) \\
&= \frac{\beta}{\delta}(\gamma - \gamma_1). \qquad (3.1.8)
\end{aligned}
$$

Let

$$
\Delta(\beta/\delta, 0) = \mathrm{tr}^2(A(\beta/\delta, 0)) - 4|A(\beta/\delta, 0)|
$$

Then by (3.1.5) and (3.1.7) with $(x, y) = (\beta/\delta, 0)$, we have

$$\begin{aligned}
&\Delta(\beta/\delta, 0) = \frac{\beta^2}{\delta^2}\left[\gamma - \frac{\delta}{\beta}(\sigma - \alpha\delta + \delta)\right]^2 - 4\beta\left[\frac{\delta}{\beta}(\sigma - \alpha\delta) - \gamma\right] \\
&= \frac{1}{\delta^2}\left[(\alpha\delta^2 + \beta\gamma - \delta(\sigma + \delta))^2\right] + 4\beta\gamma - 4\delta(\sigma - \alpha\delta) \\
&= \frac{1}{\delta^2}\left[(\alpha\delta^2 + \beta\gamma - \delta(\sigma + \delta))^2 + 4\delta^2\beta\gamma - 4\delta^3(\sigma - \alpha\delta)\right] \\
&= \frac{1}{\delta^2}\left[(\alpha\delta^2 + \beta\gamma - \delta(\sigma + \delta))^2 + 4\delta^2\beta\gamma - 4\delta^3\sigma + 4\delta^2\alpha\delta^2)\right] \\
&= \frac{1}{\delta^2}\left[(\alpha\delta^2 + \beta\gamma)^2 - 2\delta(\sigma + \delta)(\alpha\delta^2 + \beta\gamma) + \delta^2(\sigma + \delta)^2 + 4\delta^2(\alpha\delta^2 + \beta\gamma) - 4\sigma\delta^3\right] \\
&= \frac{1}{\delta^2}\left[(\alpha\delta^2 + \beta\gamma)^2 + (4\delta^2 - 2\delta\sigma - 2\delta^2)(\alpha\delta^2 + \beta\gamma) + \delta^2(\sigma + \delta)^2 - 4\sigma\delta^3\right] \\
&= \frac{1}{\delta^2}\left[(\alpha\delta^2 + \beta\gamma)^2 + (2\delta^2 - 2\delta\sigma)(\alpha\delta^2 + \beta\gamma) + \delta^2((\sigma + \delta)^2 - 4\sigma\delta)\right] \\
&= \frac{1}{\delta^2}\left[(\alpha\delta^2 + \beta\gamma)^2 + (2\delta^2 - 2\delta\sigma)(\alpha\delta^2 + \beta\gamma) + \delta^2(\sigma^2 + 2\sigma\delta + \delta^2 - 4\sigma\delta)\right] \\
&= \frac{1}{\delta^2}\left[(\alpha\delta^2 + \beta\gamma)^2 + 2\delta(\delta - \sigma)(\alpha\delta^2 + \beta\gamma) + \delta^2(\sigma^2 - 2\sigma\delta + \delta^2)\right] \\
&= \frac{1}{\delta^2}\left[(\alpha\delta^2 + \beta\gamma)^2 + 2\delta(\delta - \sigma)(\alpha\delta^2 + \beta\gamma) + \delta^2(\sigma - \delta)^2\right] \\
&= \frac{1}{\delta^2}\left[(\alpha\delta^2 + \beta\gamma) + \delta(\sigma - \delta)\right]^2 \geq 0.
\end{aligned} \tag{3.1.9}$$

(2) Under conditions (2) we have $0 < \gamma < \gamma_0 < \gamma_1$, by (3.1.6), we have $|A(\beta/\delta, 0)| > 0$ and by (3.1.8) we have $\mathrm{tr}(A(\beta/\delta, 0) < 0$. This together with (3.1.9) and Lemma 3.0.1 (ii) imply the result.

(3) Since $\sigma > \alpha\delta$ and $\gamma = \gamma_0$. By (3.1.6) we have $|A(\beta/\delta, 0)| = 0$.

By (3.1.7) we obtain:

$$\mathrm{tr}((\beta/\delta,0);\gamma_0) = \alpha\delta + \frac{\gamma_1\beta}{\delta} - \sigma - \delta \qquad = \alpha\delta + \frac{\delta(\sigma-\alpha\delta)}{\beta}\frac{\beta}{\delta} - \sigma - \delta$$
$$= \alpha\delta + (\sigma - \alpha\delta) - \sigma - \delta \qquad = -\delta$$

thus, $\mathrm{tr}\left(A(\beta/\delta,0)\right) \neq 0$.

To apply Lemma 3.0.2, we change the equilibrium $(\beta/\delta,0)$ to the origin $(0,0)$ by using the change of variables $x_1 = x - \beta_1$ and $y_1 = y$. Noting that $\beta_1 = \beta/\delta$, Then the first and second equation of (2.0.1), respectively, becomes:

$$\begin{cases} \dot{x}_1 = (\beta_1 + x_1)\left(\frac{\beta}{\alpha y_1+\beta_1+x_1} - \gamma y_1 - \delta\right) \\ \dot{y}_1 = y_1\left(\frac{\alpha\beta}{\alpha y_1+\beta_1+x_1} + \gamma(\beta_1+x_1) - \sigma\right) \end{cases}$$

$$\dot{x}_1 = \frac{\beta(\beta_1+x_1)}{\beta_1+x_1+\alpha y_1} - \gamma(\beta_1+x)y_1 - \delta(\beta_1+x)$$
$$\dot{y}_1 = \frac{\alpha\beta y_1}{\beta_1+x_1+\alpha y_1} + \gamma(\beta_1+x_1)y_1 - \sigma y_1$$

Note, that The tayor series expansion $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ can be used to approximate f and g around an equilibrium $(0,0)$. We have

$$\frac{1}{\beta_1+x_1+\alpha y_1} = \frac{1/\beta_1}{1-(-\frac{x_1+\alpha y_1}{\beta_1})} = \frac{1}{\beta_1}\sum_{n=0}^{\infty}(-1)^n\left(\frac{x_1+\alpha y_1}{\beta_1}\right)^n$$

Thus,

$$\begin{aligned}\frac{\alpha\beta y_1}{\beta_1 + x_1 + \alpha y_1} &= \alpha\beta/\beta_1 y_1 \sum_{n=0}^{\infty}(-1)^n\left(\frac{x_1+\alpha y_1}{\beta_1}\right)^n \\ &= \alpha\delta y_1\left[1 - \frac{x_1+\alpha y_1}{\beta_1} + \sum_{n=2}^{\infty}(-1)^n\left(\frac{x_1+\alpha y_1}{\beta_1}\right)^n\right]\end{aligned}$$

and

$$\frac{\beta(\beta_1 + x_1)}{\beta_1 + x_1 + \alpha y_1} = \frac{\beta(\beta_1 + x_1 + \alpha y_1) - \alpha\beta y_1}{\beta_1 + x_1 + \alpha y_1} \qquad = \beta - \frac{\alpha\beta y_1}{\beta_1 + x_1 + \alpha y_1}$$
$$= \beta - \alpha\delta y_1\left[1 - \frac{x_1+\alpha y_1}{\beta_1} + \sum_{n=2}^{\infty}(-1)^n\left(\frac{x_1+\alpha y_1}{\beta_1}\right)^n\right].$$

Hence,

$$
\begin{aligned}
\dot{x}_1 &= \frac{\beta(\beta_1 + x_1)}{\beta_1 + x_1 + \alpha y_1} - \gamma(\beta_1 + x)y_1 - \delta(\beta_1 + x) \\
&= \beta - \alpha\delta y_1\left[1 - \frac{x_1 + \alpha y_1}{\beta_1} + \sum_{n=2}^{\infty}(-1)^n\left(\frac{x_1 + \alpha y_1}{\beta_1}\right)^n\right] \\
&- \gamma(\beta_1 + x_1)y_1 - \delta(\beta_1 + x_1) \\
&= \beta - \alpha\delta y_1 + \alpha\delta y_1\left(\frac{x_1 + \alpha y_1}{\beta/\delta}\right) \\
&- \alpha\delta y_1\sum_{n=2}^{\infty}(-1)^n\left(\frac{x_1 + \alpha y_1}{\beta_1}\right)^n - \gamma(\beta_1 + x_1)y_1 - \delta(\beta_1 + x_1) \\
&= \beta - \alpha\delta y_1 + \frac{\alpha\delta^2}{\beta}x_1 y_1 + \frac{\alpha^2\delta^2}{\beta}y_1^2 - \gamma\beta_1 y_1 - \gamma x_1 y_1 - \delta\beta/\delta - \delta x_1 \\
&- \alpha\delta y_1\sum_{n=2}^{\infty}(-1)^n\left(\frac{x_1 + \alpha y_1}{\beta_1}\right)^n \\
&= \frac{\alpha^2\delta^2}{\beta}y_1^2 - \left(\gamma - \frac{\alpha\delta^2}{\beta}\right)x_1 y_1 - (\gamma\beta_1 + \alpha\delta)y_1 - \delta x_1 \\
&- \alpha\delta y_1\sum_{n=2}^{\infty}(-1)^n\left(\frac{x_1 + \alpha y_1}{\beta_1}\right)^n,
\end{aligned}
$$

and after substituting $\gamma = \gamma_0$

$$
\dot{x}_1 = \frac{\alpha^2\delta^2}{\beta}y_1^2 - \left(\frac{\delta\sigma - 2\alpha\delta^2}{\beta}\right)x_1 y_1 - (\delta x_1 + \sigma y_1) - \alpha\delta y_1\sum_{n=2}^{\infty}(-1)^n\left(\frac{x_1 + \alpha y_1}{\beta_1}\right)^n.
$$

Similarly,

$$
\begin{aligned}
\dot{y}_1 &= \frac{\alpha\beta y_1}{\beta_1 + x_1 + \alpha y_1} + \gamma(\beta_1 + x_1)y_1 - \sigma y_1 \\
&= \alpha\delta y_1 \left[1 - \frac{x_1 + \alpha y_1}{\beta_1} + \sum_{n=2}^{\infty} (-1)^n \left(\frac{x_1 + \alpha y_1}{\beta_1} \right)^n \right] + \gamma(\beta_1 + x_1)y_1 - \sigma y_1 \\
&= \alpha\delta y_1 - \frac{\alpha\delta^2}{\beta} y_1(x_1 + \alpha y_1) + \gamma(\beta_1 + x_1)y_1 - \sigma y_1 + \alpha\delta y_1 \sum_{n=2}^{\infty} (-1)^n \left(\frac{x_1 + \alpha y_1}{\beta_1} \right)^n \\
&= -\frac{\alpha^2\delta^2}{\beta} y_1^2 + \left(\gamma - \frac{\alpha\delta^2}{\beta} \right) x_1 y_1 + (\alpha\delta - \sigma + \gamma\beta_1)y_1 + \alpha\delta y_1 \sum_{n=2}^{\infty} (-1)^n \left(\frac{x_1 + \alpha y_1}{\beta_1} \right)^n,
\end{aligned}
$$

and after substituting $\gamma = \gamma_0$

$$
\dot{y}_1 = -\frac{\alpha^2\delta^2}{\beta} y_1^2 + \left(\frac{\delta\sigma - 2\alpha\delta^2}{\beta} \right) x_1 y_1 + \alpha\delta y_1 \sum_{n=2}^{\infty} (-1)^n \left(\frac{x_1 + \alpha y_1}{\beta_1} \right)^n.
$$

Let $x_2 = \delta x_1 + \sigma y_1$ and $y_2 = y_1$.

$$
\begin{aligned}
\dot{x}_2 &= \delta \dot{x}_1 + \sigma \dot{y}_1 \\
&= \frac{\alpha^2\delta^3}{\beta} y_1^2 - \delta\left(\frac{\delta\sigma - 2\alpha\delta^2}{\beta}\right)x_1y_1 - \delta(\delta x_1 + \sigma y_1) - \alpha\delta^2 y_1 \sum_{n=2}^{\infty}(-1)^n\left(\frac{x_1+\alpha y_1}{\beta_1}\right)^n \\
&\quad - \frac{\alpha^2\delta^2\sigma}{\beta} y_1^2 + \sigma\left(\frac{\delta\sigma - 2\alpha\delta^2}{\beta}\right)x_1y_1 + \alpha\delta\sigma y_1 \sum_{n=2}^{\infty}(-1)^n\left(\frac{x_1+\alpha y_1}{\beta_1}\right)^n \\
&= \frac{\alpha^2\delta^2}{\beta}(\delta - \sigma)y_1^2 + \left(\frac{\delta\sigma - 2\alpha\delta^2}{\beta}\right)(\sigma - \delta)x_1y_1 - \delta(\delta x_1 + \sigma y_1) \\
&\quad + (\sigma - \delta)\alpha\delta y_1 \sum_{n=2}^{\infty}(-1)^n\left(\frac{x_1+\alpha y_1}{\beta_1}\right)^n \\
&= \frac{\alpha^2\delta^2}{\beta}(\delta - \sigma)y_2^2 + \left(\frac{\delta\sigma - 2\alpha\delta^2}{\beta}\right)(\sigma - \delta)\left(\frac{x_2 - \sigma y_2}{\delta}\right)y_2 - \delta x_2 \\
&\quad + (\sigma - \delta)\alpha\delta y_2 \sum_{n=2}^{\infty}(-1)^n\left(\frac{\frac{x_2 - \sigma y_2}{\delta} + \alpha y_1}{\beta/\delta}\right)^n \\
&= \frac{\alpha^2\delta^2}{\beta}(\delta - \sigma)y_2^2 - \frac{(\sigma - \delta)\sigma(\sigma - 2\alpha\delta)}{\beta}y_2^2 + \frac{(\sigma - \delta)(\sigma - 2\alpha\delta)}{\beta}x_2y_2 - \delta x_2 \\
&\quad + (\sigma - \delta)\alpha\delta y_2 \sum_{n=2}^{\infty}(-1)^n\left(\frac{x_2 - (\sigma - \alpha\delta)y_1}{\beta}\right)^n \\
&= \frac{(\delta - \sigma)}{\beta}\left(\sigma(\sigma - 2\alpha\delta) + \alpha^2\delta^2\right)y_2^2 + \frac{(\sigma - \delta)(\sigma - 2\alpha\delta)}{\beta}x_2y_2 - \delta x_2 \\
&\quad + (\sigma - \delta)\alpha\delta y_2 \sum_{n=2}^{\infty}(-1)^n\left(\frac{x_2 - (\sigma - \alpha\delta)y_1}{\beta}\right)^n \\
&= \frac{(\delta - \sigma)(\sigma - \alpha\delta)^2}{\beta}y_2^2 + \frac{(\sigma - \delta)(\sigma - 2\alpha\delta)}{\beta}x_2y_2 - \delta x_2 \\
&\quad + (\sigma - \delta)\alpha\delta y_2 \sum_{n=2}^{\infty}(-1)^n\left(\frac{x_2 - (\sigma - \alpha\delta)y_1}{\beta}\right)^n
\end{aligned}
$$

and

$$
\begin{aligned}
\dot{y}_2 &= \dot{y}_1 \\
&= -\frac{\alpha^2\delta^2}{\beta} y_2^2 + \left(\frac{\delta\sigma - 2\alpha\delta^2}{\beta}\right)\left(\frac{x_2 - \sigma y_2}{\delta}\right) y_2 + \alpha\delta y_2 \sum_{n=2}^{\infty} (-1)^n \left(\frac{\frac{x_2 - \sigma y_2}{\delta} + \alpha y_1}{\beta/\delta}\right)^n \\
&= -\frac{\alpha^2\delta^2}{\beta} y_2^2 - \left(\frac{\delta\sigma - 2\alpha\delta^2}{\beta}\right)\frac{\sigma}{\delta} y_2^2 + \left(\frac{\delta\sigma - 2\alpha\delta^2}{\delta\beta}\right) x_2 y_2 + \alpha\delta y_2 \sum_{n=2}^{\infty} (-1)^n \left(\frac{x_2 - (\sigma - \alpha\delta) y_1}{\beta}\right)^n \\
&= -\frac{\alpha^2\delta^2}{\beta} y_2^2 - \left(\frac{\sigma - 2\alpha\delta}{\beta}\right)\sigma y_2^2 + \left(\frac{\sigma - 2\alpha\delta}{\beta}\right) x_2 y_2 + \alpha\delta y_2 \sum_{n=2}^{\infty} (-1)^n \left(\frac{x_2 - (\sigma - \alpha\delta) y_1}{\beta}\right)^n \\
&= \left(\frac{-\alpha^2\delta^2 - \sigma^2 + 2\alpha\delta\sigma}{\beta}\right) y_2^2 + \left(\frac{\sigma - 2\alpha\delta}{\beta}\right) x_2 y_2 \\
&\quad + \alpha\delta y_2 \sum_{n=2}^{\infty} (-1)^n \left(\frac{x_2 - (\sigma - \alpha\delta) y_1}{\beta}\right)^n \\
&= -\frac{(\sigma - \alpha\delta)^2}{\beta} y_2^2 + \left(\frac{\sigma - 2\alpha\delta}{\beta}\right) x_2 y_2 + \alpha\delta y_2 \sum_{n=2}^{\infty} (-1)^n \left(\frac{x_2 - (\sigma - \alpha\delta) y_1}{\beta}\right)^n
\end{aligned}
$$

Let

$$
\begin{aligned}
p(x_2, y_2) := \frac{(\delta - \sigma)(\sigma - \alpha\delta)^2}{\beta} y_2^2 &+ \frac{(\sigma - \delta)(\sigma - 2\alpha\delta)}{\beta} x_2 y_2 \\
&+ (\sigma - \delta)\alpha\delta y_2 \sum_{n=2}^{\infty} (-1)^n \left(\frac{x_2 - (\sigma - \alpha\delta) y_1}{\beta}\right)^n
\end{aligned}
$$

and

$$q(x_2,y_2) := -\frac{(\sigma-\alpha\delta)^2}{\beta}y_2^2 + \left(\frac{\sigma-2\alpha\delta}{\beta}\right)x_2y_2 + \alpha\delta y_2\sum_{n=2}^{\infty}(-1)^n\left(\frac{x_2-(\sigma-\alpha\delta)y_2}{\beta}\right)^n$$

then (2.0.1) can be, after the above transformations, written as

$$\begin{cases} \dot{x}_2 = -\delta x_2 + p(x_2,y_2) \\ \dot{y}_2 = q(x_2,y_2) \end{cases} \tag{3.1.10}$$

Let $x_3 = y_2$ and $y_3 = x_2$, then

$$\begin{cases} \dot{x}_3 = q(y_3,x_3) \\ \dot{y}_3 = -\delta y_3 + p(y_3,x_3) \end{cases} \tag{3.1.11}$$

From our definition of q we have that $a_{20} = -\frac{(\sigma-\alpha\delta)^2}{\beta} \neq 0$ since, $\sigma > \alpha\delta$. Moreover, $\varrho = -\delta \neq 0$. Thus, by Lemma 3.0.2 the result follows.

□

3.2 Analysis of the Second Positive Boundary Equilibrium

Consider the equilibrium point $(0, \beta/\sigma)$

By (3.0.6), (3.0.7) and (3.0.8) at $(0,\beta/\sigma)$ we have:

$$A(0,\beta/\sigma) = \begin{pmatrix} \frac{\sigma}{\alpha} - \frac{\beta\gamma}{\sigma} - \delta & 0 \\ -\frac{\sigma}{\alpha} + \frac{\beta\gamma}{\sigma} & -\sigma \end{pmatrix}, \tag{3.2.1}$$

$$|A(0, \beta/\sigma)| = -\frac{\sigma^2}{\alpha} + \gamma\beta + \delta\sigma,$$

and

$$\mathrm{tr}(0, \beta/\sigma) = \frac{\sigma}{\alpha} - \frac{\beta\gamma}{\sigma} - \delta - \sigma$$

respectively.

Lemma 3.2.1. Suppose $\alpha > 0$, $\beta > 0$, $\delta > 0$, then the following assertions hold.

(1) If $\sigma > \alpha\delta$ and $0 < \gamma < \gamma_3$, then $|A(0, \beta/\sigma)| < 0$.

(2) If one of the following conditions hold,

(*i*) If $\sigma > \alpha\delta$ and $\gamma > \gamma_3$,

(*ii*) If $0 < \sigma \leq \alpha\delta$ and $\gamma > 0$,

then $|A(0, \beta/\sigma)| > 0$.

(3) If $\sigma > \alpha\delta$ and $\gamma = \gamma_3$, then $|A((0, \beta/\sigma)| = 0$.

Proof. By (3.2.1) we obtain:

$$\begin{aligned} |A(0, \beta/\sigma)| &= -\sigma\left(\frac{\sigma}{\alpha} - \frac{\beta\gamma}{\sigma} - \delta\right) && = -\frac{\sigma^2}{\alpha} + \gamma\beta + \delta\sigma \\ &= \beta\left(\gamma - \frac{\sigma}{\alpha\beta}(\sigma - \alpha\delta)\right) && = \beta(\gamma - \gamma_3). \end{aligned}$$

(1) Under condition (1) we have $\sigma > \alpha\delta$ and $0 < \gamma < \gamma_3$. The result follows by (3.2.6).

(2) If (i) holds we have $\sigma > \alpha\delta$ and $\gamma > \gamma_3$ The result follows by (3.2.6). Similarly, If (ii) holds $0 < \sigma \leq \alpha\delta$ and $\gamma > 0$, the result follows by (3.2.5).

(3) we have $\sigma > \alpha\delta$ and $\gamma = \gamma_3$, the result follows by (3.2.6). □

Lemma 3.2.2. Suppose $\beta > 0$, $\delta > 0$, then the following assertions hold.

(1) If one of the following conditions hold,

(i) If $\alpha \geq 1$ and $\sigma > 0$ and $\gamma > 0$,

(ii) If $0 < \alpha < 1$ and $0 < \sigma \leq \sigma_0$ and $\gamma > 0$,

(iii) If $0 < \alpha < 1$ and $\sigma > \sigma_0$ and $\gamma > \gamma_2$,

then $\operatorname{tr}\big(A(0, \beta/\sigma)\big) < 0$.

(2) If $0 < \alpha < 1$ and $\sigma > \sigma_0$ and $0 < \gamma < \gamma_2$, then $\operatorname{tr}\big(A(0, \beta/\sigma)\big) > 0$.

(3) If $0 < \alpha < 1$ and $\sigma > \sigma_0$ and $\gamma = \gamma_2$, then $\operatorname{tr}\big(A(0, \beta/\sigma)\big) = 0$.

Proof. By (3.0.8) with $(x, y) = (0, \beta/\sigma)$, we have

$$\operatorname{tr}(0, \beta/\sigma) = \tfrac{\sigma}{\alpha} - \tfrac{\beta\gamma}{\sigma} - \delta - \sigma = \tfrac{\beta}{\sigma}\bigg(\tfrac{\sigma}{\alpha\beta}(\sigma - \alpha\delta - \alpha\sigma) - \gamma\bigg) = \tfrac{\beta}{\sigma}\big(\gamma_2 - \gamma\big)$$

$$\begin{aligned} \operatorname{tr}\big(A(0, \beta/\sigma)\big) &= \frac{\sigma}{\alpha} - \frac{\beta\gamma}{\sigma} - \delta - \sigma \\ &= \frac{\sigma}{\alpha}(1 - \alpha) - (\delta + \frac{\beta}{\sigma}\gamma) \end{aligned} \tag{3.2.2}$$

$$\begin{aligned} \operatorname{tr}\big(A(0, \beta/\sigma)\big) &= \frac{\beta}{\sigma}\bigg(\frac{\sigma}{\alpha\beta}(\sigma - \alpha\delta - \alpha\sigma) - \gamma\bigg) \\ &= \frac{\beta}{\sigma}\bigg(\frac{\sigma}{\alpha\beta}(\sigma(1 - \alpha) - \alpha\delta) - \gamma\bigg) \\ &= \frac{\beta}{\sigma}\bigg(\frac{\sigma(1 - \alpha)}{\alpha\beta}(\sigma - \frac{\alpha\delta}{(1 - \alpha)}) - \gamma\bigg) \\ &= \frac{\beta}{\sigma}\bigg(\frac{\sigma(1 - \alpha)}{\alpha\beta}(\sigma - \sigma_0) - \gamma\bigg) \end{aligned} \tag{3.2.3}$$

(1) If (i) holds, we have $\alpha \geq 1$ and $\sigma > 0$ and $\gamma > 0$. It follows that by (3.2.2), tr $\big(A(0, \beta/\sigma)\big) < 0$. If (ii) holds we have $0 < \alpha < 1$ and $0 < \sigma \leq \sigma_0$ and $\gamma > 0$. It follows that by (3.2.3), tr $\big(A(0, \beta/\sigma)\big) < 0$. If (iii) holds $\gamma > \gamma_2$, the result follows by (3.2.9).

(2) Since $\gamma < \gamma_2$ the result follows by (3.2.9).

(3) Since $\gamma = \gamma_2$ the result follows by (3.2.9). □

Theorem 3.2.3. Suppose that $\alpha > 0$, $\beta > 0$ and $\delta > 0$, then the following assertions hold.

(1) If $\sigma > \alpha\delta$ and $0 < \gamma < \gamma_3$,

then the equilibrium point $(0, \beta/\sigma)$ is a saddle of (2.0.1).

(2) If one of the following conditions hold,

(i) If $\sigma > \alpha\delta$ and $\gamma > \gamma_3$,

(ii) If $0 < \sigma \leq \alpha\delta$ and $\gamma > 0$,

then the equilibrium point $(0, \beta/\sigma)$ is a stable node of (2.0.1).

(3) If $\sigma > \alpha\delta$ and $\gamma = \gamma_3$,

then the equilibrium point $(0, \beta/\sigma)$ is a saddle node of (2.0.1).

Proof. By (3.0.7) with $(x, y) = (0, \beta/\sigma)$ we have,

$$
\begin{aligned}
|A(0, \beta/\sigma)| &= -\frac{\sigma^2}{\alpha} + \gamma\beta + \delta\sigma && (3.2.4) \\
&= \beta\left(\gamma - \frac{\sigma}{\alpha\beta}(\sigma - \alpha\delta)\right) && (3.2.5) \\
&= \beta(\gamma - \gamma_3) && (3.2.6)
\end{aligned}
$$

(1) Since $\gamma - \gamma_3 < 0$. By (3.2.6), we have $|A(0, \beta/\sigma)| < 0$. The result follows from Lemma (3.0.1) (i).

By (3.0.8) with $(x, y) = (0, \beta/\sigma)$, we have

$$\begin{aligned}
\operatorname{tr}\big(A(0,\beta/\sigma)\big) &= \frac{\sigma}{\alpha} - \frac{\gamma\beta}{\sigma} - \sigma - \delta && (3.2.7)\\
&= \frac{\beta}{\sigma}\left[\frac{\sigma}{\alpha\beta}(\sigma - \alpha\delta - \alpha\sigma) - \gamma\right] && (3.2.8)\\
&= \frac{\beta}{\sigma}\big(\gamma_2 - \gamma\big). && (3.2.9)
\end{aligned}$$

Let

$$\Delta(0, \beta/\sigma) = \operatorname{tr}^2\big(A(0,\beta/\sigma)\big) - 4|A(0,\beta/\sigma)|$$

Then by (3.2.4) and (3.2.7) with $(x, y) = (0, \beta/\sigma)$, we have

$$
\begin{aligned}
\Delta(0,\beta/\sigma) &= \Big(-\sigma+\frac{\sigma}{\alpha}-\frac{\gamma\beta}{\sigma}-\delta\Big)^2-4(-\frac{\sigma^2}{\alpha}+\gamma\beta+\delta\sigma)\\
&= \frac{1}{\sigma^2}\Big(-\sigma^2+\frac{\sigma^2}{\alpha}-\gamma\beta-\delta\sigma\Big)^2-4(-\frac{\sigma^2}{\alpha}+\gamma\beta+\delta\sigma)\\
&= \frac{1}{\sigma^2}\Big((-\sigma^2+\frac{\sigma^2}{\alpha})-(\gamma\beta+\delta\sigma)\Big)^2-4(\gamma\beta+\delta\sigma)+4\frac{\sigma^2}{\alpha}\\
&= \frac{1}{\sigma^2}\Big((\gamma\beta-\delta\sigma)^2-2(-\sigma^2+\frac{\sigma^2}{\alpha})(\gamma\beta-\delta\sigma)+(-\sigma^2+\frac{\sigma^2}{\alpha})^2\Big)-4(\gamma\beta+\delta\sigma)+4\frac{\sigma^2}{\alpha}\\
&= \frac{1}{\sigma^2}\Big[(\gamma\beta-\delta\sigma)^2+(2\sigma^2-2\frac{\sigma^2}{\alpha})(\gamma\beta-\delta\sigma)+(-\sigma^2+\frac{\sigma^2}{\alpha})^2-4\sigma^2(\gamma\beta+\delta\sigma)+4\frac{\sigma^4}{\alpha}\Big]\\
&= \frac{1}{\sigma^2}\Big[(\gamma\beta-\delta\sigma)^2+(2\sigma^2-2\frac{\sigma^2}{\alpha}-4\sigma^2)(\gamma\beta-\delta\sigma)+(-\sigma^2+\frac{\sigma^2}{\alpha})^2+4\frac{\sigma^4}{\alpha}\Big]\\
&= \frac{1}{\sigma^2}\Big[(\gamma\beta-\delta\sigma)^2+(-2\sigma^2-2\frac{\sigma^2}{\alpha})(\gamma\beta-\delta\sigma)+\sigma^4(1-\frac{1}{\alpha})^2+4\frac{\sigma^4}{\alpha}\Big]\\
&= \frac{1}{\sigma^2}\Big[(\gamma\beta-\delta\sigma)^2-2\sigma^2(1+\frac{1}{\alpha})(\gamma\beta-\delta\sigma)+\sigma^4(1+\frac{1}{\alpha})^2\Big]\\
&= \frac{1}{\sigma^2}\Big[(\gamma\beta-\delta\sigma)-\sigma^2(1+\frac{1}{\alpha})\Big]^2\geq 0
\end{aligned}
\tag{3.2.10}
$$

(2) Under conditions (i) we have $\gamma>\gamma_3>\gamma_2$. By (3.2.6), and (3.2.9) we have $|A(0,\beta/\sigma)|>0$ and $\operatorname{tr}\big(A(0,\beta/\sigma)\big)<0$ respectively. This together with (3.2.10) and Lemma 3.0.1 (ii) imply the result. Similarly under conditions (ii), we have $0<\sigma\leq\alpha\delta$ and $\gamma>0$. By (3.2.5) and (3.2.8), we have $|A(0,\beta/\sigma)|>0$ and $\operatorname{tr}\big(A(0,\beta/\sigma)\big)<0$ respectively. The result follows by (3.2.10) and Lemma 3.0.1 (ii).

(3) Since $\sigma>\alpha\delta$ and $\gamma=\gamma_3$. By (3.2.6) we have $|A(0,\beta/\sigma)|=0$.

By (3.2.8) we obtain:

$$
\begin{aligned}
\operatorname{tr}\big(A(0,\beta/\sigma)\big) &= \frac{\beta}{\sigma}\left[\frac{\sigma}{\alpha\beta}(\sigma-\alpha\delta-\alpha\sigma)-\frac{\sigma}{\alpha\beta}(\sigma-\alpha\delta)\right] \\
&= \frac{\beta}{\sigma}\left(\frac{\sigma}{\alpha\beta}(-\alpha\sigma)\right) \\
&= -\sigma,
\end{aligned} \tag{3.2.11}
$$

thus $\operatorname{tr}\big(A(0,\beta/\sigma)\big) \neq 0$.

To apply Lemma 3.0.2, we change the equilibrium $(0,\beta/\sigma)$ to the origin $(0,0)$ by using the change of variables $x = x_1$ and $y = y_1 + \beta_2$. Noting that $\beta_2 = \beta/\sigma$, the system (2.0.1) becomes:

$$
\begin{cases}
\dot{x}_1 = x_1\left(\frac{\beta}{\alpha(y_1+\beta_2)+x_1} - \gamma(y_1+\beta_2) - \delta\right) \\
\dot{y}_1 = (y_1+\beta_2)\left(\frac{\alpha\beta}{\alpha(y_1+\beta_2)+x} + \gamma x - \sigma\right)
\end{cases}
$$

$$
\begin{cases}
\dot{x}_1 = \frac{\beta x_1}{x_1+\alpha(y_1+\beta_2)} - \gamma(y_1+\beta_2)x_1 - \delta x_1 \\
\dot{y}_1 = \frac{\alpha\beta(y_1+\beta_2)}{x_1+\alpha(y_1+\beta_2)} + \gamma(y_1+\beta_2)x_1 - \sigma(y_1+\beta_2)
\end{cases} \tag{3.2.12}
$$

Note that

$$
\frac{1}{x_1+\alpha y_1+\alpha\beta_2} = \frac{1}{\alpha\beta_2}\sum_{n=0}^{\infty}\left(\frac{x_1+\alpha y_1}{\alpha\beta_2}\right)^n \tag{3.2.13}
$$

Thus,

$$
\begin{aligned}
\frac{\alpha\beta(y_1+\beta_2)}{\alpha\beta_2+x_1+\alpha y_1} &= \beta - \frac{\beta x_1}{\alpha\beta_2+x_1+\alpha y_1} \\
&= \beta - \frac{\sigma}{\alpha}x_1\sum_{n=0}^{\infty}(-1)^n\left(\frac{x_1+\alpha y_1}{\alpha\beta_2}\right)^n \\
&= \beta - \frac{\sigma}{\alpha}x_1\left[1 - \frac{x_1+\alpha y_1}{\alpha\beta_2} + \sum_{n=2}^{\infty}(-1)^n\left(\frac{x_1+\alpha y_1}{\alpha\beta_2}\right)^n\right]
\end{aligned}
$$

and

$$\frac{\beta x_1}{\alpha\beta_2 + x_1 + \alpha y_1} = \frac{\sigma}{\alpha}x_1\left[1 - \frac{x_1 + \alpha y_1}{\alpha\beta_2} + \sum_{n=2}^{\infty}(-1)^n\left(\frac{x_1 + \alpha y_1}{\alpha\beta_2}\right)^n\right].$$

Hence the second equation of (3.2.12) becomes,

$$\begin{aligned}
\dot{y}_1 &= \frac{\alpha\beta(y_1 + \beta_2)}{\alpha\beta_2 + x_1 + \alpha y_1} + \gamma(\beta_2 + y_1)x_1 - \sigma(\beta_2 + y_1) \\
&= \beta - \frac{\sigma}{\alpha}x_1\left[1 - \frac{x_1 + \alpha y_1}{\alpha\beta_2} + \sum_{n=2}^{\infty}(-1)^n\left(\frac{x_1 + \alpha y_1}{\alpha\beta_2}\right)^n\right] + \gamma(\beta_2 + y_1)x_1 - \sigma(\beta_2 + y_1) \\
&= \beta - \frac{\sigma}{\alpha}x_1 + \frac{\sigma^2}{\alpha\beta}x_1y_1 + \frac{\sigma^2}{\alpha^2\beta}x_1^2 + \gamma(\beta_2 + y_1)x_1 - \sigma(\beta_2 + y_1) - \frac{\sigma}{\alpha}x_1\sum_{n=2}^{\infty}(-1)^n\left(\frac{x_1 + \alpha y_1}{\alpha\beta_2}\right)^n \\
&= \frac{\sigma^2}{\alpha^2\beta}x_1^2 + (\gamma + \frac{\sigma^2}{\alpha\beta})x_1y_1 + (\gamma\beta_2 - \frac{\sigma}{\alpha})x_1 - \sigma y_1 - \frac{\sigma}{\alpha}x_1\sum_{n=2}^{\infty}(-1)^n\left(\frac{x_1 + \alpha y_1}{\alpha\beta_2}\right)^n
\end{aligned}$$

Since $\gamma = \gamma_3 = \frac{\sigma}{\alpha\beta}(\sigma - \alpha\delta) = \frac{\sigma^2}{\alpha\beta} - \frac{\delta\sigma}{\beta}$

$$\begin{aligned}
\dot{y}_1 &= \frac{\sigma^2}{\alpha^2\beta}x_1^2 + \left(\frac{\sigma^2}{\alpha\beta} - \frac{\delta\sigma}{\beta} + \frac{\sigma^2}{\alpha\beta}\right)x_1y_1 + \left(\frac{\sigma}{\alpha\beta}(\sigma - \alpha\delta)\frac{\beta}{\sigma} - \frac{\sigma}{\alpha}\right)x_1 - \sigma y_1 \\
&\quad - \frac{\sigma}{\alpha}x_1\sum_{n=2}^{\infty}(-1)^n\left(\frac{x_1 + \alpha y_1}{\alpha\beta_2}\right)^n \\
&= \frac{\sigma^2}{\alpha^2\beta}x_1^2 + \left(\frac{2\sigma^2 - \alpha\delta\sigma}{\alpha\beta}\right)x_1y_1 - (\delta x_1 + \sigma y_1) - \frac{\sigma}{\alpha}x_1\sum_{n=2}^{\infty}(-1)^n\left(\frac{x_1 + \alpha y_1}{\alpha\beta_2}\right)^n
\end{aligned}$$

and

$$\frac{\beta x_1}{\alpha\beta_2 + x_1 + \alpha y_1} = \frac{\sigma}{\alpha} x_1 \left[1 - \frac{x_1 + \alpha y_1}{\alpha\beta_2} + \sum_{n=2}^{\infty} (-1)^n \left(\frac{x_1 + \alpha y_1}{\alpha\beta_2}\right)^n\right]$$

$$\begin{aligned}
\dot{x}_1 &= \frac{\beta x_1}{\alpha\beta_2 + x_1 + \alpha y_1} - \gamma(\beta_2 + y_1)x_1 - \delta x_1 \\
&= \frac{\sigma}{\alpha} x_1 \left[1 - \frac{x_1 + \alpha y_1}{\alpha\beta_2} + \sum_{n=2}^{\infty} (-1)^n \left(\frac{x_1 + \alpha y_1}{\alpha\beta_2}\right)^n\right] - \gamma(\beta_2 + y_1)x_1 - \delta x_1 \\
&= \frac{\sigma}{\alpha} x_1 - (\frac{\sigma}{\alpha} x_1) \frac{x_1 + \alpha y_1}{\alpha\beta_2} - \gamma\beta_2 x_1 - \gamma y_1 x_1 - \delta x_1 + \frac{\sigma}{\alpha} x_1 \sum_{n=2}^{\infty} (-1)^n \left(\frac{x_1 + \alpha y_1}{\alpha\beta_2}\right)^n \\
&= \frac{\sigma}{\alpha} x_1 - \frac{\sigma^2}{\alpha^2\beta} x_1^2 - \frac{\sigma^2}{\alpha\beta} x_1 y_1 - \gamma\beta_2 x_1 - \gamma y_1 x_1 - \delta x_1 + \frac{\sigma}{\alpha} x_1 \sum_{n=2}^{\infty} (-1)^n \left(\frac{x_1 + \alpha y_1}{\alpha\beta_2}\right)^n \\
&= -\frac{\sigma^2}{\alpha^2\beta} x_1^2 - \left(\gamma + \frac{\sigma^2}{\alpha\beta}\right) x_1 y_1 + \left(-\gamma\beta_2 + \frac{\sigma}{\alpha} - \delta\right) x_1 + \frac{\sigma}{\alpha} x_1 \sum_{n=2}^{\infty} (-1)^n \left(\frac{x_1 + \alpha y_1}{\alpha\beta_2}\right)^n \\
&= -\frac{\sigma^2}{\alpha^2\beta} x_1^2 - \left(\frac{\sigma^2}{\alpha\beta} - \frac{\delta\sigma}{\beta} + \frac{\sigma^2}{\alpha\beta}\right) x_1 y_1 - \left(\frac{\sigma}{\alpha\beta}(\sigma - \alpha\delta)\frac{\beta}{\sigma} - \frac{\sigma}{\alpha} + \delta\right) x_1 \\
&\quad + \frac{\sigma}{\alpha} x_1 \sum_{n=2}^{\infty} (-1)^n \left(\frac{x_1 + \alpha y_1}{\alpha\beta_2}\right)^n \\
&= -\frac{\sigma^2}{\alpha^2\beta} x_1^2 + \left(\frac{\alpha\delta\sigma - 2\sigma^2}{\alpha\beta}\right) x_1 y_1 + \frac{\sigma}{\alpha} x_1 \sum_{n=2}^{\infty} (-1)^n \left(\frac{x_1 + \alpha y_1}{\alpha\beta_2}\right)^n
\end{aligned}$$

Let $y_2 = \delta x_1 + \sigma y_1$ and $x_1 = x_2$.

$$
\begin{aligned}
\dot{y}_2 &= \delta\dot{x}_1 + \sigma\dot{y}_1 \\
&= \delta\Bigg[-\frac{\sigma^2}{\alpha^2\beta}x_1^2 + \left(\frac{\delta\sigma - 2\sigma^2/\alpha}{\beta}\right)x_1y_1 + \frac{\sigma}{\alpha}x_1\sum_{n=2}^{\infty}(-1)^n\left(\frac{x_1+\alpha y_1}{\alpha\beta_2}\right)^n\Bigg] \\
&\quad + \sigma\Bigg[\frac{\sigma^2}{\alpha^2\beta}x_1^2 - \left(\frac{\delta\sigma - 2\sigma^2/\alpha}{\beta}\right)x_1y_1 - (\delta x_1 + \sigma y_1) - \frac{\sigma}{\alpha}x_1\sum_{n=2}^{\infty}(-1)^n\left(\frac{x_1+\alpha y_1}{\alpha\beta_2}\right)^n\Bigg] \\
&= \frac{\sigma^2}{\alpha^2\beta}(\sigma-\delta)x_2^2 + \left(\frac{\delta\sigma - 2\sigma^2/\alpha}{\beta}\right)(\delta-\sigma)\left(\frac{y_2-\delta x_2}{\sigma}\right)x_2 - \sigma y_2 \\
&\quad + (\delta-\sigma)\frac{\sigma}{\alpha}x_2\sum_{n=2}^{\infty}(-1)^n\left(\frac{x_2+\alpha\left(\frac{y_2-\delta x_2}{\sigma}\right)}{\alpha\beta/\sigma}\right)^n \\
&= \frac{\sigma^2}{\alpha^2\beta}(\sigma-\delta)x_2^2 + \left(\frac{\delta\sigma - 2\sigma^2/\alpha}{\beta}\right)(\delta-\sigma)\left(y_2x_2/\sigma - \delta/\sigma x_2^2\right) - \sigma y_2 \\
&\quad + (\delta-\sigma)\frac{\sigma}{\alpha}x_2\sum_{n=2}^{\infty}(-1)^n\left(\frac{(\sigma-\alpha\delta)x_2+\alpha y_2}{\alpha\beta}\right)^n \\
&= (\sigma-\delta)\left[\frac{\sigma^2}{\alpha^2\beta} + \left(\frac{\delta\sigma - 2\sigma^2/\alpha}{\beta}\right)\delta/\sigma\right]x_2^2 + \left(\frac{\delta\sigma - 2\sigma^2/\alpha}{\beta}\right)(\delta-\sigma)\left(y_2x_2/\sigma\right) - \sigma y_2 \\
&\quad + (\delta-\sigma)\frac{\sigma}{\alpha}x_2\sum_{n=2}^{\infty}(-1)^n\left(\frac{(\sigma-\alpha\delta)x_2+\alpha y_2}{\alpha\beta}\right)^n \\
&= \frac{\sigma-\delta}{\beta}\left(\sigma^2/\alpha^2 + \delta^2 - 2\sigma\delta/\alpha\right)x_2^2 + \left(\frac{\delta\sigma - 2\sigma^2/\alpha}{\beta\sigma}\right)(\delta-\sigma)y_2x_2 - \sigma y_2 \\
&\quad + (\delta-\sigma)\frac{\sigma}{\alpha}x_2\sum_{n=2}^{\infty}(-1)^n\left(\frac{(\sigma-\alpha\delta)x_2+\alpha y_2}{\alpha\beta}\right)^n \\
&= \frac{\sigma-\delta}{\beta}\left(\sigma/\alpha - \delta\right)^2 x_2^2 + \left(\frac{\delta - 2\sigma/\alpha}{\beta}\right)(\delta-\sigma)y_2x_2 - \sigma y_2 \\
&\quad + (\delta-\sigma)\frac{\sigma}{\alpha}x_2\sum_{n=2}^{\infty}(-1)^n\left(\frac{(\sigma-\alpha\delta)x_2+\alpha y_2}{\alpha\beta}\right)^n \\
&= \frac{(\sigma-\delta)(\sigma-\alpha\delta)^2}{\alpha^2\beta}x_2^2 + \frac{(\sigma-\delta)(2\sigma-\alpha\delta)}{\alpha\beta}y_2x_2 \\
&\quad - \sigma y_2 + (\delta-\sigma)\frac{\sigma}{\alpha}x_2\sum_{n=2}^{\infty}(-1)^n\left(\frac{(\sigma-\alpha\delta)x_2+\alpha y_2}{\alpha\beta}\right)^n
\end{aligned}
$$

and

$$
\begin{aligned}
\dot{x}_2 &= \dot{x}_1 \\
&= -\frac{\sigma^2}{\alpha^2\beta}x_2^2 + \left(\frac{\delta\sigma - 2\sigma^2/\alpha}{\beta}\right)\left(\frac{y_2 - \delta x_2}{\sigma}\right)x_2 + \frac{\sigma}{\alpha}x_2\sum_{n=2}^{\infty}(-1)^n\left(\frac{(\sigma - \alpha\delta)x_2 + \alpha y_2}{\alpha\beta}\right)^n \\
&= -\frac{(\delta - \sigma/\alpha)^2}{\beta}x_2^2 + \left(\frac{\delta\sigma - 2\sigma^2/\alpha}{\beta}\right)x_2y_2 + \frac{\sigma}{\alpha}x_2\sum_{n=2}^{\infty}(-1)^n\left(\frac{(\sigma - \alpha\delta)x_2 + \alpha y_2}{\alpha\beta}\right)^n
\end{aligned}
$$

Let

$$
p(x_2, y_2) := -\frac{(\sigma - \alpha\delta)^2}{\alpha^2\beta}x_2^2 + \left(\frac{\alpha\delta\sigma - 2\sigma^2}{\alpha\beta}\right)x_2y_2 + \frac{\sigma}{\alpha}x_2\sum_{n=2}^{\infty}(-1)^n\left(\frac{(\sigma - \alpha\delta)x_2 + \alpha y_2}{\alpha\beta}\right)^n
$$

and

$$
\begin{aligned}
q(x_2, y_2) := &\frac{(\sigma - \delta)(\sigma - \alpha\delta)^2}{\alpha^2\beta}x_2^2 + \frac{(\sigma - \delta)(2\sigma - \alpha\delta)}{\alpha\beta}y_2x_2 \\
&+ (\delta - \sigma)\frac{\sigma}{\alpha}x_2\sum_{n=2}^{\infty}(-1)^n\left(\frac{(\sigma - \alpha\delta)x_2 + \alpha y_2}{\alpha\beta}\right)^n
\end{aligned}
$$

then (2.0.1) can be written as

$$
\begin{cases}
\dot{x}_2 = p(x_2, y_2) \\
\dot{y}_2 = -\sigma y_2 + q(x_2, y_2)
\end{cases}
\tag{3.2.14}
$$

From our definition of p and q respectively we have that $a_{20} = -\frac{(\sigma-\alpha\delta)^2}{\alpha^2\beta} \neq 0$ since, $\sigma > \alpha\delta$. Moreover, $\varrho = -\sigma \neq 0$. Thus, by 3.0.2 the result follows.

□

3.3 Analysis of the Positive Interior Equilibrium

Consider the interior equilibrium (x^*, y^*)

Theorem 3.3.1. If $\alpha > 0$, $\beta > 0$, $\delta > 0$, $\sigma > \alpha\delta$ and $\gamma_0 < \gamma < \gamma_3$, then (x^*, y^*) is locally asymptotically stable.

Proof. By (2.0.9) and (2.0.12) we have the following expressions

$$\begin{aligned} \sigma y^* - \delta x^* &= \frac{\beta}{\gamma(\sigma-\alpha\delta)}\left[\sigma\left(\gamma - \frac{\delta}{\beta}(\sigma-\alpha\delta)\right) - \alpha\delta\left(\frac{\sigma}{\alpha\beta}(\sigma-\alpha\delta) - \gamma\right)\right] \\ &= \frac{\beta}{\gamma(\sigma-\alpha\delta)}\left[(\sigma+\alpha\delta)\gamma - \frac{\sigma\delta}{\beta}(\sigma-\alpha\delta) - \frac{\delta\sigma}{\beta}(\sigma-\alpha\delta)\right] \\ &= \frac{\beta}{\gamma(\sigma-\alpha\delta)}\left((\sigma+\alpha\delta)\gamma - \frac{2\sigma\delta}{\beta}(\sigma-\alpha\delta)\right) \end{aligned} \tag{3.3.1}$$

$$
\begin{aligned}
\sigma y^* + \delta x^* &= \frac{\beta}{\gamma(\sigma - \alpha\delta)}\left[\sigma\left(\gamma - \frac{\delta}{\beta}(\sigma - \alpha\delta)\right) + \alpha\delta\left(\frac{\sigma}{\alpha\beta}(\sigma - \alpha\delta) - \gamma\right)\right] \\
&= \frac{\beta}{\gamma(\sigma - \alpha\delta)}\left[(\sigma - \alpha\delta)\gamma - \frac{\sigma\delta}{\beta}(\sigma - \alpha\delta) + \frac{\delta\sigma}{\beta}(\sigma - \alpha\delta)\right] \\
&= \frac{\beta}{\gamma(\sigma - \alpha\delta)}\left[(\sigma - \alpha\delta)\gamma\right] \\
&= \beta
\end{aligned}
\tag{3.3.2}
$$

$$
x^* + \alpha y^* = \frac{1}{\gamma}\left(\sigma - \frac{\alpha\beta\gamma}{\sigma - \alpha\delta} + \frac{\alpha\beta\gamma}{\sigma - \alpha\delta} - \alpha\delta\right) = \frac{1}{\gamma}(\sigma - \alpha\delta)
$$

$$
\frac{\alpha\beta}{(x^* + \alpha y^*)^2} = \frac{\alpha\beta\gamma^2}{(\sigma - \alpha\delta)^2} \tag{3.3.3}
$$

By (3.0.8) and substituting (2.0.8), (2.0.12) and (3.3.3) we obtain the following

$$
\begin{aligned}
\operatorname{tr}(A(x^*,y^*)) &= \left(\gamma + \frac{\alpha\beta}{(x+\alpha y)^2}\right)x + \left(\frac{\alpha\beta}{(x+\alpha y)^2} - \gamma\right)y - \sigma - \delta \\
&= \frac{\alpha\beta}{(x+\alpha y)^2}(x+y) + \gamma(x-y) - \sigma - \delta \\
&= \frac{\alpha\beta\gamma}{(\sigma-\alpha\delta)^2}\left(\sigma - \frac{\alpha\beta\gamma}{\sigma-\alpha\delta} + \frac{\beta\gamma}{\sigma-\alpha\delta} - \delta\right) + \sigma - \frac{\alpha\beta\gamma}{\sigma-\alpha\delta} - \frac{\beta\gamma}{\sigma-\alpha\delta} + \delta - \sigma - \delta \\
&= \frac{\alpha\beta\gamma}{(\sigma-\alpha\delta)^2}\left(\sigma - \delta - \frac{\beta\gamma}{\sigma-\alpha\delta}(\alpha-1)\right) - \frac{\beta\gamma}{\sigma-\alpha\delta}(\alpha+1) \\
&= \frac{\alpha\beta\gamma}{(\sigma-\alpha\delta)^2}\left[\sigma - \delta - \frac{\beta\gamma}{\sigma-\alpha\delta}(\alpha-1) - \frac{(\sigma-\alpha\delta)}{\alpha}(\alpha+1)\right] \\
&= \frac{\alpha\beta^2\gamma}{(\sigma-\alpha\delta)^3}\left[\frac{(\sigma-\alpha\delta)}{\beta}\left(\sigma - \delta - \frac{(\sigma-\alpha\delta)}{\alpha}(\alpha+1)\right) - \gamma(\alpha-1)\right] \\
&= \frac{\alpha\beta^2\gamma}{(\sigma-\alpha\delta)^3}\left[\frac{(\sigma-\alpha\delta)}{\alpha\beta}\left(\alpha\sigma - \alpha\delta - \alpha\sigma + \alpha^2\delta - \sigma + \alpha\delta\right) - \gamma(\alpha-1)\right] \\
&= \frac{\alpha\beta^2\gamma}{(\sigma-\alpha\delta)^3}\left[\frac{(\sigma-\alpha\delta)}{\alpha\beta}\left(\alpha^2\delta - \sigma\right) - \gamma(\alpha-1)\right] \\
&= \frac{\alpha\beta^2\gamma}{(\sigma-\alpha\delta)^3}\left[\alpha\frac{\delta}{\beta}(\sigma-\alpha\delta) - \frac{\sigma}{\alpha\beta}(\sigma-\alpha\delta) - \gamma(\alpha-1)\right] \\
&= \frac{\alpha\beta^2\gamma}{(\sigma-\alpha\delta)^3}\left[\alpha\gamma_0 - \gamma_3 - \alpha\gamma + \gamma\right] \\
&= \frac{\alpha\beta^2\gamma}{(\sigma-\alpha\delta)^3}\left[\alpha(\gamma_0 - \gamma) + \gamma - \gamma_3\right]
\end{aligned} \tag{3.3.4}
$$

Let

$$
\tau = \frac{\alpha\beta^2}{(\sigma-\alpha\delta)^2}
$$

By (3.0.7) and substituting (3.3.1) and (3.3.2) and (3.3.3) we obtain the following

$$
\begin{aligned}
|A(x^*,y^*)| &= \gamma(\sigma y^* - \delta x^*) - \frac{\alpha\beta}{(x^*+\alpha y^*)^2}(\delta x^* + \sigma y^*) + \delta\sigma \\
&= \frac{\beta}{(\sigma-\alpha\delta)}\left((\sigma+\alpha\delta)\gamma - \frac{2\sigma\delta}{\beta}(\sigma-\alpha\delta)\right) - \frac{\alpha\beta^2}{(\sigma-\alpha\delta)^2}\gamma^2 + \delta\sigma \\
&= \frac{\alpha\beta^2}{(\sigma-\alpha\delta)^2}\left[-\gamma^2 + \frac{(\sigma-\alpha\delta)}{\alpha\beta}\left((\sigma+\alpha\delta)\gamma - \frac{2\sigma\delta}{\beta}(\sigma-\alpha\delta)\right) + \delta\sigma\frac{(\sigma-\alpha\delta)^2}{\alpha\beta^2}\right] \\
&= \tau\left[-\gamma^2 + \frac{\sigma^2-\alpha^2\delta^2}{\alpha\beta}\gamma - \frac{2\sigma\delta}{\alpha\beta^2}(\sigma-\alpha\delta)^2 + \delta\sigma\frac{(\sigma-\alpha\delta)^2}{\alpha\beta^2}\right] \\
&= \tau\left[-\gamma^2 + \frac{\sigma^2-\alpha^2\delta^2}{\alpha\beta}\gamma - \delta\sigma\frac{(\sigma-\alpha\delta)^2}{\alpha\beta^2}\right] \\
&= -\tau\left[\gamma^2 - \frac{\sigma^2-\alpha^2\delta^2}{\alpha\beta}\gamma + \delta\sigma\frac{(\sigma-\alpha\delta)^2}{\alpha\beta^2}\right] \\
&= -\tau\left[\left(\gamma - \frac{\sigma^2-\alpha^2\delta^2}{2\alpha\beta}\right)^2 - \frac{(\sigma-\alpha\delta)^2(\sigma+\alpha\delta)^2}{4\alpha^2\beta^2} + \delta\sigma\frac{(\sigma-\alpha\delta)^2}{\alpha\beta^2}\right] \\
&= -\tau\left[\left(\gamma - \frac{\sigma^2-\alpha^2\delta^2}{2\alpha\beta}\right)^2 + \frac{(\sigma-\alpha\delta)^2}{4\alpha^2\beta^2}\left(4\alpha\delta\sigma - (\sigma+\alpha\delta)^2\right)\right] \\
&= -\tau\left[\left(\gamma - \frac{\sigma^2-\alpha^2\delta^2}{2\alpha\beta}\right)^2 - \frac{(\sigma-\alpha\delta)^4}{4\alpha^2\beta^2}\right] \\
&= -\tau\left(\gamma - \frac{\sigma}{\alpha\beta}(\sigma-\alpha\delta)\right)\left(\gamma - \frac{\delta}{\beta}(\sigma-\alpha\delta)\right) \\
&= -\tau(\gamma-\gamma_3)(\gamma-\gamma_0) \\
&= \tau(\gamma_3-\gamma)(\gamma-\gamma_0)
\end{aligned}
\tag{3.3.5}
$$

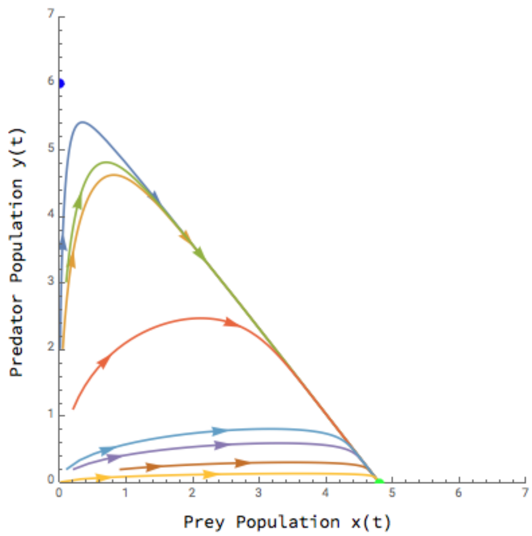
From Theorem 2.0.1 we have (x^*,y^*) is a positive equilibrium point of (2.0.1) under the conditions $\sigma - \alpha\delta > 0$ and $\gamma_0 < \gamma < \gamma_3$. Hence, by (3.3.4) and (3.3.5) we have $\mathrm{tr}(A(x^*,y^*)) < 0$ and $|A(x^*,y^*)| > 0$ respectively. It follows that by Lemma 3.0.1 *(iv)*, the interior equilibrium (x^*,y^*) is locally asymptotically stable. $\square$

Chapter 4

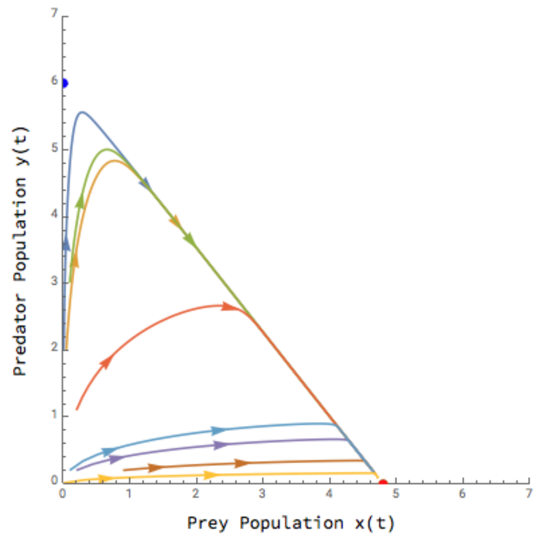
Numerical Simulations

4.1 Discussion

Among predator prey models, my focus was on one that exhibits intraguild predation. However, there are models that incorporate different behaviors observed in nature that affect the balance. For example, the Holling type IV functional response which describes the interaction between predator and prey when the prey exhibit group defense. A harvesting rate could be added to a model in which either or both species are subject to capturing. Seasonal changes in the environment affect the population dynamics. Local extinctions could be balanced by remigration, then a model would have to incorporate a migration rate. The Allee effect refers to a reduction in individual fitness at low population density. A combination of some of these behaviors have been analyzed in models before. But realistically, there is a point where adding complexity to the model loses its value because most of the data you

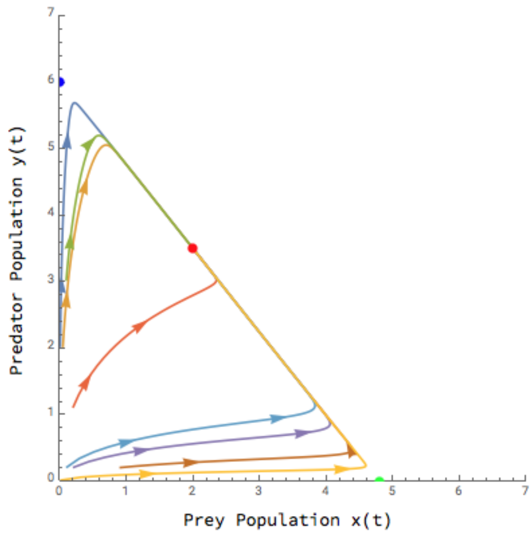


(a) $(x_1, y_1) = (4.8, 0)$ is a stable node of (2.0.1), where $\alpha = 0.5$, $\beta = 1.2$, $\delta = 0.25$, $\sigma = .2$, and $\gamma < \gamma_0$

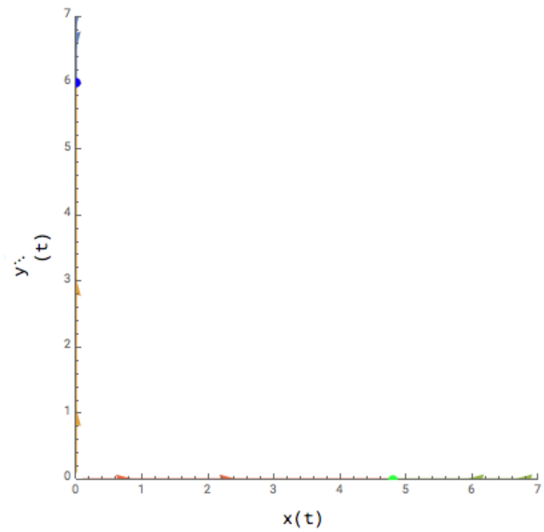


(b) $(x_1, y_1) = (4.8, 0)$ is a saddle node of (2.0.1), where $\alpha = 0.5$, $\beta = 1.2$, $\delta = 0.25$, $\sigma = .2$, and $\gamma = \gamma_0$

Figure 4.1: The phase portraits of (2.0.1) for different values of γ. Figures 4.1a and 4.1b satisfy the conditions of Theorem 3.1.3 (2) and (3), respectively. In particular, figures 4.1a and 4.1b show that the boundary equilibrium (x_1, y_1) can be a stable-node or a saddle-node respectively, depending on γ, β, α and σ and δ.

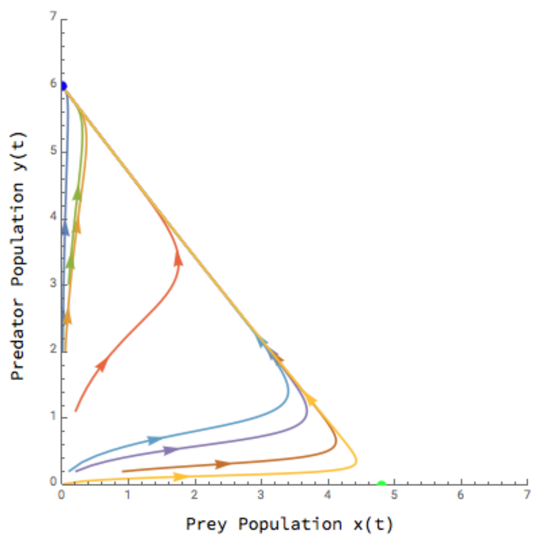


(a) (x^*, y^*) is a stable node of (2.0.1), where $\alpha = 0.5$, $\beta = 1.2$, $\delta = 0.25$, $\sigma = .2$, and $\gamma_0 < \gamma < \gamma_3$

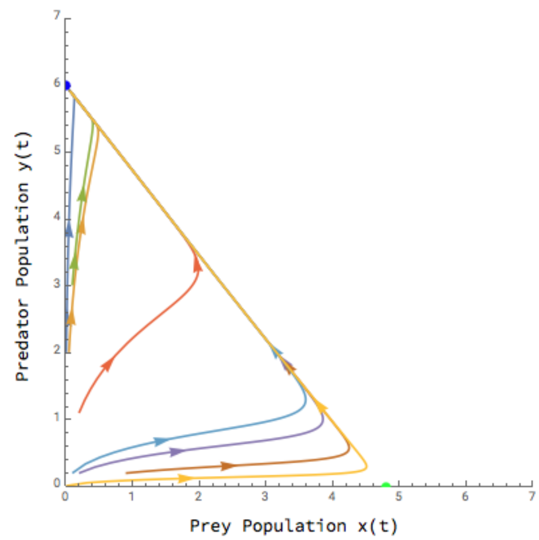


(b) (x_1, y_1) and (x_2, y_2) are both saddles of (2.0.1), where $\alpha = 0.5$, $\beta = 1.2$, $\delta = 0.25$, $\sigma = .2$, and $\gamma_0 < \gamma < \gamma_3$.

Figure 4.2: The above graphs show that the positive interior equilibrium is locally asymptotically stable for non zero initial conditions. The parameters σ, β, γ, α and δ were chosen such that they satisfy Theorem 3.3.1. Note that they also satisfy Theorem 3.1.3 (1) and Theorem 3.2.3 (1) shown in figure b) with $x = 0$ or $y = 0$ for initial conditions.



(a) $(x_2, y_2) = (0, 6)$ is a stable node of (2.0.1), where $\alpha = 0.5$, $\beta = 1.2$, $\delta = 0.25$, $\sigma = .2$, and $\gamma > \gamma_3$



(b) $(x_2, y_2) = (0, 6)$ is a saddle node of (2.0.1), where $\alpha = 0.5$, $\beta = 1.2$, $\delta = 0.25$, $\sigma = .2$, and $\gamma = \gamma_3$.

Figure 4.3: The phase portraits of (2.0.1) for different values of γ. Figures 4.3a and 4.3b satisfy the conditions of Theorem 3.2.3 (2) and (3), respectively. In particular, figures 4.3a and 4.3b show that the boundary equilibrium (x_2, y_2) can be a stable-node or a saddle-node respectively, depending on γ, β, α and σ and δ.

would need to fit the model can not be collected, even if the mathematics is possible to analyze with computational methods. In this thesis the model represents, a natural environment where the main population changes at the time are due to competition for recourses and predator prey exploitation. In the presence of limited resources, relatively small populations will increase, whereas an excessively large population will have insufficient resources to survive As we might expect when $\alpha = 1$. the system is symmetric because α represents the ratio that measures the abilities of the two species to compete for resources. As γ moves through the ranges of intervals the interior equilibrium travels linearly from the the equilibrium (x_1, y_1) to (x_2, y_2). When the interior equilibrium meets either of the boundary equilibrium it looks like a pitchfork bifurcation. We are only taking positive solutions so we are not seeing this in the numerical simulations.

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